University
LECTURE
Series

Volume 48

Computational Geometry of Positive Definite Quadratic Forms

Polyhedral Reduction Theories, Algorithms, and Applications

Achill Schürmann

American Mathematical Society

http://dx.doi.org/10.1090/ulect/048

Computational Geometry of Positive Definite Quadratic Forms

University
LECTURE
Series

Volume 48

Computational Geometry of Positive Definite Quadratic Forms

Polyhedral Reduction Theories, Algorithms, and Applications

Achill Schürmann

American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification.* Primary 11–02, 52-02, 11Hxx, 52Bxx, 52Cxx, 90Cxx; Secondary 11J70, 11R80, 20B25, 20H05.

For additional information and updates on this book, visit
www.ams.org/bookpages/ulect-48

Library of Congress Cataloging-in-Publication Data

Schürmann, Achill, 1973–
Computational geometry of positive definite quadratic forms : polyhedral reduction theories, algorithms, and applications / Achill Schürmann.
p. cm. — (University lecture series ; v. 48)
Includes bibliographical references and index.
ISBN 978-0-8218-4735-0 (alk. paper)
1. Forms, Quadratic. 2. Polyhedral functions. 3. Discrete geometry. I. Title.

QA243.S39 2009
512.7′4—dc22 2008042435

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at http://www.ams.org/

10 9 8 7 6 5 4 3 2 1 14 13 12 11 10 09

For Ida, Jannis and Kristina

Contents

Preface

The *Geometry of Positive Definite Quadratic Forms* is a rich and old subject which arose in the arithmetic studies of quadratic forms. Through the seminal works of Minkowski and Voronoi a century ago, the geometric viewpoint became predominant. The study of arithmetical and inhomogeneous minima of positive definite quadratic forms turned into a study of lattice sphere packings and coverings. Lattices and, more generally, periodic (point) sets are by now widespread in mathematics and its applications. The important monograph "Sphere packings, Lattices and Groups" [**69**] by Conway and Sloane, with its over 100 pages of references, shows exemplarily the influence on other mathematical disciplines. This becomes particularly apparent for the 24-dimensional Leech lattice and its connections to number theory, group theory, coding theory and mathematical physics. Since the complexity of problems grows with the dimension, it is no surprise that over the past decades more and more computer support was used to study higher dimensional lattices and more general structures. Still, the Geometry of Positive Definite Quadratic Forms is an essential tool, not only in the study of lattice sphere packings and coverings.

One aim of this book is to give a nearly self-contained introduction to this beautiful subject. We present the known material with new proofs, which then admit natural generalizations. These extentions of the known theory were mainly targeted to support the study of extreme periodic sets. However, it turned out that the resulting new theory has other applications as well, as for example, the classification of totally real thin number fields. On the way, always an eye is kept on computability; algorithms are developed that allow computer assisted treatments. Using tools from combinatorial, from linear and from convex optimization, many difficult problems become accessible now. This is, for example, demonstrated in the search for new currently best-known lattice sphere coverings and in the classification of 8-dimensional perfect lattices, which previously was thought to be impossible with the known methods.

Although this book deals with classical topics which have been worked on extensively by numerous authors, it shows exemplarily how computers may help to gain new insights. On the one hand it is shown how computer assisted (sometimes heuristic) exploration helps to discover new exceptional structures. In many cases these would probably not have been found without a computer. On the other hand several computer assisted proofs are given, which deal with extraordinarily large data or involve large enumerations. It is shown how proofs can be obtained from numerical results, by postprocessing of roundoff solutions. All of these aspects of computer mathematics are nowadays supported by a growing functionality of computer algebra systems and by an increasing number of reliable small programs for specific purposes. In some cases one has to combine, to supplement and to improve on existing software tools. If solutions for basic tasks are obtained they should be made accessible to the growing community of computer enthusiastic mathematicians. Underlying many of the presented computational results are in particular two such programs: A program for rigorous determinant maximization (including semidefinite programming) allowing exact certified error bounds, and secondly, a program for polyhedral representation conversion under symmetries.

Computer assisted mathematical explorations and proofs are of increasing importance in many areas of modern mathematics. Even close to the topics of this book there have been amazing developments recently. An example is the proof by Hales [**129**], [**130**] of the famous Kepler conjecture. Several exciting results have been obtained in the context of *linear and semidefinite programming bounds* for spherical codes and point sets in Euclidean spaces. There is the new sphere packing bound by Cohn and Elkies [**60**], and based on it, the proof by Cohn and Kumar [**63**] (see also [**61**]) that the Leech lattice gives the best lattice sphere packing in 24 dimensions. There is the proof of Musin [**185**] showing that the kissing number in four dimensions is 24 (see [**197**] for an excellent survey). Shortly after, Bachoc and Vallentin [**6**], gave more general, new bounds on the size of spherical codes. Their works are followed by similar approaches for other problems, using semidefinite programming. As in some parts of this book, these works involve numerical computations which are then turned into mathematical rigorous proofs. Often numerical quests and subsequent mathematical analysis lead to new mathematical insights. A fascinating example is the study of *universally optimal point configurations*, recently invoked by Cohn and Kumar [**62**] (see also [**9**] and [**264**]). Although all of this is happening literally next door to the topics of this book, I decided to keep it focused as it is. Adequate treatments will hopefully fill other books in the near future. For now I encourage the reader to study the great original works.

Acknowledgments. This book grew out of lectures held at an Oberwolfach Seminar on Sphere Packings and at the University of Magdeburg, together with parts of research articles which were previously published, in a similar or partially different form (see [**45**], [**95**], [**225**], [**226**], [**228**], [**229**]). I thank my coauthors David Bremner, Francisco Santos Leal and in particular Mathieu Dutour Sikirić and Frank Vallentin for their many contributions and their shared enthusiasm for the subject. I thank Frank, Mathieu, Henry Cohn, Slava Grishukhin, Jeff Lagarias and Jacques Martinet for their very helpful feedback on prior versions.

I am grateful to my teachers Ulrich Betke and Jörg M. Wills and thank many other colleagues for fruitful communications on topics related to this book; among them David Avis, Christine Bachoc, Eiichi Bannai, Yves Benoist, Anne-Marie Bergé, Andras and Karoly Bezdek, Karoly Börözky Jr. and Sr., Jin-Yi Cai, Bob Connelly, Renaud Coulangeon, Jesus DeLoera, Antoine and Michel Deza, Nikolai P. Dolbilin, Noam Elkies, Bob Erdahl, Komei Fukuda, Lenny Fukshansky, Rajinder J. Hans-Gill, Robert L. Griess Jr., Peter Gritzmann, Peter M. Gruber, Thomas C. Hales, Jonathan Hanke, Martin Henk, Jenö Horvath, Michael Joswig, Abhinav Kumar, Wlodzimierz Kuperberg, Peter McMullen, Oleg Musin, Gabriele Nebe, Cordian Riener, Konstantin Rybnikov, Rudolf Scharlau, Claus-Peter Schnorr, François Sigrist, Warren D. Smith, Sal Torquato, Stephanie Vance, Boris Venkov, Günter M. Ziegler, Chuanming Zong and Stefan van Zwam. I thank the Deutsche Forschungsgemeinschaft (DFG) and the AMS editors for their support.

September 2008 Achill Schürmann

Overview

This overview is intended to guide through the topics and results of this book. Definitions and explanations for used terminology can be found with help of the index. Readers not so familiar with the treated topics may perhaps start by looking into the introductory material in Chapter 1 and Sections 2.1, 3.1 and 4.1. Most of the results in this book are described in the language of positive definite quadratic forms because it is the basis for most computations. In the following summary we nevertheless primarily use “lattice-terminology”, since it is more common nowadays. References to the literature can be found in the corresponding context of the book.

Chapters 1 and 2: An introduction. The first two chapters are mainly introductory and contain, in contrast to the remaining chapters, only a few new results.

In the first chapter we briefly recall the history of the theory of positive definite quadratic forms and its natural connection to the lattice sphere packing problem. Along the way, we introduce to basic notions and results used in the following chapters. In Sections 1.2, 1.3 and 1.4 we review known results of our main aimed at applications: the sphere packing, the sphere covering and the (simultaneous) sphere packing-covering problem. For all three problems there is a notorious open question: *Do there exist dimensions in which lattices do not give optimal configurations?* This problem is one of our main motivations to extend Voronoi's (lattice) reduction theories in Chapter 3 and Chapter 4.

The second chapter deals with several aspects of Minkowski's reduction theory, which in a sense is a prototype for other polyhedral reduction theories. In Section 2.1, we start with a short discussion about reduction in general, and then introduce Minkowski's theory in Section 2.2. We in particular give a previously unknown, non-redundant description of Minkowski's polyhedral reduction domain up to dimension 7. In Section 2.3 we describe relations to Minkowski's successive minima and state a challenging conjecture concerning an improvement of a classical and central theorem in the Geometry of Numbers. In Section 2.4 we end the second chapter with an application of Minkowski reduction to multidimensional continued fraction expansions, used for simultaneous Diophantine approximations.

Chapter 3: Voronoi's first reduction theory. In Section 3.1 we start by introducing to the theory of perfect lattices, respectively to “Voronoi's first reduction theory”. Based on so-called *Ryshkov polyhedra*, we give complete proofs for the theory and explain Voronoi's algorithm. We provide some background on computational tools (such as the shortest vector problem and isometry tests for lattices), which were recently used to finish the classification of perfect lattices up to dimension 8.

Our treatment of Voronoi's theory is presented in a way such that it can be generalized naturally from a lattice theory to a theory for *m-periodic point sets*. As lattices can be viewed as linear images of $\mathbb{Z}^d$ in $\mathbb{R}^d$, m-periodic sets are linear images of a *standard periodic set*

$$\bigcup_{i=1}^{m} \boldsymbol{t_i} + \mathbb{Z}^d \qquad \text{with} \qquad \boldsymbol{t}_i \in \mathbb{R}^d. \tag{0.1}$$

In Section 3.2, we introduce a new parameter space $\mathcal{S}^{d,m}_{>0}$ to deal with m-periodic sets in $\mathbb{R}^d$ up to isometries. We introduce *generalized Ryshkov sets*, on which determinant minimization yields the densest m-periodic sphere packings.

In Section 3.3 we analyze local optima (m-extreme periodic sets) of the packing density in $\mathcal{S}^{d,m}_{>0}$. We obtain necessary and sufficient conditions. It turns out that our framework provides a new explanation for a previously by Conway and Sloane observed phenomenon; namely for the existence of uncountably many 9-dimensional 2-periodic sphere packings (fluid diamond packings) which are as dense as the densest known lattice sphere packing. We investigate the possibility of improving the densest lattices (represented as points in $\mathcal{S}^{d,m}_{>0}$) locally, to obtain a periodic non-lattice set with larger packing density. We show that this is not possible (see Corollary 3.18). More generally, we show in Theorem 3.17 that a perfect, strongly eutactic lattice is *periodic extreme*, that is, it cannot locally be improved to a denser periodic point set.

In Section 3.4 we give extensions of Voronoi's algorithm, hereby laying the theoretical foundations for systematic, computer assisted searches for dense periodic sphere packings. We develop a "$\boldsymbol{t}$-theory" which enables us to find local sphere packing optima (*$\boldsymbol{t}$-extreme sets*) among all *$\boldsymbol{t}$-periodic point sets*. These are linear images of a standard periodic set (0.1) with a fixed translational part $\boldsymbol{t} = (\boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$. For the case of a rational matrix $\boldsymbol{t}$ we show in Theorem 3.23, in analogy to the classical theory, that there exist only finitely many $\boldsymbol{t}$-extreme sets. These can be enumerated by a generalized Voronoi algorithm. On top of the $\boldsymbol{t}$-theory, we extend the theory of G-perfect and T-perfect lattices (where G is a finite subgroup of the orthogonal group and T is a linear subspace in the space of quadratic forms).

This allows us to restrict searches for dense $\boldsymbol{t}$-periodic sphere packings to ones with specific features, for example with a fixed finite symmetry group. For rational $\boldsymbol{t}$ and a finite subgroup of the orthogonal group (leaving (0.1) invariant) we show in Theorem 3.25 that there exist only finitely many local optima among sphere packings with the corresponding properties. This generalizes works of Bergé, Martinet, Sigrist and others. Our proof relies on a general observation (Lemma 3.26) regarding group actions on polyhedral subdivisions. The section ends with three examples of the G-theory, describing new results for Eisenstein, Gaussian and Hurwitz quaternionic lattices.

Chapter 4: Voronoi's second reduction theory. In the fourth chapter we generalize Voronoi's second reduction theory which is based on Delone polyhedra and Delone subdivisions of lattices. In Section 4.1 we first give the necessary definitions, some background and we explain Voronoi's theory on secondary cones (also called L-type domains). We give a simplified and generalized proof for Voronoi's theory in Section 4.2. Theorem 4.7 extends Voronoi's theory to a "$\boldsymbol{t}$-theory" for Delone subdivisions having a standard periodic vertex-set (0.1). For rational $\boldsymbol{t}$ we show that there exist only finitely many non-equivalent secondary cones (see Theorem 4.13). This gives a foundation for systematic searches for thin periodic sphere coverings. Another application is a possible search for “good” periodic quantizers.

In Section 4.3 we introduce an “equivariant G-theory” for secondary cones and, more general, a theory of T*-secondary cones*, in analogy to the theory of G-perfect and T-perfect periodic sets described in Section 3.4. As in Section 3.4 we derive from Lemma 3.26 a finiteness result for rational $\boldsymbol{t}$ and finite symmetry groups G (see Theorem 4.19). We put special efforts into an explicit description on how to obtain all Delone subdivisions of lattices and periodic sets within the G- and T-theory (see in particular Theorem 4.15 and the algorithms in Section 4.3.4). This practicability of our results is of great importance for the applications in Chapter 5.

In Section 4.4 we introduce the secondary cones of single Delone polyhedra and polyhedral complexes. We propose an algorithm to decide whether or not a given simplex with integral vertices is Delone for some positive definite quadratic form. It can possibly serve as a tool to classify lattice Delone simplices (up to $\mathsf{GL}_d(\mathbb{Z})$-equivalence) of a given dimension. We show that the number of inequivalent Delone simplices increases dramatically with the dimension, by constructing Delone simplices with relative volume increasing super-exponentially with the dimension. Previously only linear growth was known.

Chapter 5: Local analysis of coverings and applications. In the fifth chapter we harvest the fruits of Chapter 4 and apply them to obtain several results in context of the lattice sphere covering problem. The first two sections have a preparatory character.

In Section 5.1 we formulate algorithms that allow to solve (in principle) the lattice sphere covering and packing-covering problem in a given dimension by solving a finite number of convex optimization problems. We give explicit descriptions of determinant maximization and semidefinite programs. They are subject to linear matrix inequalities expressing that the covering radius of an underlying periodic set is bounded by some given constant (see Proposition 5.5). We describe the software tools we developed to find local lattice covering and packing-covering optima (or at least for generating certified bounds on their covering density or packing-covering constant).

In Section 5.2 we provide tools for a detailed local analysis. These allow to check for local optimality of a lattice, if necessary computationally. Using convex optimization software, we sometimes only have certified ranges for the actual locally optimal value. We show how this restricted information can nevertheless be used to obtain structural informations about the actual local optimizers. We derive quickly computable, local lower bounds for the lattice covering density and the packing-covering constant, applicable to all lattices having a given collection of Delone simplices. For some lattices (as seen for the Leech lattice in Section 5.5) these bounds can be tight (for the right choice of simplices attaining the covering radius). Hence they have the potential to prove local optimality of a lattice.

In Section 5.3 we explain how the tools from the previous two sections can be used to find new best known lattice coverings and packing-coverings. Our techniques enable us in particular to confirm all previously known results up to dimension 5. It is notable that these were previously obtained (without computer assistance) in many years of work and on hundreds of published pages. Beyond the previously known, we extend the knowledge up to dimension 5 by additional information on all local optima. Although the lattice covering and the lattice packing-covering problem have both not been solved in dimension 6 so far, we describe conjecturally optimal lattices obtained by heuristic methods (see Section 5.3.3). Both lattices have the property that their Delone triangulation refines the Delone subdivision of the lattice E_6^*. As a first step in direction of a proof for global optimality, we prove computationally by a branch-and-bound method that the conjectural optimal covering lattice is the unique optimum among lattices having this refinement property (see Theorem 5.28). In many cases it is a problem to obtain exact coordinates for the new lattices. Exemplarily we show how to obtain exact coordinates for the new conjectural optimal 6-dimensional packing-covering lattice. We describe a new conjecturally optimal 7-dimensional covering lattice whose corresponding positive definite quadratic forms are (to our surprise) even rational (see Section 5.3.5). We describe how we obtained new, currently best known covering lattices in dimensions $d \geq 9$. These computations are based on our new T-secondary cone theory.

In Section 5.4 we apply the theory of T-secondary cones to a problem in algebraic number theory: the classification of all totally real thin algebraic number fields. Our classification is based on a list of 17 candidates, previously given by Bayer-Fluckiger and Nebe. By proving lower bounds on the covering density of positive definite quadratic forms in an associated linear subspaces T we exclude three of the candidates. This finishes the classification.

In Section 5.5 we take a closer look at two of the most exceptional lattices (in low dimensions): The root lattice E_8 and the Leech lattice. We show that both lattices are rigid, meaning they are uniquely determined by the type of their Delone subdivision. By applying our local lower bounds we show that the Leech lattice is a local lattice covering and packing-covering optimum. This shows in particular the existence of rigid, locally optimal lattice coverings, which affirmatively answers a long standing open question of Dickson (1968). The same method can not be applied to the E_8 root lattice. In fact, a local analysis reveals that it is not even a locally optimal covering lattice. In connection with a proof of this fact, we find a new currently best known 8-dimensional covering lattice (see Section 5.5.6).

In Section 5.6 we continue the investigation of lattices similar to E_8, revealing a previously unknown phenomenon: The existence of local lattice covering maxima, for which we derive necessary and sufficient conditions. Based on them we prove computationally that the E_6 root lattice is the only lattice covering maximum in dimensions less or equal to 6, aside of $\mathbb{Z}$ (which trivially is at the same time a 1-dimensional lattice covering optimum). The E_8 root lattice itself turns out to be only "almost" a local lattice covering maximum: Almost any, but not every local change of an associated positive definite quadratic form yields a lower covering density. We baptize these lattices *covering pessima* and prove that a lattice is a covering pessimum, if the Delone polytopes attaining its covering radius are all regular cross polytopes. This is for example the case for the root lattices D_4 and E_8.

In Section 5.7 we consider local lattice covering maxima within a linear subspace T in the space of (associated) quadratic forms. We show a connection to the famous *Minkowski conjecture*, respectively to the so-called *covering conjecture*. Based on our new T-secondary cone theory we obtain an algorithm which can be used to verify or falsify the covering conjecture in a given dimension. By a recent result of Curtis McMullen the covering conjecture implies the Minkowski conjecture. Thus (in principle) the Minkowski conjecture can be verified in a given dimension using our algorithm.

Appendix A: Polyhedral representation conversion. In Appendix A we provide some background on polyhedra and in particular on a fundamental problem in polyhedral combinatorics, on which many computational results described in this book rely: The representation conversion of polyhedra with large symmetries. We start with a brief review of some basic properties and explain how to compute different polyhedral symmetries. We address the group theoretical background as well as data structures and software tools, which are necessary to deal with large orbits. Finally, we describe Decomposition methods for the representation conversion problem, which have proved to perform best in practice on the polyhedra we treated during our studies.

Conclusions and prospects. In this book we describe foundations and some applications of a *Computational Geometry of Positive Definite Quadratic Forms*. Based on generalizations of Voronoi's reduction theories, we provide new algorithms and details on their practical implementation. So far, such implementations have only been used for applications in the context of lattice sphere coverings (see Chapter 5). However, we are convinced that many more, new results can be obtained. Some interesting possible future projects are listed in Appendix B.

We think in particular that (in analogy to the lattice covering problem) many new record breaking lattices for the packing-covering constant and for the so-called *quantizer problem* can be found. We strongly believe that the theory described in this book can be used to find periodic non-lattice sets, which are "better" (with respect to any of the discussed problems) than any lattice of the same dimension. We think that it is only a matter of time and a question of sufficient computational resources, until we will see a solution of the 6-dimensional covering and the 9-dimensional sphere packing problem.

A key ingredient for computational successes described in this book is the representation conversion of polyhedra under symmetry. We think that further improvement on the available tools will not only yield further progress for the problems discussed here, but also for applications beyond the scope of this book. Furthermore, we are convinced that the future will show more and more mathematical proofs based on rigorous numerical computations and non-linear convex optimization. However, there is still plenty of groundwork to be done.

So let's do it…

Bibliography

[1] V. Alexeev, *Higher-dimensional analogues of stable curves*, Proceedings of Madrid ICM2006, Eur. Math. Soc. Pub. House, 2006, pp. 515–536.

[2] M.M. Anzin, *On the density of a lattice covering for $n = 11$ and $n = 14$*, Russian Math. Surveys **57** (2002), 407–409, translation from Uspekhi Mat. Nauk 57, No.2(344), 187–188 (2002).

[3] T.M. Apostol and M.A. Mnatsakanian, *Sums of squares of distances in m-space*, Amer. Math. Monthly **110** (2003), 516–526.

[4] A. Ash, *On eutactic forms*, Can. J. Math. **29** (1977), 1040–1054.

[5] A. Ash, D. Mumford, M. Rapoport, and Y. Tai, *Smooth compactification of locally symmetric varieties*, Lie Groups: History, Frontiers and Applications. Vol. IV., Brookline, 1975.

[6] C. Bachoc and F. Vallentin, *New upper bounds for kissing numbers from semidefinite programming*, J. Amer. Math. Soc. **21** (2008), 909–924.

[7] M.L. Balinski, *On the graph structure of convex polyhedra in n-space*, Pacific J. Math. **11** (1961), 431–434.

[8] K. Ball, *A lower bound for the optimal density of lattice packings*, Int. Math. Res. Not. **1992** (1992), 217–221.

[9] B. Ballinger, G. Blekherman, H. Cohn, N. Giansiracusa, E. Kelly, and A. Schürmann, *Experimental study of energy-minimizing point configurations on spheres*, Experiment. Math. (2008), to appear, preprint at arXiv:math/0611451.

[10] R.P. Bambah, *Lattice coverings with four-dimensional spheres*, Math. Proc. Cambridge Philos. Soc. **50** (1954), 203–208.

[11] ______, *On lattice coverings by spheres*, Proc. Nat. Acad. Sci. India **20** (1954), 25–52.

[12] R.P. Bambah, V.C. Dumir, and R.J. Hans-Gill, *Non-homogeneous problems: conjectures of Minkowski and Watson*, Number theory, Trends Math., Birkhäuser, Basel, 2000, pp. 15–41.

[13] E.P. Baranovskii, *Local density minima of a lattice covering of a four-dimensional Euclidean space by equal spheres*, Soviet Math. Dokl **6** (1965), 1131–1133, translation from Dokl. Akad. Nauk SSSR 164, 13–15 (1965).

[14] ______, *Local minima of the density of a lattice covering of four-dimensional Euclidean space by equal balls*, Siberian Math. J. **7** (1966), 779–798, translation from Sibirsk. Mat. Zh. 7, 974–1001 (1966).

[15] ______, *Simplexes of L-decompositions of Euclidean spaces*, Math. Notes **10** (1971), 827–834, translation from Mat. Zametki 10, 659–670 (1971).

[16] ______, *Volumes of L-simplexes of five-dimensional lattices*, Math. Notes **13** (1973), 460–466, translation from Mat. Zametki 13, 771–782 (1973).

[17] ______, *Partition of Euclidean spaces into L-polytopes of certain perfect lattices*, Proc. Steklov Inst. Math. **196** (1992), 29–51, translation from Tr. Mat. Inst. Steklova 196, 27–46 (1991).

[18] ______, *The perfect lattices $\Gamma(\mathfrak{A}^n)$, and the covering density of $\Gamma(\mathfrak{A}^9)$*, European J. Combin. **15** (1994), 317–323.

[19] ______, *Application of the empty sphere method to the description of an L-partition defined by a Leech lattice*, Chebyshevskiĭ Sb. **6** (2005), 30–35.

[20] E.P. Baranovskii and V.P. Grishukhin, *Non-rigidity degree of a lattice and rigid lattices*, European J. Combin. **22** (2001), 921–935.

[21] E.P. Baranovskii and S.S. Ryshkov, *Primitive five-dimensional parallelohedra*, Soviet Math. Dokl. **14** (1973), 1391–1395, translation from Dokl. Akad. Nauk SSSR 212, 532–535 (1973).

[22] ———, *The combinatorial-metric structure of L-partitions of general five-dimensional lattices*, Soviet Math. Dokl. **16** (1975), 47–51, translation from Dokl. Akad. Nauk SSSR 220, 265–268 (1975).

[23] ———, *Repartitioning complexes in n-dimensional lattices (with full description for $n \leq 6$)*, Voronoi's Impact on Modern Science, Book II (ed. P. Engel et. al.), Proc. Inst. Math. Nat. Acad. Sci. Ukraine, vol. 21, Institute of Math., Kyiv, 1998, pp. 115–124.

[24] E.S. Barnes, *The covering of space by spheres*, Canad. J. Math. **8** (1956), 293–304.

[25] ———, *The complete enumeration of extreme senary forms*, Philos. Trans. Roy. Soc. London. Ser. A. **249** (1957), 461–506.

[26] E.S. Barnes and M.J. Cohn, *On Minkowski reduction of positive quaternary quadratic forms*, Mathematika **23** (1976), 156–158.

[27] E.S. Barnes and T.J. Dickson, *Extreme coverings of n-space by spheres*, J. Austral. Math. Soc. **7** (1967), 115–127, corrigendum, ibid. 8 (1968), 638–640.

[28] E.S. Barnes and D.W. Trenerry, *A class of extreme lattice-coverings of n-space by spheres*, J. Austral. Math. Soc. **14** (1972), 247–256.

[29] H. Bass, M. Lazard, and J.P. Serre, *Sous-groupes d'indice fini dans $SL(n, Z)$*, Bull. Am. Math. Soc. **70** (1964), 385–392.

[30] H. Bass, J.W. Milnor, and J.P. Serre, *Solution of the congruence subgroup problem for $SL_n (n \geq 3)$ and $Sp_{2n}(n \geq 2)$*, Publ. Math., Inst. Hautes Étud. Sci. **33** (1967), 59–137.

[31] C. Batut, *Classification of quintic eutactic forms*, Math. Comp. **70** (2001), 395–417.

[32] C. Bavard, *Systole et invariant d'Hermite. (Systole and Hermite invariant)*, J. Reine Angew. Math. **482** (1997), 93–120.

[33] ———, *Théorie de Voronoĭ géométrique. Propriétés de finitude pour les familles de réseaux et analogues*, Bull. Soc. Math. France **133** (2005), no. 2, 205–257.

[34] E. Bayer Fluckiger, *Upper bounds for Euclidean minima of algebraic number fields*, J. Number Theory **121** (2004), 305–323.

[35] E. Bayer Fluckiger and G. Nebe, *On the Euclidean minimum of some real number fields*, J. Théor. Nombres Bordeaux **17** (2005), 437–454.

[36] A.M. Bergé and J. Martinet, *Sur la classification des réseaux eutactiques*, J. London Math. Soc. (2) **53** (1996), 417–432.

[37] A.M. Bergé, J. Martinet, and F. Sigrist, *Une généralisation de l'algorithme de Voronoi pour les formes quadratiques*, Astérisque **209** (1992), 137–158.

[38] M. Berger, *Geometry I*, Springer, Berlin, 1987.

[39] A. Bezdek, K. Bezdek, and R. Connelly, *Finite and uniform stability of sphere packings*, Discrete Comput. Geom. **20** (1998), 111–130.

[40] J. Bierbrauer and Y. Edel, *Dense sphere packings from new codes*, J. Algebr. Comb. **11** (2000), 95–100.

[41] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. Ziegler, *Oriented matroids*, Cambridge University Press, Cambridge, 1999, 2nd ed.

[42] M.N. Bleicher, *Lattice coverings of n-space by spheres*, Canad. J. Math. **14** (1962), 632–650.

[43] H.F. Blichfeldt, *The minimum values of positive quadratic forms in six, seven and eight variables*, Math. Z. **39** (1935), 1–15.

[44] K. Böröczky, *Closest packing and loosest covering of the space with balls*, Studia Sci. Math. Hungar. **21** (1986), 79–89.

[45] D. Bremner, M. Dutour Sikirić, and A. Schürmann, *Polyhedral representation conversion up to symmetries*, Proceedings of the 2006 CRM workshop on polyhedral computation (D. Bremner D. Avis and A. Deza, eds.), CRM Proceedings & Lecture Notes, AMS, Providence, 2008, to appear, preprint at arXiv:math/0702239.

[46] D. Bremner, K. Fukuda, and A. Marzetta, *Primal-dual methods for vertex and facet enumeration*, Discrete Comput. Geom. **20** (1998), 333–357.

[47] A.J. Brentjes, *Multidimensional continued fraction algorithms*, Mathematical Centre Tracts, vol. 145, Mathematisch Centrum, Amsterdam, 1981.

[48] G. Brinkmann, *Isomorphism rejection in structure generation programs*, Discrete mathematical chemistry (New Brunswick, NJ, 1998), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 51, Amer. Math. Soc., Providence, RI, 2000, pp. 25–38.

[49] H. Brown, J. Neubüser, and H. Zassenhaus, *On integral groups. III: Normalizers*, Math. Comput. **27** (1973), 167–182.

[50] G.J. Butler, *Simultaneous packing and covering in Euclidean space*, London Math. Soc. **25** (1972), 721–735.

[51] J.W.S. Cassels, *An introduction to the geometry of numbers*, Springer, Berlin, 1971, 2nd corrected print.

[52] ______, *Rational quadratic forms*, Academic Press, London, 1978.

[53] C.S. Chan and D.P. Robbins, *On the volume of the polytope of doubly stochastic matrices*, Experiment. Math. **8** (1999), 291–300.

[54] D.R. Chand and S.S. Kapur, *An algorithm for convex polytopes*, Journal of the ACM **17** (1970), 78–86.

[55] T. Christof and G. Reinelt, *Combinatorial optimization and small polytopes*, Top **4** (1996), 1–64.

[56] ______, *Decomposition and parallelization techniques for enumerating the facets of combinatorial polytopes*, Internat. J. Comput. Geom. Appl. **11** (2001), 423–437.

[57] V. Chvátal, *Linear programming*, W.H. Freeman and Company, New York, 1983.

[58] H. Cohen, *A course in computational algebraic number theory*, Springer, Berlin, 1993.

[59] H. Cohn, *New upper bounds on sphere packings. II*, Geom. Topol. **6** (2002), 329–353.

[60] H. Cohn and N. Elkies, *New upper bounds on sphere packings. I*, Ann. Math. **157** (2003), 689–714.

[61] H. Cohn and A. Kumar, *The densest lattice in twenty-four dimensions*, Electron. Res. Announc. Am. Math. Soc. **10** (2004), 58–67.

[62] ______, *Universally optimal distribution of points on spheres*, J. Amer. Math. Soc. **20** (2007), 99–148.

[63] ______, *Optimality and uniqueness of the Leech lattice among lattices*, Ann. Math. (2008), to appear, preprint at arXiv:math/0403263.

[64] M.J. Cohn, Z.D. Lomakina, and S.S. Ryshkov, *Vertices of the symmetrized Minkowski region for $n \leq 5$*, Proc. Steklov Inst. Math. **152** (1982), 213–223, translation from Tr. Mat. Inst. Steklova 152, 195–203 (1980).

[65] J.H. Conway and N.J.A. Sloane, *Low-dimensional lattices. III: Perfect forms*, Proc. R. Soc. Lond., Ser. A **418** (1988), 43–80, see also Errata (1989).

[66] ______, *The cell structures of certain lattices*, Miscellanea mathematica, Springer, Berlin, 1991, pp. 71–107.

[67] ______, *What are all the best sphere packings in low dimensions?*, Discrete Comput. Geom. **13** (1995), 383–403.

[68] ______, *The antipode construction for sphere packings*, Invent. Math. **123** (1996), 309–313.

[69] ______, *Sphere packings, lattices and groups*, Springer, New York, 1999, 3rd ed.

[70] R. Coulangeon, *Voronoï theory over algebraic number fields*, Réseaux euclidiens, designs sphériques et formes modulaires, Monogr. Enseign. Math., vol. 37, Enseignement Math., Geneva, 2001, pp. 147–162.

[71] H.S.M. Coxeter, *Integral Cayley numbers*, Duke Math. J. **13** (1946), 561–578, pp. 21–39 in Coxeter, The beauty of geometry, Dover, 1999.

[72] ______, *Extreme forms*, Canad. J. Math. **3** (1951), 391–441.

[73] H.S.M. Coxeter, L. Few, and C.A. Rogers, *Covering space with equal spheres*, Mathematika **6** (1959), 147–157.

[74] J.A. DeLoera, J. Rambau, and F. Santos Leal, *Triangulations: Applications, structures, algorithms*, Springer, Berlin, 2009, to appear.

[75] B.N. Delone, *Sur la partition régulière de l'espace a 4 dimensions*, Izv. Akad. Nauk SSSR Otdel. Fiz.-Mat. Nauk (1929), 79–110 and 145–164.

[76] ______, *The geometry of positive quadratic forms*, Uspekhi Mat. Nauk **3** (1937), 16–62, in Russian.

[77] B.N. Delone, N.P. Dolbilin, S.S. Ryshkov, and M.I. Stogrin, *A new construction of the theory of lattice coverings of an n-dimensional space by congruent balls*, Math. USSR-Izv. **4** (1970), 293–302, translattion from Izv. Akad. Nauk SSSR Ser. Mat. 34, 289–298 (1970).

[78] B.N. Delone and S.S. Ryshkov, *Solution of the problem on the least dense lattice covering of a 4-dimensional space by equal spheres*, Soviet Math. Dokl. **4** (1963), 1333–1334, translation from Dokl. Akad. Nauk SSSR 152, 523–224 (1963).

[79] A. Deza, K. Fukuda, T. Mizutani, and C. Vo, *On the face lattice of the metric polytope*, Discrete and computational geometry, Lecture Notes in Comput. Sci., vol. 2866, Springer, Berlin, 2003, pp. 118–128.

[80] A. Deza, K. Fukuda, D. Pasechnik, and M. Sato, *On the skeleton of the metric polytope*, Discrete and computational geometry (Tokyo, 2000), Lecture Notes in Comput. Sci., vol. 2098, Springer, Berlin, 2001, pp. 125–136.

[81] A. Deza and G. Indik, *A counterexample to the dominating set conjecture*, Optimization Letters **1-2** (2007), 163–169.

[82] M. Deza, V.P. Grishukhin, and M. Laurent, *The hypermetric cone is polyhedral*, Combinatorica **13** (1993), 397–411.

[83] M.M. Deza and M. Laurent, *Geometry of cuts and metrics*, Springer, Berlin, 1997.

[84] T.J. Dickson, *The extreme coverings of* 4*-space by spheres*, J. Austral. Math. Soc. **7** (1967), 490–496.

[85] ______, *A sufficient condition for an extreme covering of n-space by spheres*, J. Austral. Math. Soc. **8** (1968), 56–62.

[86] M. Dutour, *The six-dimensional Delaunay polytopes*, European J. Combin. **25** (2004), 535–548.

[87] M. Dutour, R. Erdahl, and K. Rybnikov, *Perfect Delaunay polytopes in low dimensions*, Integers **7** (2007), A39, 49 pp. (electronic).

[88] M. Dutour and F. Vallentin, *Some six-dimensional rigid forms*, Proceedings of Voronoi conference on analytic number theory and spatial tessellations, 2004, preprint at arXiv:math/0401191.

[89] M. Dutour Sikirić and V.P. Grishukhin, *The decomposition of the hypermetric cone into L-domains*, European J. Combin. (2008), to appear, preprint at arXiv:0708.0747.

[90] M. Dutour Sikirić and K. Rybnikov, *A new algorithm in geometry of numbers*, Proceedings of The 4th International Symposium on Voronoi Diagrams in Science and Engineering, Pontypridd, Wales, UK, 9-11 July 2007, 2007, preprint at arXiv:0710.4948.

[91] M. Dutour Sikirić, A. Schürmann, and F. Vallentin, *Classification of eight dimensional perfect forms*, Electron. Res. Announc. Amer. Math. Soc. **13** (2007), 21–32.

[92] ______, *Complexity and algorithms for computing Voronoi cells of lattices*, Math. Comp. (2008), to appear.

[93] ______, *The contact polytope of the Leech lattice*, 2008, in preparation.

[94] ______, *Covering maxima*, 2008, in preparation.

[95] ______, *A generalization of Voronoi's reduction theory and applications*, Duke Math. J. **142** (2008), 127–164.

[96] M.E. Dyer, *The complexity of vertex enumeration methods*, Math. Oper. Res. **8** (1983), 381–402.

[97] F.J. Dyson, *On the product of four non-homogeneous linear forms*, Annals of Math. **49** (1948), 82–109.

[98] H. Edelsbrunner, *Geometry and topology for mesh generation*, Cambridge University Press, Cambridge, 2001.

[99] P. Elbaz-Vincent, H. Gangl, and C. Soulé, *Quelques calculs de la cohomologie de* $\mathrm{GL}_N(\mathbb{Z})$ *et de la K-théorie de* $\mathbb{Z}$, C. R. Math. Acad. Sci. Paris **335** (2002), 321–324.

[100] S. Eliahoua and M. Kervaire, *A survey on modular Hadamard matrices*, Discrete Mathematics **302** (2005), 85–106.

[101] P. Engel, *New investigations of parallelohedra in R^d*, Voronoi's Impact on Modern Science, Book II (ed. P. Engel et. al.) (Kyiv), Proc. Inst. Math. Nat. Acad. Sci. Ukraine, vol. 21, Institute of Math., 1998, pp. 22–60.

[102] ______, *The contraction types of parallelohedra in* $\mathbb{E}^5$, Acta Cryst. Sect. A **56** (2000), 491–496.

[103] P. Engel and V.P. Grishukhin, *There are exactly* 222 *L-types of primitive five-dimensional lattices*, European J. Combin. **23** (2002), 275–279.

[104] R. Erdahl, *A cone of inhomogeneous second-order polynomials*, Discrete Comput. Geom. **8** (1992), 387–416.

[105] R. Erdahl and K. Rybnikov, *An infinite series of perfect quadratic forms and big Delaunay simplices in* $\mathbb{Z}^n$, Tr. Mat. Inst. Steklova **239** (2002), 170–178.

[106] ______, *On Voronoi's two tilings of the cone of metrical forms*, Rend. Circ. Mat. Palermo (2) Suppl. (2002), 279–296, IV International Conference in "Stochastic Geometry, Convex Bodies, Empirical Measures & Applications to Engineering Science", Vol. I (Tropea, 2001).

[107] G. Fejes Tóth and J.C. Lagarias, *Guest editors foreword*, Discrete Comput. Geom. **36** (2006), 1–3.

[108] L. Fejes Tóth, *Über einen geometrischen Satz.*, Math. Z. **46** (1940), 83–85.
[109] ______, *Close packing and loose covering with balls*, Publ. Math. Debrecen **23** (1976), 324–326.
[110] L. Few, *Covering space by spheres*, Mathematika **3** (1956), 136–139.
[111] K. Fukuda and A. Prodon, *Double description method revisited*, Combinatorics and computer science (Brest, 1995), Lecture Notes in Comput. Sci., vol. 1120, Springer, Berlin, 1996, pp. 91–111.
[112] A.F. Gameckii, *On the theory of covering Euclidean n-space by equal spheres*, Soviet Math. Dokl. **3** (1962), 1410–1414, translation from Dokl. Akad. Nauk SSSR 146, 991–994 (1962).
[113] ______, *The optimality of Voronoi's lattice of first type among lattices of first type of arbitrary dimension*, Soviet Math. Dokl. **4** (1963), 1014–1016, translation from Dokl. Akad. Nauk SSSR 151, 482–484 (1963).
[114] C.F. Gauss, *Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen von Ludwig August Seeber*, J. Reine. Angew. Math. **20** (1840), 312–320.
[115] E. Gawrilow and M. Joswig, *Geometric reasoning with polymake*, arXiv:math/0507273, 2005.
[116] J.R. Goldman, *The queen of mathematics*, AK Peters, Wellesley, 1998.
[117] V.P. Grishukhin, *Computing extreme rays of the metric cone for seven points*, European J. Combin. (1992), 153–165.
[118] ______, *Infinite series of extreme Delaunay polytopes*, European J. Combin. **27** (2006), 481–495.
[119] ______, *A remarkable family of six-dimensional lattices*, 2006, CEMI Russian Academy of Sciences, Working Paper # 214.
[120] H. Groemer, *Existenzsätze für Lagerungen im Euklidischen Raum*, Math. Z. **81** (1963), 260–278.
[121] P.M. Gruber, *Convex and discrete geometry*, Springer, Berlin, 2007.
[122] ______, *Application of an idea of Voronoĭ to lattice packing of convex bodies*, Discrete Comput. Geom. (2008), to appear.
[123] ______, *Geometry of the cone of positive quadratic forms*, Forum Math. (2008), to appear.
[124] P.M. Gruber and C.G. Lekkerkerker, *Geometry of numbers*, North–Holland, Amsterdam, 1987.
[125] P.M. Gruber and S.S. Ryshkov, *Facet-to-facet implies face-to-face*, European J. Combin. **10** (1989), 83–84.
[126] B. Grünbaum, *Convex polytopes*, Springer, New York, 2002, 2nd ed. prepared in cooperation with V. Kaibel, V. Klee and G.M. Ziegler.
[127] T. Grüner, R. Laue, and M. Reinhard, *Algorithms for group actions applied to graph generation*, Groups and computation, II (New Brunswick, NJ, 1995), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 28, Amer. Math. Soc., Providence, RI, 1997, pp. 113–122.
[128] H. Haiman, *A simple and relatively efficient triangulation of the n-cube*, Discrete Comput. Geom. **6** (1991), 287–289.
[129] T.C. Hales, *A proof of the Kepler conjecture*, Ann. Math **162** (2005), 1065–1185.
[130] ______, *A proof of the Kepler conjecture (unabridged version)*, Discrete Comput. Geom. **36** (2006), 1–265.
[131] R.J. Hans-Gill, M. Raka, and R. Sehmi, *On conjectures of minkowski and woods for* $n = 7$, 2007, preprint.
[132] R.J. Hans-Gill, M. Raka, R. Sehmi, and Sucheta, *A unified simple proof of a conjecture of woods for* $n \leq 6$, 2007, preprint.
[133] G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, fifth ed., The Clarendon Press Oxford University Press, New York, 1979.
[134] D. Hensley, *Continued fractions*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.
[135] C. Hermite, *Extraits de lettres de M.Ch. Hermite à M. Jacobi sur différents objets de la théorie des nombres, Deuxième lettre*, J. Reine. Angew. Math. **40** (1850), 279–290, in Oeuvres de Charles Hermite I, 100–121; letter is dated to Aug. 6th 1845.
[136] ______, *Sur l'introduction des variables continues dans la théorie des nombres*, J. Reine. Angew. Math. **41** (1851), 191–216, in Oeuvres de Charles Hermite I, 164–192.
[137] E. Hlawka, *Ausfüllung und Überdeckung konvexer Körper durch konvexe Körper*, Monatsh. Math. **53** (1949), 81–131.

[138] D.F. Holt, B. Eick, and E.A. O'Brien, *Handbook of computational group theory*, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[139] J. Horváth, *On close lattice packing of unit spheres in the space* E^n, Proc. Steklov Inst. Math. **152** (1982), 237–254, translation from Tr. Mat. Inst. Steklova 152, 216–231 (1980).

[140] ———, *Ph.D. thesis: Several problems of n–dimensional discrete geometry*, Steklov Institute of Mathematics, 1986.

[141] W.Y. Hsiang, *Least action principle of crystal formation of dense packing type and Kepler's conjecture*, World Scientific, Cambridge, 2001.

[142] K. Hulek and G.K. Sankaran, *The geometry of Siegel modular varieties*, Higher dimensional birational geometry (Kyoto, 1997), Adv. Stud. Pure Math., vol. 35, Math. Soc. Japan, Tokyo, 2002, pp. 89–156.

[143] D.-O. Jaquet–Chiffelle, *Énumération complète des classes de formes parfaites en dimension* 7, Ann. Inst. Fourier (Grenoble) **43** (1993), 21–55.

[144] ———, *Trois théorème de finitude pour les G-résaux*, J. Théor. Nombres Bordeaux **7** (1995), 165–176.

[145] D.-O. Jaquet–Chiffelle and F. Sigrist, *Classification des formes quadratiques réelles: un contre-exemple à la finitude*, Acta Arith. **68** (1994), 291–294.

[146] F. Jarre and J. Stoer, *Optimierung*, Springer, Berlin, 2003.

[147] G.A. Kabatjanskii and V.I. Levenstein, *Bounds for packings on the sphere and in space*, Probl. Inf. Transm. **14** (1978), 1–17, translation from Problemy Peredachi Informatsii, 14, 3–25 (1978).

[148] V. Kaibel and A. Schwartz, *On the complexity of polytope isomorphism problems*, Graphs Combin. **19** (2003), 215–230.

[149] P. Kaski and P.R.J. Östergård, *Classification algorithms for codes and designs*, vol. 15, Springer, Berlin, 2006.

[150] A. Kerber, *Applied finite group actions*, 2nd ed., vol. 19, Springer, Berlin, 1999.

[151] R. Kershner, *The number of circles covering a set*, Amer. J. Math. **61** (1939), 665–671.

[152] L. Khachiyan, E. Boros, K. Borys, K. Elbassioni, and V. Gurvich, *Generating all vertices of a polyhedron is hard*, Discrete Comput. Geom. **39** (2008), 174–190.

[153] A.Ya. Khinchin, *Continued fractions*, Dover Publications Inc., Mineola, NY, 1997, With a preface by B. V. Gnedenko, Reprint of the 1964 translation.

[154] M. Kneser, *Two remarks on extreme forms*, Canad. J. Math. **7** (1955), 145–149.

[155] A.N. Korkin and E.I. Zolotarev, *Sur les formes quadratiques positives*, Math. Ann. **11** (1877), 242–292.

[156] G. Kuperberg, *Notions of denseness*, Geom. Topol. **4** (2000), 277–292.

[157] J.C. Lagarias, *Geodesic multidimensional continued fractions*, Proc. London Math. Soc. (3) **69** (1994), 464–488.

[158] ———, *Bounds for local density of sphere packings and the Kepler conjecture*, Discrete Comput. Geom. **27** (2002), 165–193.

[159] J.C. Lagarias, H.W. Lenstra Jr., and C.P. Schnorr, *Korkin–Zolotarev bases and successive minima of a lattice and its reciprocal lattice*, Combinatorica **10** (1990), 333–348.

[160] J.C. Lagarias and P.A.B. Pleasants, *Local complexity of Delone sets and crystallinity*, Canad. Math. Bull. **45** (2002), 634–652.

[161] J.L. Lagrange, *Démonstartion d'un théorème d'arithmétique*, Nouv. Mém. Acad. Berlin (1770), in Oeuvres de Lagrange III, 189–201.

[162] ———, *Recherches d'arithmétique*, Nouv. Mém. Acad. Berlin (1773), 265–312, in Oeuvres de Lagrange III, 695–795.

[163] J. Larmouth, *The enumeration of perfect forms*, Computers in Number Theory, Proc. Atlas Sympos. No.2, Oxford 1969, 237-239, 1971.

[164] J. Leech, *Notes on sphere packings*, Canad. J. Math. **19** (1967), 251–267.

[165] J. Leech and N.J.A. Sloane, *New sphere packings in dimensions* 9 - 15, Bull. Am. Math. Soc. **76** (1970), 1006–1010.

[166] A.-M. Legendre, *Essai sur la théorie des nombres*, Duprat, Paris, 1798.

[167] H.W. Lenstra Jr., *Euclidean number fields of large degree*, Invent. Math. **38** (1976/77), 237–254.

[168] J.S. Leon, *Permutation group algorithms based on partitions. I. Theory and algorithms*, J. Symbolic Comput. **12** (1991), 533–583.

[169] ———, *Partitions, refinements, and permutation group computation*, Groups and computation, II (New Brunswick, NJ, 1995), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 28, Amer. Math. Soc., Providence, RI, 1997, pp. 123–158.

[170] V.I. Levenshtein, *On bounds for packings in n-dimensional Euclidean space*, Sov. Math., Dokl. **20** (1979), 417–421, translation from Dokl. Akad. Nauk SSSR 245, 1299–1303 (1979).

[171] E.M. Luks, *Permutation groups and polynomial-time computation*, Groups and computation (New Brunswick, NJ, 1991), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 11, Amer. Math. Soc., Providence, RI, 1993, pp. 139–175.

[172] K. Mahler, *A theorem on inhomogeneous diophantine inequalities*, Nederl. Akad. Wetensch., Proc. **41** (1938), 634–637.

[173] ———, *On Minkowski's theory of reduction of positive definite quadratic forms*, Q. J. Math., Oxf. Ser. **9** (1938), 259–262.

[174] J. Martinet, *Perfect lattices in Euclidean spaces*, Springer, Berlin, 2003.

[175] M. McConnell, *Generalizations of Voronoi I to symmetric spaces and arithmetic groups*, Voronoi's Impact on Modern Science (ed. P Engel et. al.), Proc. Inst. Math. Nat. Acad. Sci. Ukraine, vol. 21, Institute of Math., Kyiv, 1998, pp. 155–185.

[176] C.T. McMullen, *Minkowski's conjecture, well-rounded lattices and topological dimension*, J. Am. Math. Soc. **18** (2005), 711–734.

[177] P. McMullen and G.C. Shephard, *Convex polytopes and the upper bound conjecture*, Cambridge University Press, Cambridge, 1971.

[178] J.L. Mennicke, *Finite factor groups of the unimodular group*, Ann. Math. **81** (1965), 31–37.

[179] H. Minkowski, *Zur Theorie der positiven quadratischen Formen*, J. Reine Angew. Math. **101** (1887), 196–202, Reprint in *Gesammelte Abhandlungen*, Band I, Teubner, Leipzig, 1911.

[180] ———, *Über die positiven quadratischen Formen und über kettenbruchähnliche Algorithmen*, J. Reine Angew. Math. **107** (1891), 278–297, Reprint in *Gesammelte Abhandlungen*, Band I, Teubner, Leipzig, 1911.

[181] ———, *Diskontinuitätsbereich für arithmetische Äquivalenz*, J. Reine Angew. Math. **129** (1905), 220–274, Reprint in *Gesammelte Abhandlungen*, Band II, Teubner, Leipzig, 1911.

[182] ———, *Diophantische Approximationen*, Teubner, Leipzig, 1907, Reprint by Chelsea, New-York, 1957.

[183] ———, *Geometrie der Zahlen*, Teubner, Leipzig, 1910, 2nd ed.; Reprint by Chelsea, New-York, 1953.

[184] R.V. Moody and J. Patera, *Voronoi domains and dual cells in the generalized kaleidoscope with applications to root and weight lattices*, Canad. J. Math. **47** (1995), 573–605.

[185] O.R. Musin, *The kissing number in four dimensions*, Ann. Math. **168** (2008), 1–32.

[186] Y. Namikawa, *A new compactification of the Siegel space and degeneration of abelian varieties. I.*, Math. Ann. **221** (1976), 97–141.

[187] G. Nebe, *Gitter und Modulformen*, Jahresber. Deutsch. Math.-Verein. **104** (2002), 125–144.

[188] G. Nebe and N.J.A. Sloane, *A catalogue of lattices*, `http://www.research.att.com/~njas/lattices/`.

[189] C.E. Nelson, *The reduction of positive definite quinary quadratic forms*, Aequationes Math. **11** (1974), 163–168.

[190] Y. Nesterov and A. Nemirovskii, *Interior-point polynomial algorithms in convex programming*, vol. 13, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.

[191] T. Oda and C.S. Seshadri, *Compactifications of the generalized Jacobian variety*, Trans. Amer. Math. Soc. **253** (1979), 1–90.

[192] J. Opgenorth, *Ph.D. thesis: Normalisatoren und Bravaismannigfaltigkeiten endlicher unimodularer Gruppen*, Technical University of Aachen, 1995, also in Aachener Beiträge zur Mathematik. 16., Verlag der Augustinus Buchh., Aachen, 1996.

[193] ———, *Dual cones and the Voronoi algorithm*, Experiment. Math. **10** (2001), 599–608.

[194] D. Orden and F. Santos, *Asymptotically efficient triangulations of the d-cube*, Discrete Comput. Geom. **30** (2003), 509–528.

[195] C. Pache, *Extremality and designs in spaces of quadratic forms*, 2008, preprint at arXiv:0804.0173.

[196] R.A. Pendavingh and S.H.M. van Zwam, *New Korkin-Zolotarev inequalities*, SIAM J. Optim. **18** (2007), 364–378.

[197] F. Pfender and G.M. Ziegler, *Kissing numbers, sphere packings, and some unexpected proofs*, Notices Amer. Math. Soc. **51** (2004), 873–883.

[198] W. Plesken and B. Souvignier, *Computing isometries of lattices*, J. Symbolic Comput. **24** (1997), 327–334.

[199] H.-G. Quebbemann, *Modular lattices in Euclidean spaces*, J. Number Theory **54** (1995), 190–202.

[200] J. Rambau, *Triangulations of cyclic polytopes and higher Bruhat orders*, Mathematika **44** (1997), 162–194.

[201] V. Reiner, *Equivariant fiber polytopes*, Doc. Math. **7** (2002), 113–132.

[202] R. Remak, *Verallgemeinerung eines Minkowskischen Satzes*, Annals of Math. **49** (1928), 82–109.

[203] ———, *Über die Minkowskische Reduktion der definiten quadratischen Formen*, Compos. Math. **5** (1938), 368–391.

[204] C. Riener, *On extreme forms in dimension* 8, 2006, pp. 677–682.

[205] C.A. Rogers, *The packing of equal spheres*, Proc. Lond. Math. Soc., III. Ser. **8** (1958), 609–620.

[206] ———, *Packing and covering*, Cambridge University Press, New York, 1964.

[207] C.A. Rogers and G.C. Shephard, *The difference body of a convex body*, Arch. Math. **8** (1957), 220–233.

[208] S.S. Ryshkov, *Effectuation of a method of Davenport in the theory of coverings*, Soviet Math. Dokl. **8** (1967), 865–867, translation from Dokl. Akad. Nauk SSSR 175, 303–305 (1967).

[209] ———, *The polyhedron $\mu(m)$ and certain extremal problems of the geometry of numbers*, Soviet Math. Dokl. **11** (1970), 1240–1244, translation from Dokl. Akad. Nauk SSSR 194, 514–517 (1970).

[210] ———, *C-types of n-dimensional parallelohedra*, Soviet Math. Dokl. **14** (1973), 1314–1318, translation from Dokl. Akad. Nauk SSSR 212, 46–49 (1973).

[211] ———, *Density of an (r, R)–system*, Math. Notes **16** (1974), 855–858, translation from Mat. Zametki 1, 447–454 (1974).

[212] ———, *The reduction, in the sense of Venkov, of positive quadratic forms*, Ivanov. Gos. Univ. Učen. Zap. **89** (1974), 5–36, Studies in the geometry of positive quadratic forms.

[213] ———, *The perfect form A_n^k: the existence of lattices with a nonfundamental division simplex, and the existence of perfect forms which are not Minkowski-reducible to forms having identical diagonal coefficients*, J. Sov. Math. **6** (1976), 672–676, translation from Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova 33, 65–71 (1973).

[214] ———, *The theory of Hermite–Minkowski reduction of positive definite quadratic forms*, J. Sov. Math. **6** (1976), 651–671, translation from Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova 33, 37–64 (1973).

[215] S.S. Ryshkov and E.P. Baranovskii, *Solution of the problem of the least dense lattice covering of five-dimensional space by equal spheres*, Soviet Math. Dokl. **16** (1975), 586–590, translation from Dokl. Akad. Nauk SSSR 222, 39–42 (1975).

[216] ———, *C-types of n-dimensional lattices and 5-dimensional primitive parallelohedra (with application to the theory of coverings)*, Proc. Steklov Inst. Math. (1978), 140 pages, translation of Tr. Mat. Inst. Steklova 137 (1976).

[217] ———, *Classical methods in the theory of lattice packings*, Russian Math. Surveys **34** (1979), 1–68, translation from Uspekhi Mat. Nauk 34, No.4(208), 3–63 (1979).

[218] R. Scharlau and R. Schulze-Pillot, *Extremal lattices*, Algorithmic algebra and number theory (Heidelberg, 1997), Springer, Berlin, 1999, pp. 139–170.

[219] W. Scharlau and H. Opolka, *From Fermat to Minkowski*, Springer, New York, 1985.

[220] P. Schmutz Schaller, *Geometry of Riemann surfaces based on closed geodesics*, Bull. Am. Math. Soc., New Ser. **35** (1998), 193–214.

[221] ———, *Riemann surfaces with longest systole and an improved Voronoi's algorithm*, Arch. Math. **76** (2001), 231–240.

[222] R. Schneider, *Convex bodies: The Brunn–Minkowski–theory*, Cambridge University Press, Cambridge, 1993.

[223] A. Schürmann, *On packing spheres into containers (about Kepler's finite sphere packing problem)*, Doc. Math. **11** (2006), 393–406.

[224] ———, *Enumerating perfect forms*, 2008, preprint.

[225] ———, *Perfect, strongly eutactic lattices are periodic extreme*, 2008, preprint at arXiv:0808.2013.

[226] A. Schürmann and F. Vallentin, *Local covering optimality of lattices: Leech lattice versus root lattice E_8*, Internat. Math. Res. Notices **2005**, 1937–1955.

[227] ———, *Geometry of lattices and algorithms*, 2005, `http://www.math.uni-magdeburg.de/lattice_geometry/`.

[228] ———, *Methods in the local theory of packing and covering lattices*, Proceeding of COE Workshop on Sphere Packings, Fukuoka, Japan, 2005.

[229] ———, *Computational approaches to lattice packing and covering problems*, Discrete Comput. Geom. **35** (2006), 73–116.

[230] F. Schweiger, *Multidimensional continued fractions*, Oxford Science Publications, Oxford University Press, Oxford, 2000.

[231] Á. Seress, *Permutation group algorithms*, vol. 152, Cambridge University Press, Cambridge, 2003.

[232] N.I. Shepherd-Barron, *Perfect forms and the moduli space of abelian varieties*, Invent. Math. **163** (2006), 25–45.

[233] A. Shrijver, *Theory of linear and integer programming*, Wiley, Chichester, 1986.

[234] C.L. Siegel, *Lectures on the geometry of numbers. Notes by B. Friedman. Rewritten by Komaravolu Chandrasekharan with the assistance of Rudolf Suter.*, Springer, Berlin, 1989.

[235] F. Sigrist, *G-perfect 8-dimensional lattices over the Gauss and Eisenstein integers*, 2004, preprint at `http://www.unine.ch/math/personnel/equipes/EGFinal.dvi`.

[236] ———, *Quaternionic-perfect forms in dimension* 12, 2008, preprint.

[237] B.F. Skubenko, *A new variant of the proof of the inhomogeneous Minkowski conjecture for $n = 5$*, Proc. Steklov Inst. Math. **142** (1976), 261–275, translation from Tr. Mat. Inst. Steklov 142, 240-253 (1976).

[238] ———, *A proof of Minkowski's conjecture on the product of n linear inhomogeneous forms in n variables for $n \leq 5$*, J. Sov. Math. **6** (1976), 627–650, translation from Zap. Nauchn. Sem. Leningrad. Otd. Mat. Inst. Steklova 33, 6–33 (1973).

[239] W.D. Smith, *Ph.D. thesis: Studies in computational geometry motivated by mesh generation*, Department of Applied Mathematics, Princeton University, 1988.

[240] ———, *A lower bound for the simplexity of the N-cube via hyperbolic volumes*, European J. Combin. **21** (2000), 131–137.

[241] C. Soulé, *Perfect forms and the Vandiver conjecture*, J. Reine Angew. Math. **517** (1999), 209–221.

[242] K.C. Stacey, *The enumeration of perfect septenary forms*, J. Lond. Math. Soc., II. Ser. **10** (1975), 97–104.

[243] W. Stein, *Modular forms, a computational approach*, AMS, Providence, 2007, with an appendix by Paul E. Gunnells.

[244] M.I. Stogrin, *Locally quasidensest lattice packings of spheres*, Soviet Math. Dokl. **15** (1975), 1288–1292, translation from Dokl. Akad. Nauk SSSR 218, 62–65 (1974).

[245] ———, *Regular Dirichlet-Voronoi partitions for the second triclinic group*, Proc. Steklov Inst. Math. **123** (1975), 1–116, translation from Tr. Mat. Inst. Steklova 123 (1973), 1–128.

[246] G. Swart, *Finding the convex hull facet by facet*, J. Algorithms **6** (1985), 17–48.

[247] G. Szpiro, *Does the proof stack up?*, Nature **424** (2003), 12–13.

[248] P. Tammela, *On the reduction theory of positive quadratic forms*, Soviet Math. Dokl. **14** (1973), 651–655, translation from Dokl. Akad. Nauk SSSR 209, 1299-1302 (1973).

[249] ———, *Minkowski reduction region for positive quadratic forms in seven variables*, J. Sov. Math. **16** (1981), 836–857, translation from Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova 67, 108–143 (1977).

[250] A. Terras, *Harmonic analysis on symmetric spaces and applications II*, Springer, New-York, 1988.

[251] A. Thue, *Über die dichteste Zusammenstellung von kongruenten Kreisen in einer Ebene*, Norske Vid. Selsk. Skr. **1** (1910), 1–9.

[252] K.C. Toh, *Primal-dual path-following algorithms for determinant maximization problems with linear matrix inequalities*, Comput. Optim. Appl. **14** (1999), 309–330.

[253] S. Torquato and F.H. Stillinger, *New conjectural lower bounds on the optimal density of sphere packings*, Experiment. Math. **15** (2006), 307–331.

[254] F. Vallentin, *Ph.D. thesis: Sphere coverings, lattices, and tilings (in low dimensions)*, Center for Mathematical Sciences, Munich University of Technology, 2003, `http://tumb1.biblio.tu-muenchen.de/publ/diss/ma/2003/vallentin.html`.

[255] B.L. van der Waerden, *Die Reduktionstheorie der positiven quadratischen Formen*, Acta Math. **96** (1956), 265–309.

[256] ———, *Das Minimum von $D/f_{11}f_{22}\dots f_{55}$ für reduzierte positive quinäre quadratische Formen*, Aequationes Math. **2** (1969), 233–247.

[257] S. Vance, *Mordell's inequality for Eisenstein and Hurwitz lattices*, 2008, preprint.

[258] L. Vandenberghe, S. Boyd, and S.P. Wu, *Determinant maximization with linear matrix inequality constraints*, SIAM J. Matrix Anal. Appl. **19** (1998), 499–533.

[259] A. Vardy, *A new sphere packing in 20 dimensions*, Invent. Math. **121** (1995), 119–133.

[260] B.B. Venkov, *Réseaux et designs sphériques*, Réseaux euclidiens, designs sphériques et formes modulaires, Monogr. Enseign. Math., vol. 37, Enseignement Math., Geneva, 2001, pp. 10–86.

[261] N.M. Vetchinkin, *Uniqueness of classes of positive quadratic forms, on which values of Hermite constants are reached for $6 \le n \le 8$*, Proc. Steklov Inst. Math. **152** (1982), 37–95, translation from Tr. Mat. Inst. Steklova 152, 34–86 (1980).

[262] G.F. Voronoi, *Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Premier Mémoire. Sur quelques propriétés des formes quadratiques positives parfaites*, J. Reine Angew. Math. **133** (1907), 97–178.

[263] ———, *Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Deuxième Mémoire. Recherches sur les parallélloèdres primitifs.*, J. Reine Angew. Math. **134** (1908), 198–287, and 136 (1909), 67–181.

[264] J. Wang, *Finding and investigating exact spherical codes*, 2008, preprint at arXiv:0805.0776.

[265] T. Watanabe, *A survey and a complement of fundamental Hermite constants*, Algebraic and arithmetic theory of quadratic forms, Contemp. Math., vol. 344, Amer. Math. Soc., Providence, RI, 2004, pp. 339–350.

[266] G.L. Watson, *On the minimum of a positive quadratic form in $n (\le 8)$ variables (verification of Blichfeldt's calculations)*, Proc. Camb. Philos. Soc. **62** (1966), 719.

[267] H. Weyl, *Theory of reduction for arithmetical equivalence*, Trans. Am. Math. Soc. **48** (1940), 126–164.

[268] A.C. Woods, *Covering six space with spheres*, J. Number Theory **4** (1972), 157–180.

[269] R.T. Worley, *The Voronoi region of E_6^**, J. Austral. Math. Soc. Ser. A **43** (1987), 268–278.

[270] G.M. Ziegler, *Lectures on polytopes*, Springer, New York, 1997.

[271] C. Zong, *Sphere packings*, Springer, New York, 1999.

[272] ———, *From deep holes to free planes*, Bull. Amer. Math. Soc. **39** (2002), 533–555.

[273] ———, *What is known about unit cubes*, Bull. of the Amer. Math. Soc. **42** (2005), 181–211.

Software

[274] `CARAT` *by W. Plesken et. al.*, ver.2.0., `http://www.math.rwth-aachen.de/carat/`.

[275] `cdd` *and* `cddplus` *by K. Fukuda*, ver. 0.94., `http://www.ifor.math.ethz.ch/~fukuda/cdd_home/`.

[276] `GAP` — *Groups, Algorithms, Programming - a system for computational discrete algebra*, ver. 4.4., `http://www.gap-system.org/`.

[277] `KANT/KASH` *by M.E. Pohst et. al.*, ver. 2.5., `http://www.math.tu-berlin.de/~kant/kash.html`.

[278] `lrs` *by D. Avis*, ver. 4.2., `http://cgm.cs.mcgill.ca/~avis/C/lrs.html`.

[279] `MAGMA` — *high performance software for Algebra, Number Theory, and Geometry*, ver. 2.13., `http://magma.maths.usyd.edu.au/`.

[280] `MAXDET` *by S.P. Wu and L. Vandenberghe and S. Boyd*, `http://stanford.edu/~boyd/old_software/MAXDET.html`.

[281] `nauty` *by B.D. McKay*, ver. 2.2., `http://cs.anu.edu.au/people/bdm/nauty/`.

[282] `pd` *by A. Marzetta*, `http://www.cs.unb.ca/profs/bremner/pd/`.

[283] `Polyhedral`*, a GAP package by M. Dutour Sikirić*, `http://www.liga.ens.fr/~dutour/Polyhedral/`.

[284] PORTA *by T. Christof and A. Löbel*, ver. 1.4.0., http://www.zib.de/Optimization/Software/Porta/.

[285] rmd *(rigorous MAXDET) by A. Schürmann and F. Vallentin*, ver. 1.0., see http://www.math.uni-magdeburg.de/lattice_geometry/.

[286] scc *(secondary cone cruiser) by A. Schürmann and F. Vallentin*, ver. 1.0., see http://www.math.uni-magdeburg.de/lattice_geometry/.

[287] shvec *by F. Vallentin*, ver. 1.0., see http://www.math.uni-magdeburg.de/lattice_geometry/.

Index

Notations

In the following we list some of the notations used at different places of this book and which may not be commonly known (among mathematicians). The listing is in a mixed alphabetical (by initial sound) and thematic order. The second column lists the page of the first occurrence.

$\langle \cdot, \cdot \rangle$	2, 37	inner product on $\mathcal{S}^d$ or $\mathcal{S}^{d,m}$
$\lVert \cdot \rVert$	5	Euclidean norm
Aut	33, 46, 92	autom. group of PQF, periodic form or Delone subdiv.
$\mathrm{Aut}_{\boldsymbol{t}}$	49	$\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-restricted automorphism group of PQF
B^d	5	Euclidean unit ball in $\mathbb{R}^d$
$\mathrm{BR}_{L,\beta}(Q)$	86	matrix with property $\lvert \mathrm{BR}_{L,\beta}(Q) \rvert \geq 0 \Leftrightarrow \mu_L(Q) \leq \beta$
$\mathcal{B}_{L,\leq\beta}$, $\mathcal{B}_{L,=\beta}$	92	sets of PQFs with $\mu_L(Q) \leq \beta$ and $\mu_L(Q) = \beta$
$\mathrm{cone}(M)$	29	conic hull of a set M
$\mathbf{D}_\alpha$	92	set of PQFs Q with $\det Q \geq \alpha$
δ	7, 9, 38	packing density of discrete set, PQF or periodic form
δ_d	8	density of the densest lattice sphere packing in $\mathbb{R}^d$
$\delta_{d,m}$	38	density of the densest m-periodic sphere packing in $\mathbb{R}^d$
δ_d^*	9	density of the densest sphere packing in $\mathbb{R}^d$
det	6, 38	determinant of lattice, PQF (matrix) or periodic form
$\boldsymbol{\Delta}(\mathcal{D})$	57, 60	secondary cone of Delone subdivision $\mathcal{D}$
$\boldsymbol{\Delta}_T(\mathcal{D})$	67	T-secondary cone $\boldsymbol{\Delta}(\mathcal{D}) \cap T$ with $T \subseteq \mathcal{S}^d$ linear
$\boldsymbol{\Delta}(\Lambda, P)$	74	secondary cone of a polyhedron P w.r.t. vertex-set Λ
$\mathrm{dens}\,\Lambda$	10	point density of a discrete set Λ
$\mathrm{Del}_\Lambda(Q)$	57	Delone subdivision of PQF Q w.r.t. vertex-set Λ
$\Gamma_d(N)$	46	principal congruence subgroup of level N
γ	13, 84	packing-covering constant of discrete set or PQF
γ_d	13	smallest lattice packing-covering constant in $\mathbb{R}^d$
γ_d^*	13	smallest packing-covering constant in $\mathbb{R}^d$
$\mathsf{GL}_d(\mathbb{Z})$	2	$\{U \in \mathbb{Z}^{d \times d} : \lvert \det U \rvert = 1\}$

$\mathsf{GL}_d(\mathbb{R})$	4	$\{A \in \mathbb{R}^{d\times d} : \det A \neq 0\}$
$\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$	46	automorphism group of standard periodic set $\Lambda_{\boldsymbol{t}}$
$g_{L,\beta}$	94	gradient of smooth surface $\mathcal{B}_{L,=\beta}$
$\mathcal{H}_d$	6	Hermite's constant
$\lambda(L), \lambda(\Lambda)$	6, 9	packing radius of lattice or discrete set
$\lambda(Q), \lambda(X)$	3, 38	(generalized) arith. minimum of a PQF or periodic form
λ_i	21	i-th successive minimum of lattice or PQF
$\Lambda_{\boldsymbol{t}}$	65	standard periodic vertex-set defined by $\boldsymbol{t}$
$\mathcal{M}_d, \mathcal{M}_d^+$	19	Minkowski's reduction domains
$\operatorname{Min} Q, \operatorname{Min} X$	3 , 38	representatives of (general.) arith. minimum $\lambda(Q), \lambda(X)$
$\mu(\Lambda)$	11	covering radius of discrete set Λ
$\mu(Q), \mu_\Lambda(Q)$	83, 84	inhom. minimum of PQF w.r.t. $\mathbb{Z}^d$, respectively Λ
$\mu_P(Q)$	86, 87	squared circumradius of P with respect to Q
$N_{V,\boldsymbol{w}}$	57	quadratic form defining regulator
$N_{L,L'}$	57	quadratic form def. regulator for adj. Del. simplices L, L'
$\mathcal{N}_{\Lambda,P}(Q)$	125	normal cone of Baranovskii cone $\boldsymbol{\Delta}(\Lambda, P)$ at Q
$\mathsf{O}_d(\mathbb{R})$	5	$\{O \in \mathbb{R}^{d\times d} : O^t O = \mathrm{id}_d\}$
$\mathcal{P}(Q), \mathcal{P}(X)$	31, 40	dual cones of (generalized) Voronoi domain $\mathcal{V}(Q), \mathcal{V}(X)$
$\mathcal{P}_\lambda$	27	Ryshkov polyhedron
$\mathcal{P}_{m,\lambda}$	38	generalized Ryshkov set
$\mathcal{P}_{\lambda,\boldsymbol{t}}$	47	Ryshkov polyhedron of standard periodic set $\Lambda_{\boldsymbol{t}}$
$p_{i,j,\boldsymbol{v}}(X)$	38	polynomial $Q[\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v}]$ for periodic form $X = (Q, \boldsymbol{t})$
$\mathcal{S}^d$	1	$\{Q \in \mathbb{R}^{d\times d} : Q^t = Q\}$
$\mathcal{S}^d_{>0}$	2	$\{Q \in \mathcal{S}^d : Q$ positive definite$\}$
$\mathcal{S}^d_{\geq 0}$	4	$\{Q \in \mathcal{S}^d : Q$ positive semi-definite$\}$
$\tilde{\mathcal{S}}^d_{\geq 0}$	35	rational closure of $\mathcal{S}^d_{>0}$
$\mathcal{S}^{d,m}_{>0}$	37	parameter space for m-periodic point sets in $\mathbb{R}^d$
$\mathcal{S}^{d,m}$	37	Euclidean space $\mathcal{S}^d \times \mathbb{R}^{d\times(m-1)}$ containing $\mathcal{S}^{d,m}_{>0}$
Θ	11, 83	covering density of discrete set or PQF
Θ_d	11	density of thinnest lattice covering in $\mathbb{R}^d$
Θ_d^*	11	density of thinnest covering in $\mathbb{R}^d$
Θ_P	123	covering density function of Delone polytope P
T_G	34	linear space of G-invariant quadratic forms
$\mathcal{V}(Q), \mathcal{V}(X)$	29, 40	(generalized) Voronoi domain
$\mathcal{W}_{\mathcal{D},\leq\beta}$	92	set of PQFs in $\boldsymbol{\Delta}(\mathcal{D})$ with $\mu(Q) \leq \beta$

Titles in This Series

48 **Achill Schürmann,** Computational geometry of positive definite quadratic forms, 2009
47 **Ernst Kunz (with the assistance of and contributions by David A. Cox and Alicia Dickenstein),** Residues and duality for projective algebraic varieties, 2008
46 **Lorenzo Sadun,** Topology of tiling spaces, 2008
45 **Matthew Baker, Brian Conrad, Samit Dasgupta, Kiran S. Kedlaya, and Jeremy Teitelbaum (David Savitt and Dinesh S. Thakur, Editors),** p-adic geometry: Lectures from the 2007 Arizona Winter School, 2008
44 **Vladimir Kanovei,** Borel equivalence relations: structure and classification, 2008
43 **Giuseppe Zampieri,** Complex analysis and CR geometry, 2008
42 **Holger Brenner, Jürgen Herzog, and Orlando Villamayor (Juan Elias, Teresa Cortadellas Benítez, Gemma Colomé-Nin, and Santiago Zarzuela, Editors),** Three Lectures on Commutative Algebra, 2008
41 **James Haglund,** The q,t-Catalan numbers and the space of diagonal harmonics (with an appendix on the combinatorics of Macdonald polynomials), 2008
40 **Vladimir Pestov,** Dynamics of infinite-dimensional groups. The Ramsey–Dvoretzky–Milman phenomenon, 2006
39 **Oscar Zariski,** The moduli problem for plane branches (with an appendix by Bernard Teissier), 2006
38 **Lars V. Ahlfors,** Lectures on Quasiconformal Mappings, Second Edition, 2006
37 **Alexander Polishchuk and Leonid Positselski,** Quadratic algebras, 2005
36 **Matilde Marcolli,** Arithmetic noncommutative geometry, 2005
35 **Luca Capogna, Carlos E. Kenig, and Loredana Lanzani,** Harmonic measure: Geometric and analytic points of view, 2005
34 **E. B. Dynkin,** Superdiffusions and positive solutions of nonlinear partial differential equations, 2004
33 **Kristian Seip,** Interpolation and sampling in spaces of analytic functions, 2004
32 **Paul B. Larson,** The stationary tower: Notes on a course by W. Hugh Woodin, 2004
31 **John Roe,** Lectures on coarse geometry, 2003
30 **Anatole Katok,** Combinatorial constructions in ergodic theory and dynamics, 2003
29 **Thomas H. Wolff (Izabella Łaba and Carol Shubin, editors),** Lectures on harmonic analysis, 2003
28 **Skip Garibaldi, Alexander Merkurjev, and Jean-Pierre Serre,** Cohomological invariants in Galois cohomology, 2003
27 **Sun-Yung A. Chang, Paul C. Yang, Karsten Grove, and Jon G. Wolfson,** Conformal, Riemannian and Lagrangian geometry, The 2000 Barrett Lectures, 2002
26 **Susumu Ariki,** Representations of quantum algebras and combinatorics of Young tableaux, 2002
25 **William T. Ross and Harold S. Shapiro,** Generalized analytic continuation, 2002
24 **Victor M. Buchstaber and Taras E. Panov,** Torus actions and their applications in topology and combinatorics, 2002
23 **Luis Barreira and Yakov B. Pesin,** Lyapunov exponents and smooth ergodic theory, 2002
22 **Yves Meyer,** Oscillating patterns in image processing and nonlinear evolution equations, 2001
21 **Bojko Bakalov and Alexander Kirillov, Jr.,** Lectures on tensor categories and modular functors, 2001
20 **Alison M. Etheridge,** An introduction to superprocesses, 2000
19 **R. A. Minlos,** Introduction to mathematical statistical physics, 2000
18 **Hiraku Nakajima,** Lectures on Hilbert schemes of points on surfaces, 1999

TITLES IN THIS SERIES

17 **Marcel Berger,** Riemannian geometry during the second half of the twentieth century, 2000

16 **Harish-Chandra,** Admissible invariant distributions on reductive p-adic groups (with notes by Stephen DeBacker and Paul J. Sally, Jr.), 1999

15 **Andrew Mathas,** Iwahori-Hecke algebras and Schur algebras of the symmetric group, 1999

14 **Lars Kadison,** New examples of Frobenius extensions, 1999

13 **Yakov M. Eliashberg and William P. Thurston,** Confoliations, 1998

12 **I. G. Macdonald,** Symmetric functions and orthogonal polynomials, 1998

11 **Lars Gårding,** Some points of analysis and their history, 1997

10 **Victor Kac,** Vertex algebras for beginners, Second Edition, 1998

9 **Stephen Gelbart,** Lectures on the Arthur-Selberg trace formula, 1996

8 **Bernd Sturmfels,** Gröbner bases and convex polytopes, 1996

7 **Andy R. Magid,** Lectures on differential Galois theory, 1994

6 **Dusa McDuff and Dietmar Salamon,** J-holomorphic curves and quantum cohomology, 1994

5 **V. I. Arnold,** Topological invariants of plane curves and caustics, 1994

4 **David M. Goldschmidt,** Group characters, symmetric functions, and the Hecke algebra, 1993

3 **A. N. Varchenko and P. I. Etingof,** Why the boundary of a round drop becomes a curve of order four, 1992

2 **Fritz John,** Nonlinear wave equations, formation of singularities, 1990

1 **Michael H. Freedman and Feng Luo,** Selected applications of geometry to low-dimensional topology, 1989

Starting from classical arithmetical questions on quadratic forms, this book takes the reader step by step through the connections with lattice sphere packing and covering problems. As a model for polyhedral reduction theories of positive definite quadratic forms, Minkowski's classical theory is presented, including an application to multidimensional continued fraction expansions. The reduction theories of Voronoi are described in great detail, including full proofs, new views, and generalizations that cannot be found elsewhere. Based on Voronoi's second reduction theory, the local analysis of sphere coverings and several of its applications are presented. These include the classification of totally real thin number fields, connections to the Minkowski conjecture, and the discovery of new, sometimes surprising, properties of exceptional structures such as the Leech lattice or the root lattices.

Throughout this book, special attention is paid to algorithms and computability, allowing computer-assisted treatments. Although dealing with relatively classical topics that have been worked on extensively by numerous authors, this book is exemplary in showing how computers may help to gain new insights.

For additional information and updates on this book, visit

www.ams.org/bookpages/ulect-48

ISBN 978-0-8218-4735-0

9 780821 847350

ULECT/48