University
LECTURE
Series

Volume 61

Lectures on the Theory of Pure Motives

Jacob P. Murre
Jan Nagel
Chris A. M. Peters

American Mathematical Society

http://dx.doi.org/10.1090/ulect/061

University
LECTURE
Series

Volume 61

Lectures on the Theory of Pure Motives

Jacob P. Murre
Jan Nagel
Chris A. M. Peters

American Mathematical Society
Providence, Rhode Island

EDITORIAL COMMITTEE

Jordan S. Ellenberg
William P. Minicozzi II (Chair)
Benjamin Sudakov
Tatiana Toro

2010 *Mathematics Subject Classification.* Primary 14-02, 14C15, 14C25, 19E15.

For additional information and updates on this book, visit
www.ams.org/bookpages/ulect-61

Library of Congress Cataloging-in-Publication Data

Library of Congress Cataloging-in-Publication Data has been applied for by the AMS. See www.loc.gov/publish/cip/.

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294 USA. Requests can also be made by e-mail to `reprint-permission@ams.org`.

© 2013 by the American Mathematical Society. All rights reserved.
Printed in the United States of America.

♾ The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 18 17 16 15 14 13

Contents

Introduction v

Chapter 1. Algebraic Cycles and Equivalence Relations 1
1.1. Algebraic Cycles 1
1.2. Equivalence Relations 3

Appendix A: Survey of Some of the Main Results on Chow Groups 13
A-1. Divisors 13
A-2. Classical Results on the Picard and Albanese Varieties 14
A-3. Higher Codimension 16

Appendix B: Proof of the Theorem of Voisin–Voevodsky 19

Chapter 2. Motives: Construction and First Properties 23
2.1. Correspondences 23
2.2. (Pure) Motives 25
2.3. Examples 27
2.4. Further Remarks and Properties 28
2.5. Chow Groups and Cohomology Groups of Motives 29
2.6. Relations Between the Various Categories of Motives 30
2.7. Motives of Curves 30
2.8. Manin's Identity Principle 33

Chapter 3. On Grothendieck's Standard Conjectures 35
3.1. The Standard Conjectures 35
3.2. Jannsen's Theorem 39

Chapter 4. Finite Dimensionality of Motives 43
4.1. Introduction 43
4.2. Preliminaries on Group Representations 43
4.3. Action of the Symmetric Group on Products 44
4.4. Dimension of Motives 46
4.5. The Sign Conjecture and Finite Dimensionality 47
4.6. Curves 47

Chapter 5. Properties of Finite Dimensional Motives 51
5.1. Sums and Tensor Products 51
5.2. Smash Nilpotence of Morphisms 54
5.3. Morphisms Between Finite Dimensional Motives 55
5.4. Surjective Morphisms and Finite Dimensionality 56
5.5. Finite Dimensionality and Nilpotence 59

5.6. Applications and Conjectures 61

Chapter 6. Chow-Künneth Decomposition; the Picard and Albanese Motive 67
6.1. The Künneth and the Chow-Künneth Decomposition 67
6.2. Picard and Albanese Motives 70
6.3. The Case of Surfaces 78

Appendix C: Chow-Künneth Decomposition in a Special Case 83

Chapter 7. On the Conjectural Bloch-Beilinson Filtration 85
Introduction 85
7.1. Bloch-Beilinson Filtration 85
7.2. Another Set of Conjectures 86
7.3. Some Evidence for the Conjectures; an Overview 88
7.4. Threefolds of Type $S \times C$ 89
7.5. On Some Results of Jannsen and Their Consequences 93
7.6. Products of Two Surfaces 96
Relations Between the Conjectures 103

Chapter 8. Relative Chow-Künneth Decomposition 105
8.1. Relative Motives 105
8.2. Perverse Sheaves and the Decomposition Theorem 109
8.3. Relative Chow-Künneth Decomposition 112
8.4. Further Examples 113
8.5. From Relative to Absolute 115

Appendix D: Surfaces Fibered over a Curve 119

Chapter 9. Beyond Pure Motives 123
Two Auxilliary Notions 124
9.1. Motivic Euler Characteristics 124
9.2. Motives Via Weights 127
9.3. Voevodsky's Triangulated Category of Motives 131

Appendix E: The Category of Motivic Complexes 135

Bibliography 139

Index of Notation 145

Index 147

Introduction

The *theory of motives* was created by Grothendieck in the mid-sixties, starting from around 1964. In a letter to Serre dated August, 16 1964 Grothendieck mentioned for the first time the notion of "motives"; see [**Col-Se**, pages 173 and 275]. At that time this notion was still rather vague for him, but he was already beginning to see a precise "yoga" for such a theory (see the letter cited above) and his "motivation"(!) (or at least one of his motivations) must have been the following.

In the early sixties Grothendieck, with the help of Artin and Verdier, had developed étale cohomology theory. From that moment on there existed a cohomology theory for every prime number ℓ different from the characteristic p of the underlying field. Moreover, in characteristic zero there exist also the classical Betti and de Rham theory and for positive characteristic Grothendieck already had the outline for the crystalline cohomology theory.

Hence there was an abundance of "good" (so-called Weil) cohomology theories! But all these theories have similar properties, and in characteristic zero there even are comparison theorems between them: the famous de Rham isomorphism theorem between Betti and de Rham theory, and the Artin isomorphism between Betti and étale cohomology.

There should be a deeper reason behind this! In order to explain and understand this, Grothendieck envisioned a "universal" cohomology theory for algebraic varieties: the *theory of motives*. Grothendieck expected that there should exist a suitable $\mathbb{Q}$-linear semisimple abelian tensor category with "*realization*" functors to all Weil cohomology theories.[1]

The best way to see what Grothendieck had in mind is to quote his own words. In section 16 (les motifs - ou la cœur dans la cœur) of the "En guise d'avant propos" of his "Récoltes et Semailles" [**Groth85**], Grothendieck writes the following:

... Contrairement à ce qui se passait en topologie ordinaire, on se trouve donc placé là devant une abondance déconcertante de théories cohomologiques différentes. On avait l'impression très nette qu'en un sens, qui restait d'abord très flou, toutes ces théories devaient 'revenir au même', qu'elles 'donnaient les mêmes résultats'. C'est pour parvenir à exprimer cette intuiton de 'parenté' entre théories cohomologiques différentes que j'ai dégagé la notion de 'motif' associé à une variété algébrique. Par ce terme j'entends suggérer qu'il s'agit du 'motif commun' (ou de la raison commune) sous-jacent à cette multitude d'invariants cohomologiques différents associés à la variété, à l'aide de la multitude de toutes les théories cohomologiques possibles à priori. [...] . Ainsi, le motif associé à une variété algébrique constituerait

[1](M.) I remember that during a private conversation in October or November 1964 Grothendieck told me that he was now developing a new theory that would finally explain the (similar) behaviour of all the different cohomology theories.

l'invariant cohomologique 'ultime', 'par excellence', dont tous les autres (associés aux différentes théories cohomologiques possibles) se déduiraient, comme autant d''incarnations' musicales, ou de 'réalisations' différentes [...].

For nice "elementary" introductions to the ideas and the concept of motives we recommend the papers of Serre [**Serre91**], Mazur [**Mazur**] and the recent preprint of Milne [**Mil09**].

Grothendieck has constructed what we now call the category of "*pure motives*"; these are objects constructed from smooth, projective varieties by means of the theory of algebraic cycles modulo a suitable equivalence relation (roughly speaking motives are a kind of direct summands of algebraic varieties). The equivalence relation Grothendieck had in mind was numerical equivalence (although his construction works for every good equivalence relation). According to him [**Groth69a**, page 198]:

the theory of motives is a systematic theory of the "arithmetic properties" of algebraic varieties as embodied in their group of classes for numerical equivalence

Grothendieck was well aware that finally one needs a more general theory; see his letter to Illusie, reproduced in the appendix of [**Jann94**, p. 296]. He envisaged a category of "*mixed motives*" attached to the category of all (i.e., arbitrary) varieties defined over a field k, in the same way as in the complex case one needs not only Hodge structures for smooth compact varieties but also mixed Hodge structures for arbitrary complex varieties.

Such a theory of "mixed motives" has, up to now, not yet been constructed in a satisfactory way, although important progress has been made on the one hand by the construction of triangulated categories of motives by (independently) Hanamura [**Hana95, Hana04**], Levine [**Lev98**] and Voevodsky [**Maz-Vo-We, Voe00**], and on the other hand by Nori [**Nori**], who constructed a very original candidate for a category of mixed motives (cf. also [**Lev05**, p. 462] and [**Hub-MüS**, Appendix]).

Returning to the category of pure motives: its *construction* is (contrary to some widespread misunderstanding!) entirely *unconditional* and in fact – except for its originality – surprisingly simple! However the question whether these motives have the *required good properties* depends on *conjectures* for algebraic cycles; these are the famous "*standard conjectures*" of Grothendieck formulated in [**Groth69a**], and discussed also in the two papers of Kleiman [**Klei68**], [**Klei94**]. Partially these conjectures center around very deep existence problems for algebraic cycles.

Although up to now very little progress has been made on these conjectures (with one exception: a beautiful result by Jannsen [**Jann92**], see Lecture 3), the "*yoga of motives*" has proved to be a very powerful and influential tool in the development of algebraic geometry and number theory, especially in questions of arithmetical algebraic geometry.

The influence of this yoga is nowadays formidable! To give only some examples:

- in his 1970 lecture [**Del70**] at the ICM congress in Nice, where he outlined his famous work on Hodge theory, Deligne already mentions the inspiration from Grothendieck's theory of motives; see also [**Groth69b**];
- in the papers of Deligne, Milne, Ogus and Shih in [**Del-Mi-Og-Sh**] the influence of this yoga is evident;

— in 1991 the AMS organized a three week conference on motives, resulting in the appearance of two thick volumes of proceedings full of results and ideas originating from motives [**Jann-Kl-Se**];
— the ideas behind the celebrated work of Voevodsky clearly stem from the yoga of motives.

In fact we could make a very long list! [2]

Grothendieck himself has – unfortunately – published only one paper on motives, namely [**Groth69a**]. But in 1967 he has given a series of lectures on his theory at the IHES and – fortunately – some of the attendants of these lectures have reported on this. In fact, these lectures are at the origin of the well-known paper by Manin [**Manin**], of Demazure's Bourbaki lecture [**Dem**] and also of Kleiman's paper [**Klei70**].[3] [4] [5]

This book deals mainly with pure motives, except for the last two chapters where we have tried to give at least some ideas on relative motives and on further developments in the direction of mixed motives. Also we concentrate only on the geometric aspects of the theory. For the arithmetic aspects we suggest the beautiful book by André [**Andr**].

Grothendieck was aiming for a theory built on *numerical equivalence* of algebraic cycles because, as he did foresee and as was later proved by Jannsen, this – and only this – gives an abelian semisimple category. In the present book we mainly work with motives modulo *rational equivalence*, so-called "*Chow motives*", the reason being that such motives not only yield information on the cohomology of the underlying variety but also on the Chow groups themselves.

The structure of the book is as follows. We start with a short outline of algebraic cycles and Chow groups because, as we mentioned earlier, these are at the basis of the entire theory. To chapter 1 we have added two appendices; in the first one we give a short survey of the main known results for divisors and algebraic cycles of higher codimension, and in the second one we discuss a theorem on the relation between algebraic and (smash) nilpotent equivalence obtained independently by Voisin and Voevodsky (for the definition of these notions, see chapter 1). Then in chapter 2 we describe Grothendieck's construction of the category of pure motives and some examples, in particular motives of curves. Here we see a striking fact: the category of motives contains as a full subcategory the category of abelian varieties up to isogeny!

In chapter 3 we discuss the standard conjectures followed by the treatment of the celebrated theorem of Jannsen, which shows that the category of motives modulo numerical equivalence is an abelian semisimple category. In chapters 4 and 5 we discuss the remarkable concept of "finite-dimensionality" for motives,

[2] in the introduction of "Récoltes et Semailles" [**Groth85**, Introduction (II), p. xviii] Grothendieck states: "Et le "yoga des motifs" auquel m'a conduit cette réalité longtemps ignorée est peut-être le plus puissant instrument de découverte que j'aie dégagé dans cette première période de ma vie de mathématicien."

[3] Manin [**Manin**, p. 444] states: "I learned from the theory of motives from lectures that Grothendieck gave at IHES in the spring of 1967 ... This paper is no more than the fruit of assiduous meditation on these beautiful ideas"

[4] See also footnote 1 in Serre's paper [**Serre91**]

[5] There are also unpublished manuscripts of Grothendieck with further ideas on motives; which –as we have been informed – have a rich content.

due independently to S.-I. Kimura and O'Sullivan. This idea is undoubtedly very important. Kimura and O'Sullivan conjectured that every (Chow) motive is finite dimensional, but for the moment this has only been proved for varieties that are dominated by products of algebraic curves (for the precise statement see chapters 4 and 5); this class contains in particular all abelian varieties. In chapter 4 we treat the definition of finite-dimensionality and the case of an algebraic curve. (The finite-dimensionality of the motive of a curve basically goes back to a theorem of Šermenev in 1971, but to avoid misunderstanding: Šermenev did not have the idea of finite-dimensionality.) In chapter 5 we then discuss the surprising properties resulting from finite-dimensionality; results mainly due to Kimura and O'Sullivan. These results are obtained using representation theory of the symmetric group (partitions, Young diagrams etc.). The next chapter, chapter 6, is on the construction of the so-called Picard and Albanese motives, the distribution of the Chow groups over these motives and on the motive of an algebraic surface. The results in chapter 6 are true unconditionally, but in chapter 7 we discuss a set of conjectures on algebraic cycles supplementing, but mainly independent of, the Standard Conjectures. We first state the Bloch-Beilinson conjectures on the filtration on Chow groups, and next the related conjectures by the first named author. We discuss a theorem of Jannsen saying that these two sets of conjectures are equivalent, and we discuss examples of varieties for which these conjectures – or part of them – can be proved unconditionally, in particular we discuss the results in the case of the product of two surfaces. (This is partly based on joint work of the first author with Kahn and Pedrini.)

After these chapters, all on pure motives, we have added two chapters on further developments. Chapter 8 deals with relative motives; here correspondences should be replaced by relative correspondences, cohomology groups should be replaced by (perverse) direct image sheaves, and the Künneth formula should be replaced by the famous decomposition theorem of Beilinson, Bernstein, Deligne and Gabber. Finally, in chapter 9 we report briefly on the developments in the direction of mixed motives. First we present a construction due to Bittner (and Looijenga) of the so-called "motivic Euler characteristic", then we discuss the "motivic weight complex" of Gillet-Soulé (see also [**Gu-Na**]), and finally we give a very short presentation of the construction of Voevodsky's triangulated category of mixed motives. These last two chapters are only intended to give some idea of these concepts and developments; for a thorough treatment of the material in these chapters we advise the reader to consult the original paper of Corti-Hanamura [**Cor-Ha00**] for relative motives and the books of André [**Andr**], Levine [**Lev98**] and Mazza-Voevodsky-Weibel [**Maz-Vo-We**] for mixed motives.

Acknowledgment.

We would like to thank Stefan Müller-Stach, Morihiko Saito, Charles Vial and the referee for their comments on preliminary versions of the manuscript.

History of the origin of the book and acknowledgments (by the first author):

This book stems from lectures on pure motives that I have given over a period of years, starting with a lecture in 1988 at the University of Tokyo and followed by lectures – or series of lectures – at many other places, of which I mention in particular the lectures at the 2001 summer school in Grenoble (see the proceedings

[**MüS-Pe**]) and those at the Tata Institute of Fundamental Research in Mumbai in 2008.

I would like to thank the numerous people who have helped me before, during and after these lectures with advice and comments. Especially important and valuable for me have been the many discussions on algebraic cycles and motives over the years with Spencer Bloch and Uwe Jannsen and later, during our joint work [**Kahn-M-P**], with Bruno Kahn and Claudio Pedrini.

Finally, last but certainly not least, I would like to thank my two coauthors of the book. Without their help and encouragement these lectures would never have been transformed into a book; their help has been indispensable!

Bibliography

Ab-K-M-W. Abramovich, D., K. Karu, K. Matsuki and J., Włodarczyk: Torification and factorization of birational maps, J. Amer. Math. Soc. **15**, 531–572 (2002)

Andr. André, Y.: *Une introduction aux motifs*, Panoramas et synthèses **17**, Soc. Math. de France, Paris, (2004)

Andr-Ka-O'S. André, Y., B. Kahn and P. O'Sullivan: Nilpotence, radicaux et structures monoïdales, Rend. Sem. Mat. Univ. Padova **108**, 107–291 (2002)

Ang-MüS98. del Angel, P. L. and S. Müller-Stach: Motives of uniruled 3-folds, Compositio Math. **112**, 1–16 (1998)

Ang-MüS00. del Angel, P. L. and S. Müller-Stach: On Chow motives of 3-folds, Trans. Amer. Math. Soc. **352,** 1623–1633 (2000)

Bal-Sc. Balmer, P. and M. Schlichting: Idempotent completion of triangulated categories, J. Algebra **236**, 819–834 (2001)

BanRol. del Baño Rollin, S.: On motives and moduli spaces of stable vector bundles over a curve, PhD thesis (1998). Available at `http://www.ma1.upc.edu/recerca/preprints/preprints-1998/Fitxers/9803bano.ps/at_download/file`

Bar-Hu-Pe-VdV. Barth, W., K. Hulek, C. Peters and A. Van de Ven: *Compact Complex Surfaces, Second Enlarged Edition*, Ergebnisse der Math., **3**, Springer Verlag, Berlin, etc. (2003)

Beau77. Beauville, A.: Variétés de Prym et jacobiennes intermédiaires, Ann. Sci. École Norm. Sup. **10**, 309–391 (1977)

Beau83. Beauville, A.: Quelques remarques sur la transformation de Fourier dans l'anneau de Chow d'une variété abélienne, in Lect. Notes in Math. **1016**, Springer Verlag, Berlin, etc., 238–260 (1983)

Beau86. Beauville, A.: Sur l'anneau de Chow d'une variété abélienne, Math. Ann. **273**, 647–651 (1986)

Beil. Beilinson, A.: Higher regulators and values of L-functions, Journ. Soviet Math **80**, 2036–2070 (1985)

Beil-Ber-Del. Beilinson, A., J. Bernstein and P. Deligne: Faisceaux pervers, in *Analyse et topologie sur les espaces singuliers I*, Astérisque **100**, (1982)

Ber. Berthelot, P.: *Cohomologie cristalline*, Lect. Notes in Math. **407** Springer Verlag, Berlin, etc. (1974).

Bitt. Bittner, F.: The universal Euler characteristic for varieties of characteristic zero, Compositio Math. **140**, 1011–1032 (2004)

Blo76. Bloch, S.: Some elementary theorems about algebraic cycles on Abelian varieties, Invent. Math. **37**, (1976), 215–228.

Blo79. Bloch, S.: Torsion algebraic cycles and a theorem of Roitman, Compositio Math. J. **39**, 107–127 (1979)

Blo80. Bloch, S: *Lectures on algebraic cycles*, Duke Univ. Press. **4**, (1980), second edition: New Math. Monographs **16**, Cambridge University Press (2010)

Blo-Sri. Bloch and S, V. Srinivas: Remarks on correspondences and algebraic cycles, Amer. J. Math. **105**, 1235–1253 (1983)

Bo-Lu-Ra. Bosch, S., W. Lütkebohmert and M. Raynaud: *Néron models*, Springer Verlag, Berlin etc. (1990)

Bourb. Bourbaki: *Algèbre, livre II, Chap. 8, Modules et anneaux semi-simples*, Hermann, Paris 1958.

Ca-Mi07. de Cataldo, M. and Migliorini, L.: Intersection Forms, Topology of Maps and Motivic Decomposition for Resolutions of Threefolds, in [**Nag-Pe**], (2007), 114–150

Ca-Mi09. de Cataldo, M. and Migliorini, L.: The decomposition theorem, perverse sheaves and the topology of algebraic maps, Bulletin of the Amer. Math. Soc. **46**, 535–633 (2009)

Chev58. Chevalley, C.: *Anneaux de Chow et Applications*, Sém. Chevalley (1958)

Chow54. Chow, W. L.: The Jacobian variety of an algebraic curve, Amer. J. Math. **74**, 895–909 (1954)

Chow55. Chow, W. L.: Abstract theory of the Picard and Albanese varieties, `http://mathnt.mat.jhu.edu/mathnew/Chow/Articles/ch55c.pdf`

Chow56. Chow, W. L.: On equivalence classes of cycles in an algebraic variety, Ann. of Math. **64**, 450–479 (1956)

Chow-vW. Chow, W. L. and B. van der Waerden: Zur algebraische Geometrie IX, Math. Ann. **113**, 692–704 (1937)

Clem. Clemens, C. H.: Homological equivalence, modulo algebraic equivalence, is not finitely generated, Publ. Math. IHÉS **58**, 19–38 (1984).

Cor-Ha00. Corti, A and M. Hanamura: Motivic decomposition and intersection Chow groups I, Duke Math. Journal **103**, 459–522 (2000).

Cor-Ha07. Corti, A and M. Hanamura: Motivic decomposition and intersection Chow groups II, Pure Appl. Math. Q. **3**, 181–203 (2007)

Col-Se. *Grothendieck-Serre correspondence*. Edited by Pierre Colmez and Jean-Pierre Serre, Documents Mathématiques (Paris) **2**, Soc.Math. de France, Paris, 2001.

Cu-Re. Curtis, C. W. and I. Reiner: *Representation theory of finite groups and associative algebras*, Wiley-Interscience, New York (1962)

DJ. de Jong, A. J.: A note on Weil cohomology (2007). Available at `http://www.math.columbia.edu/~dejong/seminar/note_on_weil_cohomology.pdf`

Del68. Deligne, P.: Théorème de Lefschetz et critère de dégénérescence de suites spectrales, Publ. Math. IHÉS **35**, 107–126 (1968)

Del70. Deligne, P.: Théorie de Hodge **I**, in *Actes du congrès international Nice 1970*, Gauthiers-Villars (1971) 425–430

Del71. Deligne, P.: Théorie de Hodge II, Publ. Math. IHÉS **40**, 5–58 (1971)

Del74a. Deligne, P.: Théorie de Hodge III, Publ. Math. IHÉS **44**, 5–77 (1974)

Del74b. Deligne, P.: La conjecture de Weil. I, Publ. Math. IHÉS **43**, 273–307 (1974)

Del80. Deligne, P.: La conjecture de Weil II, Publ. Math. IHÉS **52**, 137–252 (1980)

Del-Mi-Og-Sh. Deligne, P., J. S. Milne, A. Ogus and K. Shih: *Hodge cycles, motives and Shimura varieties*, Lect. Notes in Math. **900**, Springer Verlag, Berlin, etc. (1982)

Dem. Demazure, M. : Motifs des variétés algébriques, Sém. Bourbaki 365, Lect. Notes in Math. 180, Springer Verlag, Berlin, etc. **(**1971)

Den-Mu. Deninger, C. and J. P. Murre: Motivic decomposition of abelian schemes and the Fourier transform, Journal f. die reine und angew. Math. **422**, 201–219 (1991)

Elm-Ka-Me. Elman, R., N. Karpenko, and A. Merkurjev: *The algebraic and geometric theory of quadratic forms*, American Mathematical Society Colloquium Publications **56**, Amer. Math. Soc., Providence, RI (2008)

Friedl-La. Friedlander, E. and H. Blaine Lawson: Moving Algebraic Cycles of Bounded Degree, Invent. Math. **132**, 91–119 (1998)

Friedl-Su-Vo. Friedlander, E., A. Suslin, and V. Voevodsky: *Cycles, transfers, and motivic homology theories*. Ann. of Math. Studies **143**. Princeton University Press, Princeton, NJ, 2000

Ful. Fulton, W.: *Intersection Theory*, Springer Verlag, New-York etc. (1984)

Ful-Ha. Fulton, W. and J. Harris: *Representation theory. A first course*. Grad. Texts in Math. **129**. Springer-Verlag, New York, 1991.

Gil-So. Gillet, H. and C. Soulé: Descent, motives and K-theory. Gillet, Journal f. die reine u. angew. Math. **478**, 127–176 (1996)

Gor-Ha-Mu. Gordon, B. Brent, M. Hanamura and J.P. Murre: Relative Chow-Künneth projectors for modular varieties, J. f. die reine u. angew. Math. **558**, 1-14 (2003).

Griff-Ha. Griffiths, Ph. and J. Harris: *Principles of algebraic geometry*, Wiley, New York (1978)

Griff69. Griffiths, P.: On the periods of certain rational integrals I, II, Ann. Math. **90**, 460–495, 498–541, respectively (1969)

Groth58. Grothendieck, A.: Sur une note de Mattuck-Tate. J. f. die reine u. angew. Math. **200**, 208–215 (1958)

Groth62. Grothendieck, A.: *Fondements de la géométrie algébrique*, Sèm. Bourbaki 1957–1962, Secr. Math., Paris (1962)

Groth69a. Grothendieck, A.: Standard conjectures on algebraic cycles, in *Algebr. Geom. Bombay Colloquium 1968*, Oxford, 193–199 (1969)

Groth69b. Grothendieck, A.: *Hodge's general conjecture is false for general reasons*, Topology **8**, 299–303 (1969)

Groth85. Grothendieck, A.: *Récoltes et semailles: réflexions et témoignages sur un passé de mathématicien*, Montpellier, 1985 (unpublished)

Gu-Na. Guillen, F. and V. Navarro Aznar: Un critère d'extension d'un foncteur défini sur les schémas lisses, Publ. Math. IHÉS **95**, 1–91 (2002)

Hana95. Hanamura, M.: Mixed motives and algebraic cycles I, Math. Res. Lett. **2**, 811–821 (1995)

Hana04. Hanamura, M.: Mixed motives and algebraic cycles II, Invent. Math. **158**, 105-179 (2004)

Harts. Hartshorne, R.: *Algebraic Geometry*, Springer Verlag, Berlin Heidelberg etc., Grad. Texts in Math. **52** (1977)

Hub-MüS. Huber, A. and S. Müller-Stach: On the relation between Nori Motives and Kontsevich Periods, preprint `arXiv:1105.0865`

Ill. Illusie, L.: Report on crystalline cohomology, in *Algebraic geometry, Arcata 1974*, Proc. of Symp. Amer. Math. Soc. **29**, 459–478 (1975)

Jann92. Jannsen, U.: Motives, numerical equivalence and semi-simplicity, Invent. Math. **107**, 447–452 (1992)

Jann94. Jannsen, U.: Motivic sheaves and filtration on Chow groups, in [**Jann-Kl-Se**], 245–302 (1994

Jann03. Jannsen, U.: Algebraic cycles and motives, Course notes, Tokyo Univ. 2003/2004 (unpublished).

Jann07. Jannsen, U.: On finite dimensional motives and Murre's conjecture, in [**Nag-Pe**], 112–142 (2007)

Jann-Kl-Se. *Motives. Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference held at the University of Washington, Seattle, Washington, July 20–August 2, 1991* (Ed. by Uwe Jannsen, Steven Kleiman and Jean-Pierre Serre) Proc. Symp. Pure Math. **55**, Part I and II, Amer. Math. Soc., Providence, RI, (1994)

Kahn-M-P. Kahn, B., J. P. Murre and C. Pedrini: On the Transcendental Part of the Motive of a Surface, in [**Nag-Pe**], 2007

Katz-Me. Katz, N. and W. Messing: Some consequences of the Riemann hypothesis for varieties over finite fields, Invent. Math. **23**, 73–77 (1974)

Kim. Kimura, K.: Murre's conjectures for certain product varieties, J. Math. Kyoto Univ. **47**, 621-629 (2007)

Kimu. Kimura, S-I: Chow groups are finite dimensional, in some sense, Math. Ann. **331**, 173–201 (2005)

Kimu-Mur. Kimura, S-I and J. P. Murre: On natural isomorphisms of finite dimensional motives and applications to Picard motives, in: *Proceedings Int. Coll. "Cycles, Motives and Shimura varieties"*, V. Srinivas (Ed.), Mumbai 2008, Naroshi Publ. House (New Delhi) and AMS (2010)

Kimu-Vi. Kimura, S.-I. and A. Vistoli: Chow rings of infinite symmetric products, Duke Math. J. **85**, 411–430 (1996)

Klei68. Kleiman, S. L.: Algebraic cycles and the Weil conjectures, in *Dix exposés sur la cohomologie des schémas*, North Holland, 359–386 (1968)

Klei70. Kleiman, S. L.: Motives in Algebraic Geometry, in *Conf. Oslo*, 53–82 (1970)

Klei94. Kleiman, S. L.: The standard conjectures, in [**Jann-Kl-Se**] 3–20 (1994)

Klei05. KLEIMAN, S. L. : The Picard Scheme, Part 5, in *Fundamental Algebraic Geometry; Grothendiecks FGA explained*, Math. Surveys and Monographs **123** Am. Math. Soc., Providence R.I. (2005)

Kü93. KÜNNEMANN, K.: A Lefschetz decomposition for Chow motives of abelian schemes, Invent. Math. **113**, 85–102 (1993)

La. LANG, S.: *Abelian varieties*, Interscience, New York (1959) and Springer Verlag, Berlin, etc.(1983)

Le-C. LE, J. AND CHEN, X.-W.: *Karoubianness of a triangulated category*, J. of Algebra **310**, 452-457 (2007)

Lev98. LEVINE, M.: *Mixed Motives* Math. Surveys and Monographs, Vol. **57**, Am. Math. Soc., Providence R.I. (1998)

Lev05. LEVINE, M.: Mixed motives, in: *Handbook of K-theory*. Vol. **1**, **2**, 429-521, Springer Verlag, Berlin, etc. (2005)

Lev08. LEVINE, M.: Six Lectures on Motives, in: Some recent developments in algebraic K-theory, ICTP Lecture Notes **23**, 131-227 (2008)

Liebm. LIEBERMAN, D.: Numerical and homological equivalence of algebraic cycles on Hodge manifolds, Amer. J. Math. **90**, 366–374 (1968)

Manin. MANIN, YU.I.: Correspondences, motives and monoidal transformations, Math. U.S.S,R. Sbornik **6**, 439–470 (1968)

Mats52. MATSUSAKA, T. : On the algebraic construction of the Picard variety I, Jap. J. Math. **21** 217–235 (1952)

Mats57. MATSUSAKA, T.: The criteria for algebraic equivalence and the torsion group, Amer. J. Math. **79** 53–66 (1957)

May. MAY, P.: *Simplicial objects in algebraic topology*, Van Nostrand (1967)

Mazur. MAZUR, B.: What is ... a motive? Notices Amer. Math. Soc. **51**, 1214–1216 (2004)

Maz-Vo-We. MAZZA, C, V. VOEVODSKY AND C. WEIBEL: *Lecture Notes on Motivic Cohomology* CMI/AMS, Claymath Publ. **7** (2006)

Mil80. MILNE, J. S.: *Etale Cohomology*, Princeton Univ. Press, Princeton (1980)

Mil82. MILNE, J. S.: Zero cycles on algebraic varieties in nonzero characteristic: Rojtman's theorem, Compositio Math. **47**, 271–287 (1982)

Mil98. MILNE, J. S.: Lectures on Etale Cohomology. Latest version (2012) available at `http://www.jmilne.org/math/CourseNotes/lec.html` (1998)

Mil09. MILNE, J. S.: Motives – Grothendieck's dream. Latest version (2012) at `http://jmilne.org/math/xnotes/MOT.pdf` (2009)

Muk. MUKAI, SH.: Duality between $D(X)$and $D(\hat{X})$ with its application to Picard sheaves, Nagoya Math. J. **81**, 153–175 (1981)

MüS-Pe. *Transcendental aspects of algebraic cycles . Proceedings of the Summer School held in Grenoble, June 18–July 6, 2001*. Edited by S. Müller-Stach and C. Peters. London Mathematical Society Lecture Note Series, **313**, Cambridge University Press, Cambridge (2004)

MüS-Sa. MÜLLER-STACH, S. AND M. SAITO: Relative Chow-Kuenneth decompositions for morphisms of threefolds `http://arxiv.org/abs/0911.3639`, to appear in J. f. die reine u. angew. Math.

Mum69. MUMFORD, D.: Rational equivalence of zero-cycles on surfaces, J. Math. Kyoto Univ. **9**, 195–204 (1969)

Mum74. MUMFORD, D.: *Abelian varieties* (with appendices by C.P. Ramanujam and Yuri Manin) [Second edition] Oxford Univ. Press, for the Tata Institute of Fundamental Research, Bombay (1974)

Mur90. MURRE, J. P. : On the motive of an algebraic surface, Journal f. die reine u. angew. Math. **409**, 190–204 (1990)

Mur93. MURRE, J. P.: On a conjectural filtration on the Chow groups of an algebraic variety I and II, Indag. Mathem. **4**, 177–188 and 189–201(1993)

Nag-Pe. NAGEL, J. AND C. PETERS: *Algebraic Cycles and Motives*, Proceedings of the Annual EAGER Conference And Workshop On the Occasion of J. Murre's 75–th Birthday August 30– September 3, 2004, Leiden University, the Netherlands, Lond. Math. Soc. Lect. Notes Series, Cambridge University Press **343** & **344** (2007)

Nag-Sa. NAGEL, J, AND SAITO, M.: Relative Chow-Kunneth decompositions for conic bundles and Prym varieties, Int. Math. Res. Not. **16**, 2978–3001(2009)

Ner. NÉRON, A.: La théorie de la base pour les diviseurs sur les variétés algébriques, in *Deuxième Colloque de Géométrie Algébrique, Liège, 1952* 119–126, Georges Thone, Liège; Masson & Cie, Paris, (1952)

Nori. M. NORI: *Lectures at TIFR*, unpublished 32 pages.

Pe-St. PETERS, C. AND J. STEENBRINK: *Mixed Hodge Theory*, Ergebnisse der Math, Wiss. **52**, Springer Verlag, Berlin, etc. (2008)

Pie. PIERCE, R.S: *Associative Algebras*, Grad. Text in Math. **88**, Springer Verlag, Berlin, etc. (1982)

Rob. ROBERTS, J.: Chow's moving lemma, Appendix 2 to: *"Motives" (Algebraic geometry, Oslo 1970 (Proc. Fifth Nordic Summer School in Math.)*, pp. 53–82, Wolters-Noordhoff, Groningen, (1972)

Roi. ROJTMAN, A.A.: The torsion of the group of 0-cycles modulo rational equivalence, Ann. of Math. (2) **111** 553–569, (1980)

Sam55. SAMUEL, P.: *Méthodes d'Algèbre abstraite en Géométrie Algébrique*, Erg. Math. **4**, Springer Verlag, Berlin etc. (1955)

Sam56. SAMUEL, P.: Rational equivalence of arbitrary cycles, Amer. J. Math. **78**, 383–400 (1956)

Sam58. SAMUEL, P.: Relations d'équivalence en géométrie algébrique, in *Proc. I.C.M. 1958, Edinburgh*, Cambridge Univ. Press, Cambridge (1960)

Scholl. SCHOLL, A. J. : Classical Motives, in [**Jann-Kl-Se**] 163–187 (1994)

Schwar. SCHWARZENBERGER, R.: Jacobians and symmetric products, Illinois J. Math. **7** 257–268 (1963)

Segre. SEGRE, B.: Intorno ad un teorema di Hodge sulla teoria della base per le curve di una superficie algebrica, Ann. Mat. Pura Appl. **16**, 157-163 (1937).

Serm. ŠERMENEV, A. M.: The motif of an Abelian variety (English. Russian original), Funct. Anal. Appl. **8** 47–53 (1974); translation from Funkts. Anal. Prilozh. **8**, No.1, 55–61 (1974)

Serre56. SERRE, J.-P.: Géometrie algébrique et géométrie analytique, Ann. Inst. Fourier **6**, 1–42 (1956)

Serre57. SERRE, J.-P.: *Algèbre locale, Multiplicités*, Springer Lecture Notes in Math, **11** Second edition Springer Verlag, Berlin, etc.(1965)

Serre77. SERRE, J.-P.: *Linear representations of finite groups* (transl. from the French by Leonard L. Scott), Springer Verlag, Berlin etc. (1977)

Serre91. SERRE, J.-P.: Motifs, in Astérisque **198-200**, 333–349 (1991)

Sesh62. SESHADRI, C. S.: Variété de Picard d'une variété complète, Ann. Mat. Pura Appl. (4) **57**, 117–142 (1962)

Sev. SEVERI, F.: *Serie, Sistemi d'Equivalenza e Corrispondenze Algebriche sulle Varietà Algebriche, Vol. 1* Edizioni Cremonense, Rome (1942)

SGA4. *Théorie des topos et cohomologie étale des schémas*, 1963–1964 (Topos theory and etale cohomology), Lect. Notes in Math, **269**, **270** and **305**, Springer Verlag, Berlin etc. (1972/3)

Verd. VERDIER, J.-L.: *Des Catégories Dérivées des Catégories Abéliennes* Astérisque **239** Soc. Math. de France, Paris; Amer. Math. Soc., Providence R.I. (1996)

Voe95. VOEVODSKY, V. : A nilpotence theorem for cycles algebraically equivalent to zero, Int. Math. Res.. Notes **4**, 1–12 (1995)

Voe00. VOEVODSKY, V. : Triangulated categories of motives over a field, in *Cycles, transfers and motivic cohomology theories (by V. Voevodsky, A. Suslin and E.M. Friedlander)*, Princeton Univ. Press, Princeton (2000)

Voi92. VOISIN, CL.: Une approche infinitésimale du théorème de H. Clemens sur les cycles d'une quintique générale de $\mathbb{P}^4$, J. Algebraic Geom. **1**, 157–174 (1992)

Voi96. VOISIN, CL.: Remarks on zero-cycles of self-products of varieties, in: *Moduli of vector bundles, Proc. of the Taniguchi symposium on vector bundles, December 1994*, ed. by M. Maruyama, pp. 265–285. Decker (1996)

Voi03. VOISIN, CL.: *Hodge Theory and Complex Algebraic Geometry II*, Cambridge Studies in Adv. Math. **77** Cambridge Univ. Press, Cambridge (2003)

Weib. WEIBEL, C.: *An introduction to homological algebra*, Cambridge Studies in Adv. Math. **38**, Cambridge Univ. Press, Cambridge (1994)

Weil46. WEIL, A.: *Foundations of algebraic geometry* 1962 Colloquium publications, Amer. Math. Soc., Providence R.I. **29**, (1946) (revised 1984)

Weil48. WEIL, A.: *Variétés abéliennes et courbes algébriques*, Hermann, Paris, (1948)

Weil54. WEIL, A.: Sur les critères d'équivalence en géométrie algébrique, Math. Ann. **128**, 95–127 (1954), also in *Collected papers, II*, Springer Verlag, Berlin, etc. (1980) (second printing)

Zar. ZARISKI, O.: *Algebraic Surfaces*; with appendices by S.S. Abhyankar, J. Lipman, and D. Mumford, 2nd supplemented edition, Springer Verlag, Berlin, etc. (1971)

Index of Notation

$\Delta^{\pm}(X)$: cycles realizing the sum of even (odd) Künneth parts of the diagonal $\Delta(X)$, 47
Δ_i^{topo}: the i-th Künneth component of the diagonal $\Delta(X)$, 36
$\Gamma_r(X)$: correspondence defined by $r \in R(\mathfrak{S}_n)$, 44
χ_{mot}^c: motivic Euler characteristic with compact support, 125
χ_{mot}: motivic Euler characteristic, 126
$\tau_{\leq k}, \tau_{\geq k}$: truncation functors, 110
$\bigwedge^n M$: n-th alternating motive on M, 45
${}^pH^k$: k-th perverse cohomology functor, 111
${}^pR^i f_*(F)$: i-th perverse image sheaf of F, 111

$\mathbf{1}$: motive of point, 27

$\mathsf{A}^i(X)$: codim-i algebraic cycle classes on X, 8
$\mathsf{A}^i_{\mathrm{prim}}(X)$: codim-$i$ primitive algebraic classes on X, 38
$\mathrm{Alb}(X)$: Albanese variety of X, 14

$\mathsf{B}^i(X)$: codim-i algebraic cycles modulo numerical equivalence, 39

$C^i_{\sim}(X)$: codim-i cycles on X modulo those equivalent to zero, 3
$C^i_{\sim}(X)_F$: codim-i cycles on X with F-coefficients modulo those equivalent to zero, 3
$C_{\sim}(X)_F$: cycles on X with F-coefficients modulo those equivalent to zero, 3
$C_{\sim}(X)$: cycles on X modulo those equivalent to zero, 3
$\mathfrak{C}^{\natural}$: pseudo-abelian completion of $\mathfrak{C}$, 26
$c(D)$ heart of t-structure (D, t), 111
CHM: category of Chow motives, 26
$\mathsf{CHM}_{\mathbb{Z}}(k)$: category of Chow motives with integral coefficients, 30
$\mathsf{CH}(X)$: the Chow group of X, 4
$\mathsf{CH}(X)_A$: Chow group of X with F-coefficients, 4
$\mathsf{CH}^*(X, n)$: higher Chow groups of level n, 133
$\mathsf{CH}^i(M)$: i–th Chow group of the motive M, 29
$\mathsf{CH}^i(X)$ i–th Chow group of X, 4
$\mathsf{CH}^i(X)_F$: i–th Chow group of $X$with F-coefficients, 4
$\mathsf{CH}^1_{\equiv}(X \times Y)$: degenerate divisors on $X \times Y$, 15
$\mathsf{CH}^2_{\equiv}(S \times S')_{\mathbb{Q}}$: degenerate codim-2 cycle classes on products, 100
$\mathsf{CH}^d_{\equiv}(X \times Y)$: degenerate degree 0 correspondences from $X = X_d$ to $Y = Y_d$, 80
$\mathsf{CH}^i_{\mathrm{alg}}(X)$: codim-$i$ cycle classes on X algebraically equivalent to zero, 6
$\mathsf{ch}(X)$: Chow motive of X, 26
$\mathsf{ch}^i(X)$: the i–th Chow-Künneth motive of X, 68
$\mathsf{ch}^2_{\mathrm{alg}}(S)$: algebraic part of the second Chow-Künneth component for a surface S, 79
$\mathsf{ch}^2_{\mathrm{trans}}(S) = t(S)$: transcendental part of the second Chow-Künneth component for a surface S, 79
$\mathrm{Cone}\, f$: the cone of a morphism f of complexes, 110
$\mathsf{Cor}_{\mathrm{fin}}(X, Y)$: finite correspondences from X to Y, 131
$\mathsf{Cor}(X, Y)$, $\mathsf{Cor}_{\sim}(X, Y)$: classes of correspondences from X to Y, 23
$c_\lambda(T)$: Young symmetrizer of T, 52

D: duality-operator, 29
$d_\lambda(X)$: correspondence defined by $e_\lambda \in R(\mathfrak{S}_n)$, 44
d_{alt}: the n-th alternator of variety of motive, 44
d_{sym}: the n-th symmetriser of variety or motive, 44
$\mathsf{Div}(X)$: group of divisors on X, 1
$\mathsf{Div}_\tau(X)$: τ–equivalence for divisors on X, 13
$\mathrm{div}(f)$: divisor of the rational function f, 4

$\mathsf{DM}^{\mathrm{gm}}_{\mathrm{eff}}(k)$: effective geometric motives over k, 132

e_λ: element in $R(\mathfrak{S}_n)$ defined by partition λ, 44
e_{alt}: the n-the alternator in $R(\mathfrak{S}_n)$, 44
e_{sym}: the n-th symmetrizer in $R(\mathfrak{S}_n)$, 44

$F^\bullet_{BB}$: Bloch-Beilinson filtration, 85
f^*: (flat) pull back of cycles, 2
f_*: push forward of cycles, 2

$G(X)$: Gysin complex of X, 127
GrVect_F: category of finite dimensional graded F-vector spaces, 7
$\mathsf{Griff}^i(X)$: i–th Grifiths group of X, 18

$H^2_{\mathrm{trans}}(X)$: transcendental 2–cohomology for surface X, 17
$H^i_{\mathrm{crys}}(X/W(k))$: i–th crystalline cohomology of X, 8
$H^i_{\mathrm{dR}}(X)$: i–th (algebraic) de Rham cohomology of X, 8
$H^i_{\mathrm{B}}(X)$: i–th Betti cohomology of X, 8
$H^i_{\mathrm{d}R}(X_{\mathrm{an}};\mathbb{C})$: i–th (analytic) de Rham cohomology of X, 8
$H^i_{\text{ét}}(X_{\bar{k}},\mathbb{Q}_\ell)$: i–th étale cohomology of X, 8
$H^p(X,\mathbb{Z}(q))$: (p,q)–th motivic cohomology group of X, 133
$h_\sim$: functor of motives of smooth varieties with respect to an equivalence $\sim$, 26

$i(V\cdot W;Z)$: intersection number of V and W along Z, 2
$\mathrm{IC}^\bullet_S$: intersection complex of S, 109
$\mathrm{IH}^*(S)$: intersection cohomology groups of S, 109

$\mathrm{J}(C)$: jacobian of the curve C, 15

$K_0\mathsf{Mot}_\sim(k)$: K-group for motives, 125
$K_0\mathsf{Var}(k)$: K-group of k-varieties, 124

$\mathbf{L}_S$: Lefschetz motive over S, 107
$\mathbf{L}_\sim$: Lefschetz motive, 28

$M(X)$: Voevodsky-motive of X, 132
$M^c(X)$: Voevodsky-motive of X with compact support, 133
$\mathsf{Mot}^{\mathrm{eff}}_\sim(k)$: the category of effective motives over k, 25
$\mathsf{Mot}_\sim(k)$: pure motives over k, 25

NM: category of Grothendieck motives, 26
$\mathsf{NS}(X)$: Néron-Severi group of X, 13

ord: the order homomorphism, 4

$\mathrm{Perv}(S)$: perverse sheaves on S, 111
$\mathrm{Pic}^0_{\mathrm{red}}(X)$: Picard variety of X, 14

$R(G)$: group ring on G with $\mathbb{Q}$-coefficients, 44

$\mathsf{SmCor}(k)$: category of smooth correspondences over k, 131
$\mathsf{SmProj}(k)$: the category of smooth projective varieties over k, 1
$\mathrm{Sym}^n(M)$: n-th symmetric motive on M, 45

$T(X)$: Albanese kernel for X , 17
$T_\ell(A)$: ℓ–adic Tate group of A, 73
$\mathbb{T}_\lambda M$: motive built from M and partition λ, 45
$\mathbf{T}_\sim$: Tate motive, 28

$V\cdot W$: intersection cycle of V and W, 2
$\mathsf{Var}(k)$: category of k-varieties, 124
$\mathcal{V}(S)$: category of varieties smooth and projective over a quasi-projective smooth base, 113

$W(X)$: weight complex of X, 127

$X\vdash Y$: correspondence from X to Y, 23

${}^{\mathsf{T}}Z$: transpose of the correspondence Z, 3
$\mathsf{Z}^i_\otimes(X)$: codim-i cycles smash nilpotently equivalent to 0, 7
$\mathsf{Z}^i(X)_F$: codim-i cycles on X with F-coefficients, 1
$\mathsf{Z}^i_{\mathrm{hom}}(X)$: codim–$i$ cycles homologically equivalent to zero, 9
$\mathsf{Z}^i_{\mathrm{num}}(X)$ codim–i cycles numerically equivalent to zero, 10
$\mathsf{Z}^i(X)$: codim-i cycles on X, 1
$\mathsf{Z}^i_{\mathrm{alg}}(X)$: codim-$i$ cycles algebraically equivalent to zero, 6
$\mathsf{Z}^i_\sim(X)$: codim-i cycles on X equivalent to zero, 3
$\mathsf{Z}^i_{\mathrm{rat}}(X)$: codim-$i$ cycles on X modulo rational equivalence, 4
$\mathsf{Z}_\sim(X)_F$: codim-i cycles on X with F-coefficients equivalent to zero, 3

Index

abelian category
  semi simple —, 39
abelian variety
  isogeny between abelian varieties, 31
Albanese
  kernel, 17, 80
  motive, 70
  variety, 14
algebraic
  classes, 35
  cycle, 1

Beilinson formula, 86
Betti cohomology, 8
Bloch
  Conjecture, 17, 64, 80, 96
Bloch-Beilinson
  conjectures, 85, 86
  filtration, 85
blow-up diagram, 125

category
  derived —, 105
  homotopy —, 105
Chow group, 4
  higher — , 133
Chow motive, 26
  birationally isomorphic varieties, 29
  of non-isomorphic varieties may be equal, 34
  surfaces, 78
Chow-Künneth
  conjecture, 68, 86
  decomposition for surfaces, 78
  decomposition, curves, 31, 68
  decomposition, examples, 69
  decomposition, relative —, 112
  motive, 68
cohomology
  Betti —, 8
  classical Weil —, 8
  crystalline —, 9
  de Rham —, 8
  étale —, 9
comparison theorems for Weil cohomology theories, 9
cone of a complex, 110
conjecture
  $N(M), N'(M)$, 63
  $S(X)$, 63
  of Kimura-O'Sullivan, 62
constructible sheaf, 109
correspondence, 3, 23
  degenerate —, 80
  degree, 23
  relative —, 107
  topological —, 37
correspondences
  finite —, 131
cosupport condition, 109
crystalline cohomology, 9
cycle class map, 8

De Rham cohomology, 8
decomposition theorem (due to Beilinson, Bernstein, Deligne, Gabber), 111
degenerate
  correspondence, 80
  divisor on a product, 15
dimension of motives, 46
  is well-defined, 56
distinguished triangle, 110
divisor, 13
  degenerate —, 15
duality for motives, 29

effective geometric motive, 131, 132
equivalence
  τ–equivalence, 13
  algebraic —, 6
  homological —, 7
  linear —, 4
  numerical —, 10
  rational —, 4, 5
  smash nilpotent —, 7
equivalence relation
  adequate ("good") —, 3
étale cohomology, 9

finite
  correspondences, 131
  dimensional motive, 46
  dimensionality and products, 53
  dimensionality and nilpotent morphisms, 59
  dimensionality and surjective morphisms, 58

Griffiths group, 18
Grothendieck
  standard conjectures, 35
  motive, 26
Gysin
  complex, 127

heart of a t-structure, 111
higher Chow groups, 133
homotopic
  morphisms, 127
homotopy
  category, 127
  equivalence, 127
  operator, 127
  property for Chow groups, 6

intersection
  cohomology group, 109
  complex, 109, 110
  number, 2
isogeny between abelian varieties, 31

Künneth
  component of the diagonal, 67
  conjecture, 36
  formula, 8

Lefschetz
  motive, 28, 107
  theorems, 8
Leray spectral sequence, 106
Lieberman's lemma, 24
  relative —, 114

Manin's identity principle, 33
Matsusaka's theorem, 13
motive
  for curve is finite dimensional, 47
  Albanese —, 70
  Chow groups, 29
  Chow-Künneth, 68
  cohomology groups, 30
  constituent of a —, 26
  curves, 68
  dimension, 46
  direct sums, 28
  effective geometric —, 131
  for complete intersection, 83
  in the sense of Voevodky, 132
  Lefschetz —, 107
  naturally isomorphic motives, 69
  of a blow-up, 34
  of a curve, 30
  of a projective bundle, 34
  of a smooth projective variety, 26
  phantom —, 61, 95
  Picard —, 70
  related to representations of the symmetric group, 45
  relative —, 107
  transcendental —, 80
motivic
  cohomology, 133
  decomposition conjecture, 113
  Euler characteristic, 126
    for pairs, 126
    with compact support, 125
moving lemma, 3
Murre conjecture
  I(X), 86
  for curves, surfaces, abelian varieties, 88
  II(X), 86
  III(X), 87
  IV(X), 87
  — divisors, 89
  — products of 2 surfaces, 96
  — results of Jannsen, 93
  — special threefolds, 89
  — zero-cycles, 89

Néron-Severi group, 13
naive motivic ring, 126

perverse
  t-structure, 111
  cohomology functors, 111
  sheaf, 109
phantom motive, 61, 95
Picard
  group, 13
  motive, 70
  scheme, 13
  variety, 14
Poincaré-duality, 8
primitive algebraic classes, 38
projector, 23
pseudo-abelian completion, 26

quasi-isomorphism, 105

realization functor, 112
relative
  Chow-Künneth decomposition, 112
  correspondence, 107
  Lieberman lemma, 114
  Manin identity principle, 114
  motive, 107
    abelian schemes, 114
    conic bundles over a surface, 115

relative cell decomposition, 114
semi-small maps, 113
surfaces over a curve, 119
resolution of singularities for field, 124

sign conjectue $S(X)$, 47
smash-nilpotent morphism, 54
smooth correspondences, 131
spectral sequence
Leray —, 106
standard conjecture
Lefschetz type conjectures
$B(X)$, $A(X, L)$, 37
$C(X)$=Künneth conjecture, 36, 68
$D(X)$, 10
of Hodge type (Hdg(X)), 38
support condition, 109

t-structure, 111
heart of a —, 111
Tate
group T_ℓ, 73
motive, 28
tensor-nilpotent morphism, 54
topological correspondence, 37
trace isomorphism, 8
transcendental
2–cohomology for surface X, 17
triangulated category, 110
of motives, 123
truncation functors, 110

vanishing lemma for symmetric and
alternating products of motives, 53
Voevodsky
Conjecture, 63, 64
motive, 132
Voevodsky-Voisin result on smash
nilpotence, 7, 19

weak factorization
holds for k, 124
weight
complex, 127, 128
spectral sequence, 130
Weil
classical — cohomology, 8
cohomology, 7
Theorem on jacobians, 15

Young
symmetrizer of a tableau, 52
tableau, 51

zero cycles, 17

The theory of motives was created by Grothendieck in the 1960s as he searched for a universal cohomology theory for algebraic varieties. The theory of pure motives is well established as far as the construction is concerned. Pure motives are expected to have a number of additional properties predicted by Grothendieck's standard conjectures, but these conjectures remain wide open. The theory for mixed motives is still incomplete.

This book deals primarily with the theory of pure motives. The exposition begins with the fundamentals: Grothendieck's construction of the category of pure motives and examples. Next, the standard conjectures and the famous theorem of Jannsen on the category of the numerical motives are discussed. Following this, the important theory of finite dimensionality is covered. The concept of Chow-Künneth decomposition is introduced, with discussion of the known results and the related conjectures, in particular the conjectures of Bloch-Beilinson type. We finish with a chapter on relative motives and a chapter giving a short introduction to Voevodsky's theory of mixed motives.

For additional information
and updates on this book, visit

www.ams.org/bookpages/ulect-61

ISBN 978-0-8218-9434-7

ULECT/61

AMS *on the* Web
www.ams.org