

University LECTURE Series



Applied
Mathematics

Volume 55

A Primer on Pseudorandom Generators

Oded Goldreich



American Mathematical Society

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American Mathematical Society
Providence, Rhode Island

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Contents

Preface	ix
1 Introduction	1
1.1 The Third Theory of Randomness	2
1.2 Organization of the Primer	4
1.3 Standard Conventions	5
1.4 The General Paradigm	6
1.4.1 Three fundamental aspects	6
1.4.2 Notational conventions	7
1.4.3 Some instantiations of the general paradigm	8
Notes	8
Exercises	9
2 General-Purpose Pseudorandom Generators	11
2.1 The Basic Definition	11
2.2 The Archetypical Application	12
2.3 Computational Indistinguishability	15
2.3.1 The general formulation	15
2.3.2 Relation to statistical closeness	16
2.3.3 Indistinguishability by multiple samples	16
2.4 Amplifying the Stretch Function	19
2.5 Constructions	21
2.5.1 Background: one-way functions	21
2.5.2 A simple construction	23
2.5.3 An alternative presentation	23
2.5.4 A necessary and sufficient condition	24
2.6 Non-uniformly Strong Pseudorandom Generators	25
2.7 Stronger (Uniform-Complexity) Notions	27
2.7.1 Fooling stronger distinguishers	27
2.7.2 Pseudorandom functions	27
2.8 Conceptual Reflections	29
Notes	30
Exercises	31

3	Derandomization of Time-Complexity Classes	35
3.1	Defining Canonical Derandomizers	35
3.2	Constructing Canonical Derandomizers	37
3.2.1	The construction and its consequences	38
3.2.2	Analyzing the construction	40
3.2.3	Construction 3.4 as a general framework	41
3.3	Reflections Regarding Derandomization	43
	Notes	43
	Exercises	44
4	Space-Bounded Distinguishers	47
4.1	Definitional Issues	47
4.2	Two Constructions	50
4.2.1	Sketches of the proofs of Theorems 4.2 and 4.3	51
4.2.2	Derandomization of space-complexity classes	54
	Notes	56
	Exercises	56
5	Special Purpose Generators	59
5.1	Pairwise Independence Generators	60
5.1.1	Constructions	60
5.1.2	A taste of the applications	62
5.2	Small-Bias Generators	63
5.2.1	Constructions	64
5.2.2	A taste of the applications	65
5.2.3	Generalization	66
5.3	Random Walks on Expanders	66
5.3.1	Background: expanders and random walks on them	67
5.3.2	The generator	68
	Notes	69
	Exercises	69
	Concluding Remarks	77
	Appendices	79
A	Hashing Functions	79
A.1	Definitions	79
A.2	Constructions	80
A.3	The Leftover Hash Lemma	81
B	On Randomness Extractors	83
B.1	Definitions	84
B.2	Constructions	85

C A Generic Hard-Core Predicate	89
D Using Randomness in Computation	93
D.1 A Simple Probabilistic Polynomial-Time Primality Test	93
D.2 Testing Polynomial Identity	95
D.3 The Accidental Tourist Sees It All	96
E Cryptographic Applications of Pseudorandom Functions	99
E.1 Secret Communication	99
E.2 Authenticated Communication	101
F Some Basic Complexity Classes	103
Bibliography	107
Index	113

Preface

*Indistinguishable things are identical.*¹

G.W. Leibniz (1646–1714)

This primer to the theory of pseudorandomness presents a fresh look at the *question of randomness*, which arises from a complexity theoretic approach to randomness. The crux of this (complexity theoretic) approach is the postulate that a distribution is random (or rather pseudorandom) if it cannot be distinguished from the uniform distribution by *any efficient procedure*. Thus, (pseudo)randomness is not an inherent property of an object, but is rather subjective to the observer.

At the extreme, this approach says that the question of whether the world is actually deterministic or allows for some free choice (which may be viewed as a source of randomness) is irrelevant. *What matters is how the world looks to us and to various computationally bounded devices*. That is, if some phenomenon looks random, then we may treat it as if it is random. Likewise, if we can generate sequences that cannot be distinguished from the uniform distribution by any efficient procedure, then we can use these sequences in any efficient randomized application instead of the ideal coin tosses that are postulated in the design of this application.

The pivot of the foregoing approach is the notion of *computational indistinguishability*, which refers to pairs of distributions that cannot be distinguished by efficient procedures. The most fundamental incarnation of this notion associates efficient procedures with polynomial-time algorithms, but other incarnations that restrict attention to different classes of distinguishing procedures also lead to important insights. Likewise, the *effective generation* of pseudorandom objects, which is of major concern, is actually a general paradigm with numerous useful incarnations (which differ in the computational complexity limitations imposed on the generation process).

Following the foregoing principles, we briefly outline some of the key elements of the theory of pseudorandomness. Indeed, the key concept is that of a pseudorandom generator, which is an efficient deterministic procedure that stretches short random seeds into longer pseudorandom sequences. Thus, a generic formulation of pseudorandom generators consists of specifying three fundamental aspects – the *stretch measure* of the generators; the class of distinguishers that the generators are

¹This is Leibniz's *Principle of Identity of Indiscernibles*. Leibniz admits that counterexamples to this principle are conceivable but will not occur in real life because God is much too benevolent. We thus believe that he would have agreed to the theme of this text, which asserts that *indistinguishable things should be considered as if they were identical*.

supposed to fool (i.e., the algorithms with respect to which the *computational indistinguishability* requirement should hold); and the resources that the generators are allowed to use (i.e., their own *computational complexity*).

The archetypical case of pseudorandom generators refers to efficient generators that fool any feasible procedure; that is, the potential distinguisher is any probabilistic polynomial-time algorithm, which may be more complex than the generator itself (which, in turn, has time-complexity bounded by a fixed polynomial). These generators are called general-purpose, because their output can be safely used in any efficient application. Such (general-purpose) pseudorandom generators exist if and only if there exist functions (called one-way functions) that are easy to evaluate but hard to invert.

In contrast to such (general-purpose) pseudorandom generators, for the purpose of derandomization (i.e., converting randomized algorithms into corresponding deterministic ones), a relaxed definition of pseudorandom generators suffices. In particular, for such a purpose, one may use pseudorandom generators that are somewhat more complex than the potential distinguisher (which represents a randomized algorithm to be derandomized). Following this approach, adequate pseudorandom generators yield a full derandomization of probabilistic polynomial-time algorithms (e.g., $BPP = P$), and such generators can be constructed based on the assumption that some exponential-time solvable problems (i.e., problems in \mathcal{E}) have no sub-exponential size circuits.

Indeed, both the general-purpose pseudorandom generators and the aforementioned “derandomizers” demonstrate that randomness and computational difficulty are related. This trade-off is not surprising in light of the fact that the very definition of pseudorandomness refers to computational difficulty (i.e., the difficulty of distinguishing the pseudorandom distribution from a truly random one).

Finally, we mention that it is also beneficial to consider pseudorandom generators that fool space-bounded distinguishers and generators that exhibit some limited random behavior (e.g., outputting a pairwise independent or a small-bias sequence). Such (special-purpose) pseudorandom generators can be constructed without relying on any computational complexity assumptions, because the behavior of the corresponding (limited) distinguishers can be analyzed even at the current historical time. Nevertheless, such (special-purpose) pseudorandom generators offer numerous applications.

Note: The study of pseudorandom generators is part of complexity theory (cf. e.g., [24]), and some basic familiarity with complexity theory will be assumed in the current text. In fact, the current primer is an abbreviated (and somewhat revised) version of [24, Chap. 8]. Nevertheless, we believe that there are merits to providing a separate treatment of the theory of pseudorandomness, since this theory is of natural interest to various branches of mathematics and science. In particular, we hope to reach readers that may not have a general interest in complexity theory at large and/or do not wish to purchase a book on the latter topic.

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Index

Author Index

Ajtai, M., 69
 Blum, M., 31
 Chaitin, G.J., 1, 29
 Goldreich, O., 31
 Goldwasser, S., 30, 31
 Håstad, J., 31
 Impagliazzo, R., 31, 43, 44
 Kolmogorov, A., 1, 29
 Komlos, J., 69
 Levin, L.A., 31
 Luby, M., 31
 Micali, S., 30, 31
 Naor, J., 69
 Naor, M., 69
 Nisan, N., 43, 56
 Reingold, O., 56
 Shannon, C.E., 1
 Solomonoff, R.J., 1
 Szemerédi, E., 69
 Trevisan, L., 86
 Wigderson, A., 43, 44
 Yao, A.C., 30, 43
 Zuckerman, D., 56

Archetypical case of pseudorandom
 generator, 9–34

Blum-Micali Construction, 24

Boolean Circuits, 26
 constant-depth, 42
 Natural Proofs, 28

Chebyshev's Inequality, 81

Complexity classes
 BPL, 51, 54–57, 104
 BPP, 25–27, 35–39, 51, 93–95, 104
 E, 104
 EXP, 104
 L, 105

NL, 54, 105
 NP, 43, 103, 104
 P, 103
 quasi-P, 42
 RL, 54, 96–97, 105
 RP, 104
 SC, 54

Computational Indistinguishability, 6, 11,
 13, 15–19, 30
 multiple samples, 16–19
 non-triviality, 16
 The Hybrid Technique, 17–19, 24,
 30, 40
 vs statistical closeness, 16

Computational Learning Theory, 28

Computational problems
 Primality Testing, 93–95
 Testing polynomial identity, 95–96
 Undirected Connectivity, 96–97

Conceptual discussion of derandomization,
 43–45

Conceptual discussion of pseudorandom-
 ness, 29–34

Derandomization, 25–27, 35–45
 high end, 39
 low end, 39

Discrepancy sets, 66

Expander Graphs, 66, 67
 random walk, 67–75

Expander random walks, 66–75

Extractors, *see* Randomness Extractors,
 see Randomness Extractors

Fourier coefficients, 63

General paradigm of pseudorandomness,
 1–9, 77–78

- General-purpose pseudorandom
 - generator, 9–34
 - application, 12–15
 - construction, 20–25
 - definition, 11–12
 - stretch, 19–20, 23–24
- Hashing, 79–82
 - Extraction Property, 85
 - highly independent, 80
 - Leftover Hash Lemma, 80–82
 - Mixing Property, 51, 81
 - pairwise independent, 80–82
 - Universal, 25, 80
- Hitting, 67–75
- Information Theory, 1
- Interactive Proof systems
 - constant-round, 42, 44
 - public-coin, 42
- Kolmogorov Complexity, 1, 29
- Linear Feedback Shift Registers, 64
- Nisan-Wigderson Construction,
 - 38–44, 77
- NP-Completeness, 104
- One-Way Functions, 16
 - Hard-Core Predicates, 31
- Pairwise independence generator, 60–63
- Probabilistic Log-Space, 96–97
- Probabilistic Polynomial-Time, 93–97
- Probability Theory
 - conventions, 4–6
- Pseudorandom Functions, 27–29, 31
- Pseudorandom Generators
 - Connection to Extractors, 86–87
 - Nisan-Wigderson Construction, 86, 87
- Randomness Extractors, 44, 83–87
 - Connection to Pseudorandomness, 86–87
 - from few independent sources, 84
 - Seeded Extractors, 83–84
 - using Weak Random Sources, 83–84
- Reductions
 - Karp-Reductions, 104
 - Polynomial-time Reductions, 103
 - Reducibility Argument, 18, 40, 89
- Small bias generator, 63–66
- Space-Bounded Distinguishers, 47–57
- Special purpose pseudorandom generator,
 - 59–75
- Statistical difference, 5, 16
- Time-constructible, 36
- Turing machines
 - with advice, 26
- Universal sets, 66
- Unpredictability, 23–24, 31, 40
- Variation distance, *see* Statistical difference

Titles in This Series

- 55 **Oded Goldreich**, A primer on pseudorandom generators, 2010
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Detail from a drawing by Joshua Grissit

This primer surveys the theory of pseudorandomness, starting with the general paradigm, and discussing various incarnations while emphasizing the case of general-purpose pseudorandom generators (withstanding any polynomial-time distinguisher). Additional topics include the “derandomization” of arbitrary probabilistic polynomial-time algorithms, pseudorandom generators withstanding space-bounded distinguishers, and several natural notions of special-purpose pseudorandom generators.

The primer assumes basic familiarity with the notion of efficient algorithms and with elementary probability theory, but provides a basic introduction to all notions that are actually used. As a result, the primer is essentially self-contained, although the interested reader is at times referred to other sources for more detail.



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