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CHAPTER VI

Teichmüller Spaces

A. Preliminaries

Let S be a Riemann surface whose universal covering $\widetilde{S}$ is conformally isomorphic to the upper halfplane H. The cover transformations of $\widetilde{S}$ over S are represented by linear transformations of H upon itself which form a discontinuous subgroup Γ of the group Ω of all such transformations. We can write

$$S = \Gamma\backslash H \text{ (orbits)}$$

and the canonical mapping

$$\pi \colon H \to \Gamma\backslash H$$

is a complex analytic projection of H on S.

Conjugate subgroups represent conformally equivalent Riemann surfaces. Indeed, if $\Gamma_0 = B_0\Gamma B_0^{-1}$, $B_0 \in \Omega$ then $z \to B_0 z$ maps orbits of Γ on orbits of Γ_0 (for $B_0 A z = (B_0 A B_0^{-1}) B_0 z$). Therefore B_0 determines a one-one conformal mapping of $S_0 = \Gamma_0\backslash H$ on S.

Conversely, let there be given a topological mapping

$$g \colon S_0 \to S.$$

It can be lifted to a topological mapping $\tilde{g} \colon \widetilde{S}_0 \to \widetilde{S}$ which obviously satisfies

$$\pi \circ \tilde{g} = g \circ \pi_0.$$

$$\begin{array}{ccc} H & \xrightarrow{\ \tilde{g}\ } & H \\ {\scriptstyle \pi_0}\big| & & \big|{\scriptstyle \pi} \\ S_0 & \xrightarrow{\ g\ } & S \end{array}$$

If g is conformal, so is $\tilde{g}$, and we have $\tilde{g} = B_0 \in \Omega$ and $\Gamma_0 = B_0\Gamma B_0^{-1}$.

The classes of conformally equivalent Riemann surfaces correspond to classes of conjugate discontinuous subgroups of Ω (without elliptic fixpoints).

But even if g is not conformal, it is still true that

$$A = \tilde{g} \circ A_0 \circ \tilde{g}^{-1} \in \Gamma$$

whenever $A_0 \in \Gamma_0$, for

$$\begin{aligned} \pi \circ A &= \pi \circ \tilde{g} \circ A_0 \circ \tilde{g}^{-1} \\ &= g \circ \pi_0 \circ A_0 \circ \tilde{g}^{-1} \\ &= g \circ \pi_0 \circ \tilde{g}^{-1} \\ &= \pi. \end{aligned}$$

In other words, $\tilde{g}$ defines an isomorphism θ such that

$$A_0^\theta = \tilde{g} \circ A_0 \circ \tilde{g}^{-1}.$$

It is not quite unique, for we may replace $\tilde{g}$ by $B \circ \tilde{g} \circ B_0$ where $B \in \Gamma$, $B_0 \in \Gamma_0$. This changes θ into θ' with

$$A_0^{\theta'} = B \circ \tilde{g} \circ (B_0 A_0 B_0^{-1}) \circ \tilde{g}^{-1} \circ B^{-1}$$

which means that we compose θ with inner automorphisms of Γ_0 and Γ. We say that θ and θ' are equivalent isomorphisms.

LEMMA. *g_1 and g_2 determine equivalent isomorphisms θ_1 and θ_2 if and only if they are homotopic.*

PROOF. If g_1, g_2 are homotopic they can be deformed into each other via $g(t)$, say, which depends continuously on t. We can then find $\tilde{g}(t)$ so that it varies continuously with t, and since

$$A_0^{\theta(t)}(z) = \tilde{g}(t) \circ A_0 \circ \tilde{g}(t)^{-1}$$

has values in a discrete set it must actually be constant.

Conversely, suppose g_1, g_2 determine equivalent θ_1, θ_2. By changing $\tilde{g}_1, \tilde{g}_2$ we may suppose that $\theta_1 = \theta_2$, and hence

$$\tilde{g}_2^{-1} \tilde{g}_1 A_0 = A_0 \tilde{g}_2^{-1} \tilde{g}_1.$$

Define $\tilde{g}(t, z)$ as the point which divides the noneuclidean line segment between $\tilde{g}_1(z)$ and $\tilde{g}_2(z)$ in the ratio $t : (1 - t)$.

Because

$$\begin{aligned} \tilde{g}_1(A_0 z) &= A\tilde{g}_1(z) \\ \tilde{g}_2(A_0 z) &= A\tilde{g}_2(z) \end{aligned} \quad (A = A_0^\theta)$$

it follows that

$$\tilde{g}(t, A_0 z) = A\tilde{g}(t, z).$$

Hence $g(t) = \pi \circ \tilde{g}(t) \circ \pi_0^{-1}$ is a mapping from S_0 to S, and we have shown that g_1 and g_2 are homotopic.

Definition of $T(S_0)$ (the Teichmüller space):

Consider all pairs (S, f) where S is a Riemann surface and f is a sense-preserving q.c. mapping of S_0 onto S. We say that $(S_1, f_1) \sim (S_2, f_2)$ if $f_2 \circ f_1^{-1}$ is homotopic to a conformal mapping of S_1 on S_2. The equivalence classes are the *points* of $T(S_0)$, and (S_0, I) is called the initial point of $T(S_0)$.

Every f determines a q.c. mapping $\tilde{f}$ of H on itself, and thereby an isomorphism θ of Γ_0. Two isomorphisms correspond to the same Teichmüller point if and only if they differ by an inner automorphism of Ω.

The space $T(S_0)$ has a natural Teichmüller metric: the distance of (S_1, f_1), (S_2, f_2) is $\log K$ where K is the smallest maximal dilatation of a mapping homotopic to $f_2 \circ f_1^{-1}$.

Let us compare $T(S_0)$ and $T(S_1)$. Let g be a q.c. mapping of S_0 on S_1. The mapping

$$(S, f) \to (S, f \circ g)$$

induces a mapping of $T(S_1)$ onto $T(S_0)$. Indeed, if $(S, f) \sim (S', f')$, then $(S, f \circ g) \sim (S', f' \circ g)$. This mapping is clearly isometric.

B. Beltrami Differentials

A q.c. mapping $f\colon S_0 \to S$ induces a mapping $\tilde{f}$ of H on itself which satisfies

$$\tilde{f} \circ A_0 = A \circ \tilde{f} \tag{1}$$

for $A = A_0^\theta$. Conversely, if $\tilde{f}$ satisfies (1) it induces a mapping f.*

From (1) we obtain

$$(A' \circ f) f_z = (f_z \circ A_0) A_0'$$
$$(A' \circ f) f_{\bar{z}} = (f_{\bar{z}} \circ A_0) \overline{A_0'}$$

and thus the complex dilatation μ_f satisfies

$$\mu_f = (\mu_f \circ A_0) \overline{A_0'} / A_0'$$

or

$$\mu(A_0 z) = \mu(z) A_0'(z) / \overline{A_0'(z)}. \tag{2}$$

A measurable and essentially bounded function μ which satisfies (2) for all $A_0 \in \Gamma_0$ is called a *Beltrami differential* with respect to Γ_0. Another way to express the condition is to say that

$$\mu(z) \frac{d\bar{z}}{dz}$$

is invariant under Γ_0.

Conversely, if μ_f does satisfy (2), then

$$\mu_{f \circ A_0} = \mu_f$$

and it follows that $f \circ A_0$ is an analytic function of f, or that

$$A = f \circ A_0 \circ f^{-1}$$

is analytic, and hence a linear transformation.

The linear space of Beltrami differentials will be denoted by $B(\Gamma_0)$ and its open unit ball with respect to the L^∞ norm is denoted by $B_1(\Gamma_0)$.

For every $\mu \in B_1(\Gamma_0)$ we know that there exists a corresponding f^μ which maps H on itself. We normalize it so that it leaves $0, 1, \infty$ fixed. It is then unique.

We set

$$A^\mu = f^\mu \circ A_0 \circ (f^\mu)^{-1}$$

and write Γ^μ for the corresponding group, θ^μ for the isomorphism. Since θ^μ represents a point in Teichmüller space we have actually defined a mapping

$$B_1(\Gamma_0) \to T(S_0).$$

It is clearly continuous from L^∞ to the Teichmüller metric.

There is an obvious equivalence relation: $\mu_1 \sim \mu_2$ if θ^{μ_1} and θ^{μ_2} are equivalent isomorphisms. It is very difficult to recognize this equivalence by direct comparison of μ_1 and μ_2. Because we cannot solve the global problem we shall be content to solve the local problem for infinitesimal deformations.

Before embarking on this road we shall discuss a different approach which has several advantages. The mapping f^μ was obtained by extending μ to the lower halfplane by symmetry. If instead we extend μ to be identically zero in the lower halfplane we get a new mapping that we shall call f_μ (again normalized by fixpoints at 0, 1, ∞).

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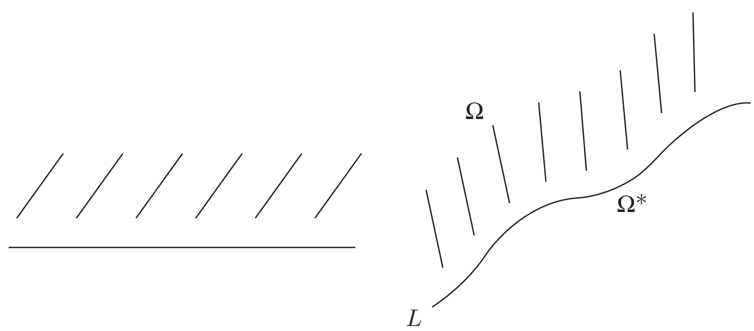
* For typographical simplicity both mappings will henceforth be denoted by f.

Clearly, f_μ gives a q.c. mapping of the upper halfplane and a conformal mapping of the lower halfplane. The real axis is mapped on a line L that permits a q.c. reflection.



It is again true that

$$A_\mu = f_\mu \circ A_0 \circ f_\mu^{-1}$$

is conformal, and hence a linear transformation (it is conformal in Ω and Ω^*, and it is q.c., hence conformal). We get a new group Γ_μ which is discontinuous on $\Omega \cup \Omega^*$. We call it a Fuchsoid group. We also get a surface $\Gamma_\mu \backslash \Omega = S$ and a q.c. mapping $S_0 \to S$ as well as a conformal mapping $\overline{S_0} \to \Gamma_\mu \backslash \Omega^*$ where $\overline{S_0}$ has the conjugate complex structure of S_0.

Observe that f^μ and f_μ are defined for all $\mu \in L^\infty$ with $\|\mu\|_\infty < 1$ ($\mu \in B_1$) even when the group Γ_0 reduces to the trivial group. Our first result is

LEMMA 1. *$f^\mu = f^\nu$ on the real axis if and only if $f_\mu = f_\nu$ on the real axis, and hence in H^*.*

PROOF. 1) If $f_\mu = f_\nu$ on the real axis, then the regions $f_\mu(H)$ and $f_\nu(H)$ are the same and therefore

$$f_\mu \circ (f^\mu)^{-1} = f_\nu \circ (f^\nu)^{-1},$$

for both are normalized conformal mappings of H on the same region.

2) Suppose $f^\mu = f^\nu$ on the real axis. The mapping $h = (f^\nu)^{-1} \circ f^\mu$ reduces to the identity on the real axis, so h can be extended to a q.c. mapping of the whole plane by putting $h(z) = z$ in H^*. Consider the q.c. mapping $A = f_\nu \circ h \circ (f_\mu)^{-1}$. In $f_\mu(H^*)$, $A = f_\nu \circ (f_\mu)^{-1}$ is conformal. In $f_\mu(H)$,

$$A = f_\nu \circ (f^\nu)^{-1} \circ f^\mu \circ (f_\mu)^{-1}$$

is conformal. Thus A is a linear transformation, and the normalization makes it the identity. This means $f_\nu = f_\mu$ in H^*. □

We shall now make the assumption that Γ_0 is *of the first kind*, which means that it is not discontinuous at any point on the real axis. It is then known that the orbits on the real axis are dense. In particular the fixpoints are dense. Under these conditions we can prove

LEMMA 2. *μ_1 and μ_2 in $B_1(\Gamma_0)$ determine the same Teichmüller point if and only if $f^{\mu_1} = f^{\mu_2}$ on the real axis.*

PROOF. If $f^{\mu_1} = f^{\mu_2}$ on the real axis, then $A^{\mu_1} = A^{\mu_2}$ on the real axis and hence identically. Therefore $\theta^{\mu_1} = \theta^{\mu_2}$, and μ_1 and μ_2 determine the same Teichmüller point.

Conversely, suppose θ^{μ_1} and θ^{μ_2} are equivalent isomorphisms. This means there is a linear transformation $S \in \Omega$ such that

$$A^{\mu_2} \circ S = S \circ A^{\mu_1} \text{ for all } A \text{ in } \Gamma_0.$$

We conclude that S maps the fixpoints of A^{μ_1} on the fixpoints of A^{μ_2} (attractive on attractive). Since they correspond to each other we see that

$$S \circ f^{\mu_1} = f^{\mu_2}$$

on the real axis. By the normalization, S must be the identity. □

COROLLARY. *If Γ_0 is of the first kind, μ_1 and μ_2 determine the same Teichmüller point if and only if*

$$f_{\mu_1} = f_{\mu_2} \text{ in } H^*.$$

This means we may *identify* the Teichmüller space $T(S_0)$ with the space of conformal mappings of H^* of the form f_μ, $\mu \in B_1(\Gamma_0)$.

An even better characterization is by consideration of the Schwarzian derivative

$$\{f_\mu, z\} = \frac{f_\mu'''}{f_\mu'} - \frac{3}{2}\left(\frac{f_\mu''}{f_\mu'}\right)^2.$$

Let us recall its properties with respect to composition. Consider

$$F(z) = f(\zeta(z))$$

and let primes mean differentiation. We get

$$\begin{aligned} F'(z) &= f'(\zeta)\zeta'(z), \\ \frac{F''}{F'} &= \frac{f''(\zeta)}{f'(\zeta)}\zeta' + \frac{\zeta''}{\zeta'}, \\ \frac{F'''}{F'} - \left(\frac{F''}{F'}\right)^2 &= \left(\frac{f'''(\zeta)}{f'(\zeta)} - \left(\frac{f''(\zeta)}{f'(\zeta)}\right)^2\right)\zeta'^2 \\ &\qquad + \frac{f''(\zeta)}{f'(\zeta)}\zeta'' + \frac{\zeta'''}{\zeta'} - \left(\frac{\zeta''}{\zeta'}\right)^2, \\ \{F, z\} &= \{f, \zeta\}\zeta'(z)^2 + \{\zeta, z\}. \end{aligned}$$

For a better formulation, let us denote the Schwarzian by $[f]$. The formula reads

$$[f \circ g] = ([f] \circ g)(g')^2 + [g].$$

There are two special cases. For $f = A$, a linear transformation,

$$[A \circ g] = [g]$$

and for $g = A$

$$[f \circ A] = ([f] \circ A)(A')^2.$$

On setting $\phi_\mu = [f_\mu]$,

$$\begin{aligned} (\phi_\mu \circ A)A'^2 &= [f_\mu \circ A] = [A_\mu \circ f_\mu] \\ &= [f_\mu] = \phi_\mu. \end{aligned}$$

We see that ϕ_μ satisfies

$$(\phi_\mu \circ A)(A')^2 = \phi_\mu$$

which makes it a *quadratic differential* ($\phi\, dz^2$ is invariant).

The following theorem is due to Nehari:

LEMMA 3. *If f is schlicht in the halfplane H^*, then* $|[f]| \leq \frac{3}{2} y^{-2}$.

PROOF. Suppose that $F(\zeta) = \zeta + \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} + \cdots$ is schlicht for $|\zeta| > 1$. The integral $\frac{1}{2i}\int_{|\zeta|=r} \overline{F}\, dF$ measures the area enclosed by the image of $|\zeta| = r$, and is therefore positive. One computes

$$\begin{aligned}\frac{1}{2i}\int_{|\zeta|=r} \overline{F}\, dF &= \frac{1}{2i}\int \left(\overline{\zeta} + \frac{\overline{b}_1}{\overline{\zeta}} + \cdots\right)\left(1 - \frac{b_1}{\zeta^2} - \cdots\right) d\zeta \\ &= \pi\left(r^2 - \frac{|b_1|^2}{r^2} - \cdots\right).\end{aligned}$$

It follows that $|b_1| \leq 1$. (More economically, $|b_1|^2 + 2|b_2|^2 + \cdots + n|b_n|^2 + \cdots \leq 1$, which is Bieberbach's Flächensatz.)

Note that

$$\begin{aligned}F' &= 1 - \frac{b_1}{\zeta^2} + \cdots \\ F'' &= \frac{2b_1}{\zeta^3} + \cdots \\ F''' &= -\frac{6b_1}{\zeta^4} + \cdots\end{aligned}$$

gives $[F] = -\frac{6b_1}{\zeta^4} + \cdots$ and hence

$$\lim_{\zeta\to\infty} |\zeta^4[F]| \leq 6.$$

Set $\zeta = Uz = (z - \overline{z}_0)/(z - z_0)$, $z_0 = x_0 + iy_0$, $y_0 < 0$. Consider $F(\zeta) = f(U^{-1}\zeta)$. Then

$$[f] = ([F] \circ U)U'^2.$$

Here

$$\begin{aligned}U' &= \frac{-2iy_0}{(z - z_0)^2}, \\ U &\sim \frac{2iy_0}{z - z_0} \quad (z \to z_0) \\ U'^2 &\sim -\frac{1}{4y_0^2} U^4 \quad (z \to z_0).\end{aligned}$$

On going to the limit we find

$$[f](z_0) = -\frac{1}{4y_0^2} \lim [F] \cdot \zeta^4$$

and then

$$|[f]| \leq \frac{3}{2}\frac{1}{y^2}. \quad \square$$

In view of the lemma it is natural to define a norm on the quadratic differentials by

$$\|\phi\| = \sup |\phi(z)| y^2.$$

C. Δ Is Open

We have defined a mapping $\mu \to \phi_\mu$ from the unit ball $B_1(\Gamma)$ to the space $Q(\Gamma)$ of quadratic differentials with finite norm. The image of $B_1(\Gamma)$ under this mapping will be denoted by $\Delta(\Gamma)$. It is our aim to show that $\Delta(\Gamma)$ is an open subset of $Q(\Gamma)$.

The question makes sense even in the case where Γ is the trivial group consisting only of the identity. In this case the spaces will be denoted by B_1, Δ, Q.

THEOREM 1. *Δ is an open subset of Q.*

Clearly, Δ consists of the Schwarzians $[f]$ of functions f which are schlicht holomorphic in the lower halfplane and have a q.c. extension to the upper halfplane. We know already that all ϕ in Δ satisfy $\|\phi\| \le 3/2$.

LEMMA 1. *Every holomorphic ϕ with $\|\phi\| < \frac{1}{2}$ is in Δ.*

PROOF. We choose two linearly independent solutions of the differential equation

$$\eta'' = -\tfrac{1}{2}\phi\eta \tag{1}$$

which we may normalize by $\eta_1'\eta_2 - \eta_2'\eta_1 = 1$. It is easy to check that $f = \eta_1/\eta_2$ satisfies $[f] = \phi$. Observe that the solutions of (1) have at most simple zeros. Hence f has at most simple poles, and at other points $f' \neq 0$.

We want to show that f is schlicht and has a q.c. extension to the upper halfplane. To construct the extension we consider

$$F(z) = \frac{\eta_1(z) + (\overline{z} - z)\eta_1'(z)}{\eta_2(z) + (\overline{z} - z)\eta_2'(z)}, \qquad (z \in H^*).$$

We remark first that the numerator and denominator do not vanish simultaneously (because $\eta_1'\eta_2 - \eta_2'\eta_1 = 1$). Therefore F is defined everywhere, but could be ∞.

Simple computations yield

$$\begin{aligned} F_{\overline{z}} &= \frac{1}{(\eta_2 + (\overline{z} - z)\eta_2')^2} \\ F_z &= \frac{\frac{1}{2}\phi(\overline{z} - z)^2}{(\eta_2 + (\overline{z} - z)\eta_2')^2} \end{aligned} \tag{2}$$

and thus

$$\frac{F_z}{F_{\overline{z}}} = \frac{1}{2}\phi(\overline{z} - z)^2.$$

The assumption implies $|F_z| \le k|F_{\overline{z}}|$ for some $k < 1$. Hence F is q.c. but sense-reversing.

The extension will be defined by

$$\hat{f}(z) = \begin{cases} f(z) & z \in H^* \\ F(\overline{z}) & z \in H. \end{cases} \tag{3}$$

It must be shown that $\hat{f}$ is a q.c. 1-1 mapping.

This is easy if ϕ is very regular. Let us assume, specifically, that ϕ remains analytic on the real axis, and that it has a zero of at least order 4 at ∞. It is immediate that f and F agree on the real axis, and it is also evident that $\hat{f}$ is locally schlicht. The assumption at ∞ means that there are solutions of (1) whose power series expansions at ∞ being with 1 and z respectively. We have thus

$$\eta_1 = a_1 z + b_1 + O\left(\frac{1}{|z|}\right)$$
$$\eta_2 = a_2 z + b_2 + O\left(\frac{1}{|z|}\right)$$

with $a_1 b_2 - a_2 b_1 = 1$. This gives

$$F(z) = \frac{a_1\overline{z} + b_1 + O(|z|^{-1})}{a_2\overline{z} + b_2 + O(|z|^{-1})} \to \frac{a_1}{a_2}$$

which is also the limit of f.

Now the schlichtness of $\hat{f}$ follows by the monodromy theorem. We may of course, compose $\hat{f}$ with a linear transformation to make it normalized.

To prove the general case we use approximation. Put $S_n z = (2nz-i)/(iz+2n)$. Then $S_n H^* \Subset H^*$ and $S_n z \to z$ for $n \to \infty$. Set $\phi_n(z) = \phi(S_n z) S_n'(z)^2$. We have

$$y^2|\phi_n(z)| = |\phi(S_n z)|\,|S_n'(z)|^2 y^2 < |\phi(S_n z)|(\operatorname{Im} S_n(z))^2$$

and hence $\|\phi_n\| \le \|\phi\|$. Now ϕ_n has all the regularity properties. We can find mappings $\hat{f}_n$ with $[\hat{f}_n] = \phi_n$ in H^* and uniformly bounded dilatations. By compactness there exists a subsequence which converges to a solution $\hat{f}_0$ of the original problem.

If $\phi_n \to \phi$ it is not hard to see that a normalized solution of $\eta'' = -\frac{1}{2}\phi_n\eta$ converges to a normalized solution of $\eta'' = -\frac{1}{2}\phi\eta$. Therefore, if we choose the same normalizations we may conclude that $\hat{f}_0 = \hat{f}$ in H and in H^*. Hence $\hat{f}$ can be extended by continuity to the real axis and is a solution of the problem with

$$\mu(z) = -2\phi(\overline{z})y^2, \quad z \in H.$$

But if $\phi \in Q(\Gamma)$ one verifies that $\mu \in B(\Gamma)$, and we conclude

LEMMA 2. *The origin of* $Q(\Gamma)$ *is an interior point of* $\Delta(\Gamma)$.

Suppose now that $\phi_0 \in \Delta$ and $[f_0] = \phi_0$ where $f_0 = f_{\mu_0}$. We assume that f_0 maps H on Ω, H^* on Ω^*. Then the boundary curve L of Ω admits a q.c. reflection λ. According to Lemma 3 of Chapter IV D, we can choose λ so that corresponding euclidean lengths have bounded ratio. This means that λ is $C(K)$-q.c. and

$$C(K)^{-1} \le |\lambda_{\overline{z}}| \le C(K)$$

provided that f_0 is K-q.c.

If $[f] = \phi$ the composition rule for Schwarzians gives

$$\phi - \phi_0 = \{f, f_0\} f_0'^2.$$

The noneucidean metric in Ω^* is such that

$$\rho(\zeta)|d\zeta| = \frac{|dz|}{-y}.$$

Therefore, $\|\phi - \phi_0\| \le \epsilon$ shows that $g = f \circ f_0^{-1}$ satisfies

$$|[g](\zeta)| \le \epsilon\rho(\zeta)^2.$$

For sufficiently small ϵ we have to show that g has a q.c. extension.

We set $\psi = [g]$ and determine normalized solutions η_1, η_2 of

$$\eta'' = -\tfrac{1}{2}\psi\eta.$$

This time we construct

$$g(\zeta) = \eta_1(\zeta)/\eta_2(\zeta), \qquad \zeta \in \Omega^*$$
$$\hat{g}(\zeta) = \frac{\eta_1(\zeta^*) + (\zeta - \zeta^*)\eta_1'(\zeta^*)}{\eta_2(\zeta^*) + (\zeta - \zeta^*)\eta_2'(\zeta^*)}, \qquad \zeta \in \Omega.$$

(Recall that $\zeta^* = \lambda(\zeta)$.) Computation gives

$$\mu_{\hat{g}}(\zeta) = \frac{\frac{1}{2}(\zeta - \zeta^*)^2 \psi(\zeta^*) \lambda_{\bar{\zeta}}(\zeta)}{1 + \frac{1}{2}(\zeta - \zeta^*)^2 \psi(\zeta^*) \lambda_\zeta(\zeta)}, \qquad \zeta \in \Omega.$$

But $|\lambda_\zeta| < |\lambda_{\bar{\zeta}}| \le C(K)$ and $|\zeta - \zeta^*| < C\rho(\zeta^*)^{-1}$. Hence

$$|\mu_{\hat{g}}| \le \frac{\epsilon \cdot C(K)}{1 - \epsilon \cdot C(K)} < 1 \tag{4}$$

as soon as ϵ is small enough.

It must again be shown that $\hat{g}$ is continuous and schlicht. There is no difficulty if L is analytic and ψ is analytic on L with a zero of order four at ∞.

The general case again requires an approximation argument. Let $f_n = f_0 \circ S_n$, where S_n is as in the proof of Lemma 1, and let L_n be the image of the real axis under f_n. L_n is an analytic curve that admits a K-q.c. reflection, and ψ is analytic on L_n.

The Poincaré density ρ_n of $\Omega_n^* = f_n(H^*)$ is $\ge \rho$, so that $|\psi| \le \epsilon\rho^2$ implies $|\psi| \le \epsilon\rho_n^2$. Therefore we can construct a sequence of normalized q.c. mappings $\hat{g}_n$ such that $[\hat{g}_n] = \psi$ in Ω_n^* and $\mu_{\hat{g}_n}$ satisfies the inequality (4). A subsequence of the $\hat{g}_n$ tends to a q.c. limit $\hat{g}$ which is equal to g in Ω^*. This proves Theorem 1. Since $\mu_{\hat{g}}$ satisfies (4) we conclude:

COROLLARY. *For every sequence of $\phi_n \in \Delta$ converging to $\phi_0 = [f_{\mu_0}] \in \Delta$, there exist $\mu_n \to \mu_0$ such that $[f_{\mu_n}] = \phi_n$.*

The proof is by writing $\phi = [\hat{g} \circ f_0]$.

We come now to the most delicate part:

THEOREM 2. *$\Delta(\Gamma)$ is an open subset of $Q(\Gamma)$.*

REMARK. This was first proved by Bers. The idea of the proof that follows is due to Clifford Earle.

Given any $\mu_0 \in B_1$, we construct a mapping

$$\beta_0 \colon \Delta \to \Delta$$

as follows: Given $\phi \in \Delta$ there exists a $\mu \in B_1$ such that $\phi = \phi_\mu$. With this μ we determine λ by

$$f^\lambda = f^\mu \circ (f^{\mu_0})^{-1} \tag{5}$$

and set $\beta_0(\phi) = \phi_\lambda$. It is unique, for if $\phi_\mu = \phi_{\mu_1}$, then f^μ has the same boundary values as f^{μ_1}. Hence f^λ has the same boundary values as f^{λ_1}, and hence $\phi_\lambda = \phi_{\lambda_1}$.

It is evident that β_0 is 1-1, and it carries ϕ_{μ_0} into zero. Moreover, β_0 is continuous.* For if $\phi_n \to \phi = [f_\mu]$ we have just proved the existence of $\mu_n \to \mu$ such that $\phi_n = [f_{\mu_n}]$. The corresponding ϕ_{λ_n} converge to ϕ_λ.

* See Editors' Note 5 on p. 83.

LEMMA (Earle). *$\phi_\mu \in Q(\Gamma)$ if and only if for every $A \in \Gamma$ there exists a linear transformation B such that*

$$f^\mu \circ A \circ (f^\mu)^{-1} = B \text{ on } \mathbb{R}.$$

PROOF. $\phi_\mu \in Q(\Gamma)$ is equivalent to $[f_\mu \circ A] = [f_\mu]$, and this is true if and only if $f_\mu \circ A \circ f_\mu^{-1} = C$, a linear transformation, in Ω_μ^*.

1) If B exists, then

$$C = f_\mu \circ A \circ f_\mu^{-1} = f_\mu \circ (f^\mu)^{-1} \circ B \circ f^\mu \circ (f_\mu)^{-1}$$

on $f^\mu(\mathbb{R})$. The first expression is holomorphic in Ω_μ^*, the second in Ω_μ. Hence C is a linear transformation.*

2) If C exists, then

$$B = f^\mu \circ A \circ (f^\mu)^{-1} = f^\mu \circ f_\mu^{-1} \circ C \circ f_\mu \circ (f^\mu)^{-1}$$

on $\mathbb{R}$. But the last expression is a conformal mapping of H on itself, hence a linear transformation. The lemma is proved. □

Assume now that $\mu_0 \in B_1(\Gamma)$. We find that β_0 maps $\Delta(\Gamma)$ on $\Delta(\Gamma^{\mu_0})$, for

$$f^\lambda \circ A^{\mu_0} \circ (f^\lambda)^{-1} = f^\mu \circ A \circ (f^\mu)^{-1} = A^\mu.$$

From the lemma we infer that $Q(\Gamma) \cap \Delta$ is mapped on $Q(\Gamma^{\mu_0}) \cap \Delta$. The origin has a neighborhood N in $Q(\Gamma^{\mu_0})$ which is contained in $\Delta(\Gamma^{\mu_0})$. Write $N = Q(\Gamma^{\mu_0}) \cap N_0 = Q(\Gamma^{\mu_0}) \cap \Delta \cap N_0$ where N_0 is a neighborhood in Δ. Then

$$\beta_0^{-1}(N) = Q(\Gamma) \cap \Delta \cap \beta_0^{-1}(N_0) = Q(\Gamma) \cap \beta_0^{-1}(N_0)$$

which is a neighborhood of ϕ_{μ_0} in $Q(\Gamma)$. From $N \subset \Delta(\Gamma^{\mu_0})$ we get $\beta_0^{-1}(N) \subset \Delta(\Gamma)$ and this proves that ϕ_{μ_0} has a neighborhood in $Q(\Gamma)$ which is contained in $\Delta(\Gamma)$.

Conclusion:

If the group Γ of cover transformations of H over S_0 is of the first kind, the Teichmüller space $T(S_0)$ is identified with $\Delta(\Gamma)$, an open subset of $Q(\Gamma)$. The Teichmüller metric defines the same topology as the norm in $Q(\Gamma)$.

Consider the case where S_0 is a compact Riemann surface of genus $g > 1$. Then $Q(\Gamma)$ has complex dimension $3g-3$. Choose a basis $\phi_1, \dots, \phi_{3g-3}$. Every $\phi \in Q(\Gamma)$ is a linear combination

$$\phi = \tau_1\phi_1 + \cdots + \tau_{3g-3}\phi_{3g-3}$$

with complex coefficients. We find that $\Delta(\Gamma)$ can be identified with a bounded open set in $\mathbb{C}^{3g-3}$.

We can show, moreover, that the parametrization by $\tau = (\tau_1 \dots, \tau_{3g-3})$ defines the Riemann surfaces of genus g as a *holomorphic family* of Riemann surfaces.

According to Kodaira and Spencer a holomorphic family may be described as follows:

There is given an $(n+1)$-dimensional complex manifold V and a holomorphic mapping $\pi\colon V \to M$ on an n-dimensional complex manifold M. Each fiber $\pi^{-1}(\tau)$, $\tau \in M$ is a Riemann surface.

The complex structures of V and M are related in the following way: There is an open covering $\{U_a\}$ of V so that for each a there is given a holomorphic homeomorphism $h_\alpha\colon U_\alpha \to \mathbb{C} \times M$ connected by $\phi_{\alpha\beta} = h_\alpha \circ h_\beta^{-1}$ (for intersecting

* Because it is quasiconformal and conformal a.e.

U_α, U_β). For any $\tau \in M$ the restriction of $\phi_{\alpha\beta}$ to $h_\beta(U_\alpha \cap U_\beta \cap \pi^{-1}(\tau))$ shall be complex analytic (in fact, these functions determine the complex structure of $\pi^{-1}(\tau)$.)

In our case, M will be the set $\Delta(\Gamma)$. Every $\tau \in \Delta(\Gamma)$ determines a $\phi = [f_\mu]$, and f_μ is uniquely defined in H^*. Hence Ω^*_μ, Ω_μ, and the group Γ_μ are determined by τ. To emphasize the dependence on τ we change the notations to ϕ_τ, Ω_τ, Γ_τ, etc. Since f_τ is determined in H^* by the differential equation $[f_\tau] = \phi_\tau$, we know that f_τ depends holomorphically on the parameter τ. For $A \in \Gamma$, the corresponding $A_\tau \in \Gamma_\tau$ is determined by the condition $f_\tau \circ A = A_\tau \circ f_\tau$ in H^*. This means that A_τ depends holomorphically on τ, a fact that will be important.

The Riemann surfaces in our family will be $S(\tau) = \Omega_\tau/\Gamma_\tau$ and V will be their union. Thus the points of V are orbits $\Gamma_\tau\zeta$ with $\zeta \in \Omega_\tau$ and $\tau \in \Delta(\Gamma)$. The projection $\pi\colon V \to M$ is defined so that $\pi^{-1}(\tau) = S(\tau)$.

Consider a point $\Gamma_{\tau_0}\zeta_0$ in V. We pick a fixed ζ_0 from the orbit and determine an open neighborhood $N(\zeta_0)$ such that $\overline{N}$ is compact, contained in Ω_{τ_0}, and does not meet its images under Γ_{τ_0}. The neighborhood $N(\epsilon, \zeta_0, \tau_0)$ of $\Gamma_{\tau_0}\zeta_0$ will consist of all $\Gamma_\tau\zeta$ such that $\|\phi_\tau - \phi_{\tau_0}\| < \epsilon$ and $\zeta \in N(\zeta_0)$. Here ϵ shall be so small that $\overline{N}$ does not meet its images under Γ_τ. This is possible because A_τ is near A_{τ_0} when ϕ_τ is near ϕ_{τ_0}. As a consequence, there is only one $\zeta \in N(\zeta_0) \cap \Gamma_\tau\zeta$ and the mapping $h\colon \Gamma_\tau\zeta \to (\zeta, \tau)$ is well-defined in $N(\epsilon, \zeta_0, \tau_0)$.

The neighborhoods $N(\epsilon, \zeta_0, \tau_0)$ shall be a base for the topology of V. We assert, in addition, that the parameter mappings $h\colon \Gamma_\tau\zeta \to (\zeta, \tau)$ defined on the basic neighborhoods make $\pi\colon V \to M$ into a holomorphic family. Indeed, if two basic neighborhoods U_0 and U_1 intersect, then in $U_0 \cap U_1$, $h_0(\Gamma_\tau\zeta) = (\zeta, \tau)$ and $h_1(\Gamma_\tau\zeta) = (A_\tau\zeta, \tau)$ for some $A_\tau \in \Gamma_\tau$. Hence the mapping $h_1 \circ h_0^{-1}$ is given by $(\zeta, \tau) \to (A_\tau\zeta, \tau)$, which we know is holomorphic in both τ and ζ. It is evident that the parameter mappings define a complex structure on V which agrees with the conformal structure of the surfaces $S(\tau)$ and makes the mapping $\pi\colon V \to M$ holomorphic.

D. The Infinitesimal Approach

We continue with a direct investigation of f^μ, A^μ that does not make use of the mappings f_μ.

For any function $F(\mu)$ and any $\nu \in L^\infty$ we set

$$\lim_{t \to 0} \frac{F(\mu + t\nu) - F(\mu)}{t} = \dot{F}(\mu)[\nu]$$

when the limit exists, and we omit the argument μ when the derivative is taken for $\mu = 0$. We are assuming that t is real.

We have already derived the representation

$$\dot{f}[\nu](\zeta) = -\frac{1}{\pi} \iint \nu(z) R(z, \zeta) dx\, dy$$

where

$$R(z, \zeta) = \frac{1}{z - \zeta} - \frac{1 - \zeta}{z} - \frac{\zeta}{z - 1}.$$

We apply the formula to the symmetric case: $\nu(\overline{z}) = \overline{\nu}(z)$, and we write more explicitly

$$\dot{f}[\nu](\zeta) = -\frac{1}{\pi}\iint_H \nu(z)R(z,\zeta)dx\,dy - \frac{1}{\pi}\iint_H \overline{\nu}(z)R(\overline{z},\zeta)dx\,dy. \tag{1}$$

It is evident that $\dot{f}[\nu]$ is linear in the real sense, but not in the complex sense. To obtain a conjugate complex linear functional we form

$$\Phi[\nu] = \dot{f}[\nu] + i\dot{f}[i\nu] \tag{2}$$

and we find

$$\Phi[\nu](\zeta) = -\frac{2}{\pi}\iint_H \overline{\nu}(z)R(\overline{z},\zeta)dx\,dy. \tag{3}$$

It is holomorphic for $\zeta \in H$. Its third derivative is

$$\Phi'''(\zeta) = \phi[\nu](\zeta) = -\frac{12}{\pi}\iint_H \frac{\overline{\nu}(z)}{(\overline{z}-\zeta)^4}dx\,dy. \tag{4}$$

If $\nu \in B(\Gamma)$ one verifies that ϕ is a quadratic differential ($\phi \in Q(\Gamma)$).

From

$$f^\mu(Az) = A^\mu f^\mu(z)$$

with $\mu = t\nu$ we obtain after differentiation

$$\dot{f}[\nu]\circ A = \dot{A}[\nu] + A'\dot{f}[\nu] \tag{5}$$

(the existence of $\dot{A}$ requires some, but not much, proof). Since $\dot{f}[\nu]_{\overline{z}} = \nu$ differentiation of (5) gives

$$(\nu\circ A)\overline{A'} = \dot{A}_{\overline{z}} + A'\nu$$

and hence $\dot{A}_{\overline{z}} = 0$ because $\nu \in B(\Gamma)$. We conclude that the $\dot{A}$ are analytic functions. Moreover, $\dot{A}/A'$ is real on the real axis and can hence be extended by symmetry to the whole plane. The explicit formula for $\dot{f}[\nu]$ shows that it is $o(|z|^2)$ at ∞. The apparent singularity of $\dot{A}/A'$ at $A^{-1}\infty$ is therefore removable, and there is at most a double pole at ∞. We conclude that

$$\frac{\dot{A}}{A'} = P_A \tag{6}$$

where P_A is a second degree polynomial. From

$$(A_1A_2)^{\cdot} = (\dot{A}_1\circ A_2) + (A_1'\circ A_2)\dot{A}_2$$
$$(A_1A_2)' = (A_1'\circ A_2)A_2'$$

we deduce that

$$P_{A_1A_2} = \frac{P_{A_1}\circ A_2}{A_2'} + P_{A_2}. \tag{7}$$

We shall say that ν is *trivial*, and we write $\nu \in N(\Gamma)$, if all $\dot{A}[\nu] = 0$. There is a whole slew of equivalent conditions:

LEMMA 1. *The following conditions are all equivalent.*

a) $\dot{A}[\nu] = 0$ *for all* $A \in \Gamma$,

b) $P_A = 0$ *for all* $A \in \Gamma$,

c) $\dot{f}[\nu] = 0$ *on* $\mathbb{R}$,

d) $\Phi[\nu] \equiv 0$,

e) $\phi[\nu] \equiv 0$,

f) $\iint_{\Gamma\backslash H} \nu\phi\, dx\, dy = 0$ *for all* $\phi \in Q(\Gamma)$.*

PROOF. a) $\Leftrightarrow$ b) by (6). c) $\Rightarrow$ a) by (5). Conversely, if $\dot{A} = 0$ then since $\dot{f}(0) = 0$ it follows that $\dot{f}(A0) = 0$ for all A. These points are dense on $\mathbb{R}$ (we are assuming that the group is of the first kind) and hence, by continuity, $\dot{f} = 0$ on $\mathbb{R}$.

The definition of Φ, together with (5) and (6), gives

$$\frac{\Phi \circ A}{A'} - \Phi = P_A[\nu] + iP_A[i\nu].$$

If $\Phi \equiv 0$, it follows that $P_A[\nu] + iP_A[i\nu] = 0$ on $\mathbb{R}$. But both polynomials are real on $\mathbb{R}$, hence identically zero, so d) $\Rightarrow$ b). Conversely, if $\dot{f}[\nu] = 0$ on $\mathbb{R}$, then Φ is purely imaginary on $\mathbb{R}$ and can be extended to be analytic in the whole plane. Since $\Phi = o(|z|^2)$ it is a first degree polynomial. But it vanishes at 0 and 1. Hence $\phi \equiv 0$ and c) $\Rightarrow$ d).

Since $\Phi''' = \phi$, d) $\Rightarrow$ e). Conversely, if $\phi = 0$, Φ is a polynomial and we reason as above to conclude $\Phi \equiv 0$.

Condition f) is the most important one. We prove it only for the case that $\Gamma\backslash H$ is compact, and we represent it as a compact fundamental polygon S with matched sides. If $\dot{A} = 0$, then $\dot{f} \circ A = A'\dot{f}$. We know further that $\dot{f}_{\bar{z}} = \nu$. By Stokes' formula we find

$$\iint_S \nu\phi\, dx\, dy = -\frac{1}{2i} \iint_S \dot{f}_{\bar{z}}\phi\, dz\, d\bar{z} = \frac{1}{2i} \int_{\partial S} \dot{f}\phi\, dz.$$

But the condition means that $\dot{f}/dz$ is invariant. Hence

$$\dot{f}\phi\, dz = \frac{\dot{f}}{dz} \cdot \phi\, dz^2$$

is invariant, and the boundary integral vanishes.

To prove the opposite, consider a real ζ so that (3) takes the form

$$\overline{\Phi(\zeta)} = -\frac{2}{\pi} \iint_H \nu(z) R(z, \zeta)\, dx\, dy.$$

We introduce a Poincaré θ-series

$$\psi(z) = \sum R(Az, \zeta) A'(z)^2.$$

* For $\Gamma\backslash H$ non-compact, the condition shall hold for all $\phi \in Q(\Gamma)$ such that $\iint_{\Gamma\backslash H} |\phi| dx\, dy < \infty$.

From the fact that $\int_H |R(z,\zeta)|dx\,dy < \infty$ it is quite easy to deduce that the series converges. We obtain now

$$\begin{aligned}\overline{\Phi(\zeta)} &= -\frac{2}{\pi}\sum\iint_{A(S)}\nu(z)R(z,\zeta)dx\,dy\\ &= -\frac{2}{\pi}\sum\iint_S \nu(Az)R(Az,\zeta)|A'(z)|^2 dx\,dy\\ &= -\frac{2}{\pi}\sum\iint \nu(z)R(Az,\zeta)A'(z)^2 dx\,dy\\ &= -\frac{2}{\pi}\iint \nu\psi\,dx\,dy\end{aligned}$$

which is zero if f) holds. Hence Φ is identically zero. The lemma is proved. □

We need the following lemma.

LEMMA 2. *Suppose ϕ is analytic in H and*

$$\sup|\phi|y^2 < \infty.$$

Then

$$\phi(\zeta) = \frac{12}{\pi}\iint_H \frac{\phi(z)y^2}{(\overline{z}-\zeta)^4}dx\,dy. \tag{8}$$

For the proof we note that

$$\begin{aligned}\frac{y^2}{(\overline{z}-\zeta)^4} &= -\frac{1}{4}\frac{(\overline{z}-z)^2}{(\overline{z}-\zeta)^4} = -\frac{1}{4}\left[\frac{1}{(\overline{z}-\zeta)^2} - \frac{2(z-\zeta)}{(\overline{z}-\zeta)^3} + \frac{(z-\zeta)^2}{(\overline{z}-\zeta)^4}\right]\\ &= -\frac{1}{4}\frac{\partial}{\partial\overline{z}}\left[-\frac{1}{(\overline{z}-\zeta)} + \frac{z-\zeta}{(\overline{z}-\zeta)^2} - \frac{1}{3}\frac{(z-\zeta)^2}{(\overline{z}-\zeta)^3}\right].\end{aligned}$$

Assume first that ϕ is still analytic on $\mathbb{R}$. Then integration by parts gives

$$\frac{12}{\pi}\iint_H \frac{\phi y^2 dx\,dy}{(\overline{z}-\zeta)^4} = -\frac{3}{2\pi i}\int_{\mathbb{R}}\left(-\frac{1}{3}\right)\frac{\phi(z)}{z-\zeta}dz = \phi(\zeta).$$

It is easy to complete the proof by applying this formula to $\phi(z+i\epsilon)$, $\epsilon > 0$, for the hypothesis guarantees absolute convergence.

Compare formula (8) with (4). We have defined an antilinear mapping

$$\Lambda\colon \nu \to \phi[\nu]$$

from $B(\Gamma)$ to $Q(\Gamma)$. On the other hand we may define

$$\Lambda^*\colon \phi \to -\overline{\phi}y^2$$

which is a mapping from $Q(\Gamma)$ to $B(\Gamma)$. Lemma 2 tells us that $\Lambda\Lambda^*$ is the identity.

By Lemma 1, e), $\nu \in N(\Gamma)$ if and only if $\Lambda\nu = 0$. From $\Lambda\Lambda^* = I$ we conclude that

$$\nu - \Lambda^*\Lambda\nu \in N(\Gamma).$$

in other words, ν is equivalent $\bmod N(\Gamma)$ to $-\overline{\phi}[\nu]y^2$. Of course this is the only $\Lambda^*\phi$ which is equivalent to ν, for if $-\phi y^2 \in N(\Gamma)$ then $\iint_{\Gamma\backslash H}|\phi(z)|^2y^2\,dx\,dy = 0$, hence $\phi = 0$.

We conclude:

Λ establishes an isomorphism of $B(\Gamma)/N(\Gamma)$ on $Q(\Gamma)$. The inverse isomorphism of $Q(\Gamma)$ on $B(\Gamma)/N(\Gamma)$ is given by Λ^*. □

A compact surface S, of genus $g > 1$, determines a group Γ generated by linear transformations $A_1, \dots, A_{2g}$ which satisfy

$$A_1 A_2 A_1^{-1} A_2^{-1} \cdots A_{2g-1} A_{2g} A_{2g-1}^{-1} A_{2g}^{-1} = I. \tag{9}$$

We say that $\{A_1, \dots, A_{2g}\}$ is a canonical set. If it belongs to a surface, A_1 and A_2 have four distinct fixpoints. By passing to a conjugate subgroup we can make A_1 have fixpoints at 0, ∞ and A_2 to have a fixpoint at 1. When this is so we say that the generating system is normalized.

Set

$$\begin{aligned} V &= \text{set of normalized canonical systems,} \\ T &= \text{set of normalized canonical systems} \\ &\qquad \text{that come from a surface of genus } g. \end{aligned}$$

It can be shown that V is a real analytic manifold of dimension $6g-6$. We are going to prove that T is an open subset of V, and that it carries a natural complex structure.

Let S determine Γ with normalized generators $(A) = (A_1, \dots, A_{2g})$. We choose a basis $\nu_1, \dots, \nu_{3g-3}$ of $B(\Gamma)/N(\Gamma)$ and set

$$\nu(\tau) = \tau_1 \nu_1 + \cdots + \tau_{3g-3}\nu_{3g-3}$$

and $\tau_k = t_k + i t'_k$. For small τ we obtain a system $(A)^{\nu(\tau)} \in T$. The points of the manifold V near (A) can be expressed by local parameters $u_1, \dots, u_{6g-6}$, and the mapping $\tau \to (A)^{\nu(\tau)}$ takes the form

$$u_j = h_j(t_1, t'_1, \dots, t_{3g-3}, t'_{3g-3}).$$

We have to show

1) the h_j are continuously differentiable,

2) the Jacobian is $\neq 0$ at $\tau = 0$.

1) has already been proved. The coefficients of the A_k are differentiable functions of the u_k. Therefore, if all $\dot{u}_k[\nu]$ are 0, so are all $\dot{A}_k[\nu]$ and hence all $\dot{A}[\nu]$. Observe that

$$\frac{\partial h_j}{\partial t_k} = \dot{u}_j[\nu_k], \qquad \frac{\partial h_j}{\partial t'_k} = \dot{u}_j[i\nu_k].$$

If the Jacobian at the origin were zero there would exist real numbers ξ_k, η_k such that

$$\sum \xi_k \frac{\partial h_j}{\partial t_k} + \eta_k \frac{\partial h_j}{\partial t'_k} = 0 \text{ all } j.$$

This would mean

$$\dot{u}_j \left[\sum (\xi_k + i\eta_k)\nu_k\right] = 0$$

and hence

$$\dot{A} \left[\sum (\xi_k + i\eta_k)\nu_k\right] = 0.$$

Hence $\sum(\xi_k + i\eta_k)\nu_k \in N(\Gamma)$, and this is possible only if all ξ_k, $\eta_k = 0$. We have proved that the Jacobian does not vanish.

The proof shows that T is an open subset of V. It also follows that if $\|\mu\|$ is small enough, then there exist unique complex numbers $\tau_1(\mu), \dots, \tau_{3g-3}(\mu)$ such that

$$A^{\tau_1(\mu)\nu_1 + \cdots + \tau_{3g-3}(\mu)\nu_{3g-3}} = A^\mu.$$

Set $\mu = t_\rho$ and differentiate with respect to t for $t = 0$. We obtain

$$\dot{A}[\dot{\tau}_1[\rho]\nu_1 + \cdots + \dot{\tau}_{3g-3}[\rho]\nu_{3g-3}] = \dot{A}[\rho].$$

This implies

$$\dot{\tau}_1[\rho]\nu_1 + \cdots + \dot{\tau}_{3g-3}[\rho]\nu_{3g-3} - \rho \in N(\Gamma).$$

Replace ρ by $i\rho$. We can then eliminate the term ρ to obtain

$$\sum_1^{3g-3} (\dot{\tau}_k[\rho] + i\dot{\tau}_k[i\rho])\nu_k \in N(\Gamma)$$

from which it follows that

$$\dot{\tau}_k[i\rho] = i\dot{\tau}_k[\rho].$$

In other words, the $\dot{\tau}_k$ are complex linear functionals, which means that the $\tau_k(\mu)$ are differentiable in the complex sense at $\mu = 0$.

From this we can prove that the coordinate mappings $(A^\mu) \to (\tau_1(\mu), \ldots, \tau_{3g-3}(\mu))$ define a complex structure on T. Indeed, we must show that on overlapping neighborhoods the coordinates are analytic functions, and it suffices to show this at the origin of a given coordinate system. Take $\mu(\tau) = \sum \tau_i \nu_i$ and $\mu_0 = \mu(\tau_0)$ near zero in $B(\Gamma)$. Define $\lambda(\tau)$ in $B(\Gamma^{\mu_0})$ by $f^{\mu(\tau)} = f^{\lambda(\tau)} \circ f^{\mu_0}$. By the formulas of section C in Chapter I, we have

$$\lambda(\tau) \circ f^{\mu_0} = \frac{\mu(\tau) - \mu_0}{1 - \overline{\mu}_0 \mu(\tau)} (f_z^{\mu_0}/|f_z^{\mu_0}|)^2,$$

so that λ depends analytically on τ.

Now choose a basis $\lambda_1, \ldots, \lambda_{3g-3}$ for $B(\Gamma^{\mu_0})/N(\Gamma^{\mu_0})$. Near (A^{μ_0}) we have coordinate functions $\sigma_1(\lambda), \ldots, \sigma_{3g-3}(\lambda)$. This means that for τ near τ_0 we can write uniquely

$$(A^{\mu(\tau)}) = ((A^{\mu_0})^{\lambda(\tau)}) = ((A^{\mu_0})^{\sum \sigma_i(\lambda(\tau))\lambda_i}).$$

Because $\sigma_i(\lambda)$ is complex analytic at $\lambda = 0$, we conclude that σ_i is a complex analytic function of τ at τ_0. This is exactly what we required.