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Volume 45

p-adic Geometry

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http://dx.doi.org/10.1090/ulect/045

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2000 *Mathematics Subject Classification*. Primary 14G22; Secondary 11F85, 14F30.

The 2007 Arizona Winter School was supported by NSF grant DMS-0602287.
The first author was supported by NSF grant DMS-0600027.
The second author was partially supported by NSF grant DMS-0600919.
The third author was partially supported by NSF grant DMS-0653023.
The fourth author was supported by NSF CAREER grant DMS-0545904,
and a Sloan Research Fellowship.
The fifth author was supported by NSF grant DMS-0245410.

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Library of Congress Cataloging-in-Publication Data

Arizona Winter School (2007 : University of Arizona)
p-adic geometry : lectures from the 2007 Arizona Winter School / Matthew Baker... [et al.] ; edited by David Savitt, Dinesh S. Thakur.
p. cm. — (University lecture series ; v. 45)
Includes bibliographical references.
ISBN 978-0-8218-4468-7 (alk. paper)
1. Arithmetical algebraic geometry–Congresses. 2. p-adic analysis–Congresses 3. Geometry, Algebraic–Congresses I. Baker, Matthew, 1973– II. Savitt, David. III. Thakur, Dinesh S. IV. Title.

QA242.5.A757 2007
516.3′5–dc22 2008023597

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10 9 8 7 6 5 4 3 2 1 13 12 11 10 09 08

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Preface

This volume contains notes which accompanied the lectures at the tenth annual Arizona Winter School, held from March 10 to 14, 2007 at the University of Arizona in Tucson. The Arizona Winter School is an intensive five-day meeting, each year organized around a different central topic in arithmetic geometry, featuring several courses by leading and emerging experts ("an annual pilgrimage," in the words of one participant). The Winter School is the main activity of the Southwest Center for Arithmetic Geometry, which was founded in 1997 by a group of seven mathematicians working in the southwest United States, and which has been supported since that time by the National Science Foundation.

The special character of the Arizona Winter School comes from its format. Each speaker proposes a project, and a month before the Winter School begins, the speaker is assigned a group of graduate students who work on the project. The speakers also provide lecture notes and a bibliography. During the actual school the speaker and and his or her group of students work every evening on the assigned project. On the last day of the workshop, the students from each group present their work to the whole school. The result is a particularly intense and focused five days of mathematical activity (for the students and speakers alike).

The topic of the Winter School in 2007 was p-adic geometry, and the speakers were Matthew Baker, Brian Conrad, Kiran Kedlaya, and Jeremy Teitelbaum. Samit Dasgupta joins Teitelbaum as a co-author. We thank the authors for their hard work before, during, and after the Winter School. We are also grateful to John Tate and Vladimir Berkovich, two pioneers of non-archimedean geometry, who agreed to describe the early history of some of their contributions to the subject. The anonymous reviewers made numerous valuable comments, and we thank them for their careful reading of this manuscript. Finally, we are indebted to the other members (past and present) of the Southwest Center; it is thanks to their efforts that the Winter School exists in its present form.

David Savitt
Dinesh Thakur

Foreword

John Tate

This book contains the notes of the four lecture series of the 2007 Arizona Winter School on non-archimedean geometry, together with a short account by Berkovich of how he developed, in difficult circumstances, the remarkable theory of the spaces which bear his name.

Brian Conrad's talks are an excellent introduction to the general theory of the original rigid analytic spaces, of Raynaud's view of them as the generic fibers of formal schemes, and Berkovich spaces. The p-adic upper half plane as rigid analytic space and several applications of it are discussed in the notes of Samit Dasgupta and Jeremy Teitelbaum. Matt Baker's lectures offer a detailed discussion of the Berkovich projective line and p-adic potential theory on that and more general Berkovich curves. Finally, Kiran Kedlaya discusses cohomologies, deRham and rigid, the comparison between them, and how to compute them.

In this foreword I want to do two things: (1) tell more than is in the brief preface about the institution which gave rise to this book, the AWS, for I think it's a great concept which would be good to try in other subjects and places, and (2) give a bit of ancient history of the book's subject, p-adic geometry, especially the part I was involved in.

It's hard to describe the AWS adequately. "Intense", "interactive", "infectious enthusiasm", "flexibility in support" give some idea of the flavor. You have to experience it to believe it can work so well. The unique and novel feature of the AWS is that after the usual set of lectures during the day that all such schools have, there are working sessions until late in the night in which students work with the help of the speakers and their assistants on projects related to the lectures. In fact, even the lecture series part of the school is not so usual, because the speakers produce a complete set of notes for their lectures a month ahead of time, with background references, so that the participants can be better prepared (this book is essentially the notes for the 2007 school). The speakers also describe ahead of time the projects which their group of students will work on at the school. These projects can involve elaborations of the theory, working out special examples, computing some quantity in various cases, writing computer code to do that, background work on foundational aspects... anything to challenge the participants and help them learn by working with the material. The projects associated to the lectures in this book, and also the lectures and projects from several earlier Winter Schools, can be found on the Arizona Winter School website. On the last day of the school students from each group make a joint presentation of the results the group has obtained.

Another key feature of the AWS is that the participants come from a wide variety of programs from all over the country, with very different levels of preparation, pressure and expectations. The organizers try to preserve this diversity in each of the four working groups. Typically a group will have an advanced student working in an area close to the group's project, it will have students with a very strong, but also not so strong background, it will have students from many different universities, some elite, some not, it will have students of both sexes, and at least one student from the southwest, the region represented by the organizers. Preferences for projects expressed by students on their applications are taken into account, but not all can be satisfied. In fact, the AWS has become so popular that not all students are officially assigned to groups. But all are encouraged to hang out and informally participate anyway. The whole scheme is flexible.

The interactions between students from different backgrounds that the AWS makes possible has a positive impact on all attending. For all students it is an opportunity to feel the exhilaration, drive and power that the students from the top programs show. On the other hand, the very nature of the AWS is to create an environment where students with lower level backgrounds are encouraged to ask basic questions. Discussion of these can be very valuable to all; it may reveal that what seemed clear was not quite so clear after all. The cross-pollination of mathematical cultures which takes place at the AWS is of benefit to everybody. Ultimately participants get a clear picture of what it means to do mathematics in the real world and this can be a significant learning experience for students and postdocs of any background.

Almost all of the participants are housed in the same hotel, and the evening sessions are held there too. There is no escape. Everyone is constantly involved in small and large discussion groups, on the lectures, on the project topics and toward the joint presentations to be given on the last day.

The school is five days of very hard work — 16 hour days — for all participants. In spite of that, or perhaps just because of the intellectual intensity, most seem to thrive. As an informal participant in several of the schools I am aware of many testimonials, oral and written, that it is a rewarding experience, both scientifically and interpersonally. The topic changes every year and the frequency with which many students return year after year is another indication of the value of the school. Work at the school has been the germ of many Ph.D. theses.

The first Winter School was held in 1998, and it has been going strong since then. The organizers (which by now include a few ex-students from early schools) deserve great credit, for the original conception, for improving it in various ways over the years, and for the mostly excellent choices of topics and speakers they have made each year. The 2007 lectures were outstanding and unusually closely related. Hence the idea to collect them in a book.

Now I would like to switch gears and discuss some old (pre-1965) history. The basic problem in creating a global theory of analytic manifolds over a non-archimedean local field K is that analytic continuation in the usual sense does not work. We can agree that a function in a "closed" disc $|x-a| \leq r$ (which is also open), or in an "open" disc $|x-a| < r$ (which is also closed), is analytic if and only if it has a power series expansion convergent in the disc, and, in fact, this turns out ultimately to be the right idea. But this is a much stricter condition than to have a power series expansion in a neighborhood of every point of the disc, because in the non-archimedean metric every disc is a *disjoint* union of smaller discs. So this is not a local definition, and there's no obvious way to globalize it.

The first to overcome this difficulty was Marc Krasner. He made a good theory of analytic functions of one variable. They were defined on certain subsets of $\mathbb{P}^1(K) = K \cup \infty$ which he called "quasiconnected". For example, the set obtained by removing from $\mathbb{P}^1(K)$ a finite set of discs of the form $|(ax+b)/(cx+d)| < r$ is quasiconnected and an analytic function on it is one which is a uniform limit of rational functions with poles in its complement. Krasner's theory had little influence on the later global higher dimensional developments described in this volume, but was valued and further developed by p-adic analysts doing the theory of p-adic differential equations, etc. I remember that Dwork was very upset that there was no mention of Krasner in the introduction to the book [BGR] referred to by Conrad in this volume, and I think he was right to be.

The earliest steps toward the subject of this book were mine. My motivation was the isomorphism $K^*/q^{\mathbb{Z}} \xrightarrow{\sim} E_q(K)$, for $q \in K, 0 < |q| < 1$, where E_q is an elliptic curve over K with invariant $j = q^{-1} + 744 + 196884q + \cdots$. I still remember the thrill and amazement I felt when it occurred to me that the classical formulas for such an isomorphism over $\mathbb{C}$ made sense p-adically when properly normalized.[1]

This uniformization of some elliptic curves made me wonder if there might not be a general theory of p-adic manifolds. Two years later in the fall of 1961, very much influenced by Grothendieck's theory of schemes, I was ready to make a serious attempt to create such a theory. In contrast to the difficult circumstances Berkovich faced as he developed his theory, my situation could not have been more favorable, with a good job at Harvard and friends like Serre and Grothendieck to help me. I recorded my progress in a series of letters to Serre. He wrote that he was keeping them carefully, but not reading them carefully. But he did find and fix a gap in my proof of the acyclicity theorem.[2] He was interested at the time in another aspect of p-adic analysis, namely Dwork's spectacular proof of the rationality of the zeta function of algebraic varieties over a finite field, and was developing a theory of Fredholm determinants in p-adic Banach spaces in order to simplify Dwork's proof. In his course that winter Serre discussed the curve E_q, p-adic Banach spaces, and Dwork's proof and his own generalization of it to some L-functions. In Serre's seminar, Houzel talked on my letters, which Serre had had typed at the IHES for limited distribution. Ten years later they were published as a paper in Inventiones math. 12 (1971).

Grothendieck was visiting Harvard at the time I was writing the letters, and his presence was a great help. By then, in contrast to the bizarre negativism he had shown in his letter to Serre quoted at the end of Brian Conrad's introduction, he had become wildly optimistic, writing, again to Serre, in Oct.'61: "...Sooner or later it will be necessary to subsume ordinary analytic spaces, rigid analytic spaces, formal schemes, and maybe schemes themselves into a single kind of structure for which all the usual theorems hold: Stein spaces, Grauert finiteness, Remmert-Grauert GAGA, maybe also Rothstein type theorems...."

[1] The notes I wrote at the time (fall '59) are published in the book *Elliptic curves, modular forms and Fermat's last theorem*, International Press, Cambridge MA, 1995. Though I published nothing earlier, the curve E_q became known thanks to others.

[2] I had carelessly thought that the two complexes which are mentioned in the proof of Lemma 8.5 of my Inventiones 1971 paper are the same. Serre explained to me that they are not at all the same, but are, in fact, homotopic, so the lemma is OK.

The "affinoid" part of the theory I was trying to make certainly benefited from discussions with Grothendieck, but where his help was essential was in how to define a global rigid space by gluing affinoids together. In that section of the letters I say "we follow fully and faithfully a plan furnished by Grothendieck" and to the best of my recollection that is not much of an exaggeration. The plan, with its "h-structures" and "special coverings" is rather complicated and I'm not sure I ever really understood it well. The definition of rigid space given in the present book is certainly much simpler and more satisfying than the one Grothendieck and I arrived at. That they are equivalent is far from obvious.

That's the end of my involvement in the history. I regret that I am too ignorant to say much about the further development of the theory. It was soon substantially simplified by the introduction of rational subdomains by Gerritzen and Grauert. Grothendieck's vision of a grand unification quoted above did not take place, but what happened was much more interesting, especially the great idea of Berkovich. To understand how the theory has evolved, the present book is an excellent place to start.

In recent decades, p-adic geometry and p-adic cohomology theories have become indispensable tools in number theory, algebraic geometry, and the theory of automorphic representations. The Arizona Winter School 2007, on which the current book is based, was a unique opportunity to introduce graduate students to this subject.

Following invaluable introductions by John Tate and Vladimir Berkovich, two pioneers of non-archimedean geometry, Brian Conrad's chapter introduces the general theory of Tate's rigid analytic spaces, Raynaud's view of them as the generic fibers of formal schemes, and Berkovich spaces. Samit Dasgupta and Jeremy Teitelbaum discuss the p-adic upper half plane as an example of a rigid analytic space and give applications to number theory (modular forms and the p-adic Langlands program). Matthew Baker offers a detailed discussion of the Berkovich projective line and p-adic potential theory on that and more general Berkovich curves. Finally, Kiran Kedlaya discusses theoretical and computational aspects of p-adic cohomology and the zeta functions of varieties. This book will be a welcome addition to the library of any graduate student and researcher who is interested in learning about the techniques of p-adic geometry.

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ISBN 978-0-8218-4468-7

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ULECT/45