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CHAPTER 5

Busemann functions

The modern approach to infinite geodesics involves Busemann functions. They were first exploited in FPP in the important papers of Hoffman [**111**, **112**], and they allow one to obtain some information about geodesics without the unproven assumptions (for example, positive curvature of the limiting shape) of Newman.

5.1. Basics of Busemann functions

Long before FPP existed, H. Busemann invented various tools to study the geometry of geodesics in certain metric spaces [**42**]. Busemann functions grew out of the attempt to understand parallelism between geodesics. We can give some basic properties in the setting of FPP. Consider a geodesic ray R, and list its vertices in order as

$$V(R) = \{x_0, x_1, \ldots\}.$$

Here, x_0 is taken as the initial vertex of the ray.

Definition 5.1. The Busemann function associated to the ray R is $f_R : \mathbb{Z}^d \to \mathbb{R}$, given by

$$f_R(x) = \lim_n \left[T(x, x_n) - T(x_0, x_n)\right].$$

This limit exists since the terms are bounded ($|T(x, x_n) - T(x_0, x_n)| \leq T(x_0, x)$) and monotone:

$$\begin{aligned} T(x, x_{n+1}) - T(x_0, x_{n+1}) &= T(x, x_{n+1}) - T(x_n, x_{n+1}) - T(x_0, x_n) \\ &\leq T(x, x_n) - T(x_0, x_n). \end{aligned}$$

The Busemann function for R measures asymptotically how far behind x_0 is relative to x when both points attempt to travel down the ray R. The next lemma collects a few properties of this function.

Lemma 5.2. *The following hold.*

(*a*) *For $m < n$, one has $f_R(x_m) - f_R(x_n) = T(x_m, x_n)$.*
(*b*) *For all x, one has $|f_R(x)| \leq T(x, x_0)$.*
(*c*) *If R_1 and R_2 are geodesic rays that coalesce (they have finite symmetric difference) and have initial points x_0 and y_0, then*

$$f_{R_1}(x) = f_{R_2}(x) - f_{R_2}(x_0) \text{ for all } x \in \mathbb{Z}^d.$$

Proof. Proofs of items (*a*) and (*b*) follow directly from the definition. As for (*c*), write the vertices of R_1 as $x_0, x_1, \ldots$ and the vertices of R_2 as $y_0, y_1, \ldots$. By coalescence, we can find $N \in \mathbb{N}$ such that for n large, one has $x_n = y_{n+N}$. For such

n,

$$\begin{aligned} T(x, x_n) - T(x_0, x_n) &= T(x, y_{n+N}) - T(y_0, y_{n+N}) \\ &\quad - T(x_0, y_{n+N}) + T(y_0, y_{n+N}) \\ &\to f_{R_2}(x) - f_{R_2}(x_0). \end{aligned}$$

□

We will often use a Busemann function that does not depend on the initial point. If R is a geodesic ray, then set

$$f_R(x, y) = f_R(x) - f_R(y).$$

We note the following properties of this Busemann function:

(1) f_R is anti-symmetric: $f_R(x, y) = -f_R(y, x)$.

(2) f_R is additive:

$$f_R(x, y) = f_R(x, z) + f_R(z, y) \text{ for } x, y, z \in \mathbb{Z}^d.$$

(3) If R_1 and R_2 are geodesic rays that coalesce, then $f_{R_1}(x, y) = f_{R_2}(x, y)$ for all $x, y \in \mathbb{Z}^d$.

PROOF. Using item (c) in Lemma 5.2 and writing x_0 for the initial point of R_1, one has

$$\begin{aligned} f_{R_1}(x) - f_{R_1}(y) &= (f_{R_2}(x) - f_{R_2}(x_0)) - (f_{R_2}(y) - f_{R_2}(x_0)) \\ &= f_{R_2}(x) - f_{R_2}(y). \end{aligned}$$

□

(4) (Translation covariance) Let θ be a translation of the lattice by an integer vector. Then

$$f_R(x, y)(\omega) = f_{\theta R}(\theta(x), \theta(y))(\theta(\omega)).$$

Here, the translated weight-configuration $\theta(\omega)$ is defined as $\tau_e(\theta(\omega)) = \tau_{\theta^{-1}e}(\omega)$ for any edge e. Furthermore θR is the translated geodesic ray.

5.2. Hoffman's argument for multiple geodesics

In this section we present Hoffman's [**111**] argument that there exist at least two infinite geodesics a.s.. This result was also proved by Garet-Marchand [**91**], but Hoffman's techniques, involving Busemann functions, led to many other results. We will use the assumptions of Hoffman, which do not require the τ_e's to be independent:

(1) $\mathbb{P}$ is ergodic under lattice translations.

(2) $\mathbb{P}$ has unique passage times: a.s., distinct paths have distinct passage times.

(3) $\mathbb{E}\tau_e^{2+\delta} < \infty$ for some $\delta > 0$.

(4) The limit shape $\mathcal{B}$ for $\mathbb{P}$ is bounded.

Recall from Definition 4.21 that two geodesic rays R_1 and R_2 are distinct if they share at most finitely many edges and vertices. Let $\mathcal{N} = \mathcal{N}(\omega)$ be the maximal number of distinct geodesic rays in the edge-weight configuration for outcome ω. It can be shown that $\mathcal{N}$ is a measurable function of the edge-weights and is invariant under lattice translations. So by ergodicity, it is a.s. constant.

THEOREM 5.3 (Hoffman [**112**], Theorem 2; Garet-Marchand [**91**], Theorem 3.2). *The number $\mathcal{N}$ is at least two.*

PROOF. By taking a subsequential limit of the geodesics from 0 to ne_1 as in Section 4.4.1, we see that $\mathcal{N} > 0$. Assume for a contradiction that $\mathcal{N} = 1$. Let $R = R(\omega)$ be any geodesic ray and define $f : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ by

$$f(x,y) = f_R(x,y) = f_{R(\omega)}(x,y)(\omega).$$

On the probability 1 set on which $\mathcal{N} = 1$, this function is well-defined and is independent of the choice of R. Indeed, if $R' = R'(\omega)$ is another geodesic ray, then it cannot be disjoint from R. Due to uniqueness of passage times, R' must coalesce with R and therefore $f_R = f_{R'}$.

One can also check that f is measurable. Due to translation covariance of f_R, the function f is also translation covariant:

$$f(x,y)(\omega) = f\big(\theta(x),\theta(y)\big)(\theta(\omega)) \text{ a.s.}$$

for any lattice translation θ. This, combined with additivity and the ergodic theorem, implies that for each $x \in \mathbb{Z}^d$,

$$\frac{1}{n} f(0,nx) = \frac{1}{n}\sum_{k=1}^{n} f((k-1)x,kx) \to \mathbb{E}f(0,x) \quad \text{a.s. and in } L^1. \tag{5.1}$$

To evaluate this limit, we use symmetry: for $k = 1,\dots d$,

$$0 = \mathbb{E}f(0,e_k) + \mathbb{E}f(e_k,0) = \mathbb{E}f(0,e_k) + \mathbb{E}f(-e_k,0)$$

and by translation invariance, this equals

$$\mathbb{E}f(0,e_k) + \mathbb{E}f(0,e_k),$$

so $\mathbb{E}f(0,e_k) = 0$. We further obtain

$$\mathbb{E}f(0,me_k) = m\mathbb{E}f(0,e_k) = 0$$

for all integers m and then by additivity,

$$\mathbb{E}f(0,x) = 0 \quad \text{for all } x \in \mathbb{Z}^d.$$

Combining with (5.1), one has

$$\frac{1}{n} f(0,nx) \to 0 \quad \text{a.s. and in } L^1.$$

To upgrade this convergence to an omni-directional statement, one can use the condition $\mathbb{E}\tau_e^{2+\delta} < \infty$ along with the bound $|f(x,y)| \le T(x,y)$ to prove a type of shape theorem for the function f. That is, for each $\epsilon > 0$,

$$\mathbb{P}\left(|f(0,x)| < \epsilon\|x\|_1 \text{ for all } x \in \mathbb{Z}^d \text{ with } \|x\|_1 \text{ large enough}\right) = 1. \tag{5.2}$$

(The proof follows the same method as in Section 2.3.) In particular, if we define R to be the (a.s. unique) limit of geodesics from 0 to ne_1, and we write its vertices in order as $x_0, x_1, \dots$, then one has

$$f(0,x_n)/\|x_n\|_1 \to 0 \quad \text{a.s..}$$

On the other hand, by property 1 of f_R, one has $f(0,x_n) = T(0,x_n)$. Therefore

$$T(0,x_n)/\|x_n\|_1 \to 0 \quad \text{a.s..}$$

This contradicts the fact that the limit shape is bounded and therefore shows $\mathcal{N} > 1$. $\square$

5.3. Directions of geodesics via Busemann functions

In [**147**], Newman introduced the following limits, in an attempt to study the local geometry of the boundary of the growing ball $B(t)$:

$$\lim_n [T(x, x_n) - T(y, x_n)] \quad \text{for } x, y \in \mathbb{Z}^2,$$

where (x_n) is a deterministic sequence diverging to infinity in a direction θ. He was able to show that under the uniformly positive curvature assumption on $\mathcal{B}$ (Definition 2.39 or (2.29)), the limit exists a.s. for Lebesgue-almost every direction θ.

In 2008, Hoffman [**112**] defined Busemann-type functions similar to the above, and introduced the idea of studying geodesics through their Busemann functions. These ideas were continued by Damron-Hanson in 2013 [**68**]. In the next two sections, we will describe some of the ideas therein.

Let H_n be the hyperplane $\{x \in \mathbb{R}^d : x \cdot e_1 = n\}$ and define the function

$$B_n(x, y) = T(x, H_n) - T(y, H_n) \quad \text{for } x, y \in \mathbb{Z}^d. \tag{5.3}$$

For purposes of illustration, we will in this section assume that

$$B(x, y) := \lim_n B_n(x, y) \quad \text{exists a.s.} \tag{5.4}$$

for $x, y \in \mathbb{Z}^d$. This assumption should be compared to assumption (2.18). It is not known how to show that limits like (5.4) exist, and this is one of the first and main obstacles when dealing with Busemann functions. We will further discuss this issue in Section 5.4, where we can get around it by considering types of weak limits. The results presented in the rest of this section are valid under assumption (5.4).

QUESTION 5.3.1. Prove that under some conditions on the edge-weights the limit (5.4) exists.

REMARK 5.4. For the generalized Busemann functions introduced in Section 2.6, the question above corresponds to Question 2.6.1. One may also want to consider Busemann limits to points rather than lines; that is, limits of the form $\lim_n [T(x, ne_1) - T(y, ne_1)]$.

LEMMA 5.5. *Assume* (5.4) *and Hoffman's conditions 1–4. One has*

$$\mathbb{E}B(x, y) = (x - y) \cdot \rho \textit{ for } x, y \in \mathbb{Z}^d,$$

where $\rho = e_1 \mu(e_1)$.

PROOF. By translation invariance,

$$\mathbb{E}B(x, y) = \mathbb{E}B(\theta(x), \theta(y)) \tag{5.5}$$

for any lattice translation θ. As B is additive and satisfies

$$\mathbb{E}B(0, -x) = -\mathbb{E}B(0, x),$$

we find that $\mathbb{E}B(x, y)$ is a linear function of $x - y$. To find this function, it suffices to find $\mathbb{E}B(0, e_i)$ for $i = 1, \ldots, d$. By symmetry, one has

$$\mathbb{E}B(0, e_i) = 0$$

for $i > 1$. To find $\mathbb{E}B(0, e_1)$, we use an averaging trick introduced by Garet-Marchand and Hoffman (and which also appears in some form in Kingman [**128**, eq. (26)]). By translating by e_1, we obtain

$$\begin{aligned}\mathbb{E}T(0, H_n) &= \sum_{k=1}^{n} \mathbb{E}[T((k-1)e_1, H_n) - T(ke_1, H_n)] \\ &= \sum_{k=1}^{n} \mathbb{E}[T(0, H_{n-k+1}) - T(e_1, H_{n-k+1})] \\ &= \sum_{k=1}^{n} \mathbb{E}B_k(0, e_1).\end{aligned}$$

Using the bound $|B_k(x,y)| \le T(x,y)$ and the dominated convergence theorem, one has

$$\mathbb{E}B(0, e_1) = \lim_n \frac{1}{n}\mathbb{E}T(0, H_n) = \mu(e_1).$$

The last equality can be shown using the shape theorem. (See, for example, [**68**, Lemma 3.6].) □

Using Lemma 5.5, along with the ergodic theorem, one has a.s. and in L^1,

$$\frac{1}{n}B(0, nx) = \frac{1}{n}\sum_{k=1}^{n} B((k-1)x, kx) \to x \cdot \rho \quad \text{for } x \in \mathbb{Z}^d.$$

As in (5.2), it is not difficult to upgrade this to the following sort of shape theorem for the Busemann function. See, for example, Damron-Hanson [**68**, Section 4], where this is done for a "reconstructed Busemann function."

LEMMA 5.6. *Assume* (5.4) *and Hoffman's conditions 1–4. For each* $\epsilon > 0$,

$$\mathbb{P}\left(|B(0,x) - x \cdot \rho| > \epsilon\|x\|_1 \text{ for only finitely many } x \in \mathbb{Z}^d\right) = 1,$$

where $\rho = e_1\mu(e_1)$.

We now show how assumption (5.4), along with an "exposed point" assumption, implies existence of geodesic rays with asymptotic direction e_1. Let S be the set

$$S = \partial\mathcal{B} \cap \{w : \rho \cdot w = 1\}.$$

Because the rightmost set is a supporting hyperplane for the limit shape at the point $e_1/\mu(e_1)$, the set S is a portion of the boundary $\partial\mathcal{B}$ containing $e_1/\mu(e_1)$. We will say that a geodesic ray with vertices $x_0, x_1, \ldots$ is asymptotically directed in S if each limit point of the sequence $(x_n/\mu(x_n))$ is contained in S. If S contains only one point, we then say that this ray has an asymptotic direction. If the point $e_1/\mu(e_1)$ is exposed (there is a supporting hyperplane for $\mathcal{B}$ that touches $\mathcal{B}$ only at $e_1/\mu(e_1)$), then by symmetry, $S = \{e_1/\mu(e_1)\}$. In that case, the following theorem implies that a.s., there is an infinite geodesic starting from 0 that has asymptotic direction e_1. It is an open problem to show without assumption (5.4) that with positive probability, there is a geodesic ray with an asymptotic direction:

QUESTION 5.3.2. Show that under general assumptions on the edge-weights (τ_e) and for some $d \ge 2$, there a.s. exists an infinite geodesic with vertices $0 = x_0, x_1, \ldots$ that has an asymptotic direction; that is, the sequence $(x_n/\mu(x_n))$ converges.

The following theorem is similar to [**68**, Theorem 5.3].

THEOREM 5.7. *Assume* (5.4) *and Hoffman's conditions 1–4. A.s., every subsequential limit of geodesics from 0 to H_n is asymptotically directed in S.*

PROOF. Let Γ be any subsequential limit of geodesics from 0 to H_n (which must be self-avoiding by our assumption of uniqueness of passage times) and label its vertices $0 = x_0, x_1, x_2, \ldots$. First note that for $m \geq 0$, one has

$$B(0, x_n) = T(0, x_n). \tag{5.6}$$

Indeed, choose (n_k) to be a subsequence such that some geodesics Γ_{n_k} from 0 to H_{n_k} converge to Γ as $k \to \infty$. For k large, the first n vertices of Γ_{n_k} coincide with those of Γ, so

$$\begin{aligned} B(0, x_n) &= \lim_{N \to \infty} [T(0, H_N) - T(x_n, H_N)] \\ &= \lim_{k \to \infty} [T(0, H_{n_k}) - T(x_n, H_{n_k})] = T(0, x_n). \end{aligned}$$

Choose z to be any limit point of the sequence $(x_n/\mu(x_n))$ so that for some subsequence (n_j), one has $x_{n_j}/\mu(x_{n_j}) \to z$. Now by Lemma 5.6, a.s. for any choice of Γ and (n_j),

$$\lim_{j \to \infty} \frac{B(0, x_{n_j})}{\mu(x_{n_j})} = \lim_{j \to \infty} \rho \cdot (x_{n_j}/\mu(x_{n_j})) = \rho \cdot z.$$

On the other hand, by the shape theorem and (5.6), this equals

$$\lim_{j \to \infty} \frac{T(0, x_{n_j})}{\mu(x_{n_j})} = 1,$$

and so $\rho \cdot z = 1$. Since $z \in \partial \mathcal{B}$, this means $z \in S$. □

COROLLARY 5.8. *Assume* (5.4), *Hoffman's conditions 1–4, and that $e_1/\mu(e_1)$ is an exposed point of $\mathcal{B}$. A.s., every subsequential limit of geodesics from 0 to H_n has asymptotic direction e_1.*

5.4. Busemann increment distributions and geodesic graphs

Here we will explain one method of getting around Question 5.3.1 to obtain results about directions of geodesic rays. This method shares some similarities with the plan of attack used in the original paper of Kingman, described in Section 2.6. (We make this relation precise below in Remark 5.16.) Instead of trying to establish existence of the limit (5.4), we will focus on a weak limit by considering translational averages. We follow Damron-Hanson [**68**]. Their results were stated only for two dimensions, but the existence of geodesics directed in sectors (Theorem 5.9) also holds for general dimensions, so we outline an argument for this expanded statement.

We will focus on i.i.d. distributions, although the theorems apply to a wide class of translation-ergodic measures (similar to Hoffman's conditions from Section 5.2). Consider the following conditions on the common distribution function F for i.i.d. edge-weights:

I. $\mathbb{E}Y^d < \infty$, where Y is the minimum of $2d$ i.i.d. edge-weights. Furthermore, $F(0) < p_c$.
II. F is continuous.

Recall that a supporting hyperplane H for the limit shape $\mathcal{B}$ at a point $z_0 \in \partial\mathcal{B}$ is a hyperplane that contains z_0 and such that $\mathcal{B}$ does not intersect both components of H^c. If there is only one such hyperplane, we say that $\partial\mathcal{B}$ is differentiable at z_0. In this case, we write H_{z_0} for this hyperplane. Write

$$S_{z_0} = H_{z_0} \cap \partial\mathcal{B}.$$

THEOREM 5.9 (Damron-Hanson [**68**], Theorem 1.1). *Assume I. If $\partial\mathcal{B}$ is differentiable at z_0, then a.s., there is a geodesic ray Γ containing the origin which is asymptotically directed in S_{z_0}. This means every limit point of the set $\{x/\mu(x) : x \in \Gamma\}$ is contained in S_{z_0}.*

Note that if z_0 is an exposed point of differentiability, then there is a geodesic ray with asymptotic direction z_0, since in that case, $S_{z_0} = \{z_0\}$. This is an improvement on Newman's theorem (Theorem 4.25) because we have replaced the global curvature condition with a local, directional condition.

The next result is related to coalescence of geodesic rays.

THEOREM 5.10 (Damron-Hanson [**68**], Theorem 1.11). *Take dimension $d = 2$, assume I and II and that $\partial\mathcal{B}$ is differentiable at z_0. With probability one, there exists a collection of geodesic rays $(\Gamma_x : x \in \mathbb{Z}^2)$ satisfying the following properties:*

1. *Each $x \in \mathbb{Z}^2$ is a vertex of Γ_x.*
2. *(Directedness) Each Γ_x is asymptotically directed in S_{z_0}.*
3. *(Coalescence) For all $x, y \in \mathbb{Z}^2$, the paths Γ_x and Γ_y coalesce.*
4. *(Finiteness of backward paths) Each $x \in \mathbb{Z}^2$ is on Γ_y for only finitely many $y \in \mathbb{Z}^2$.*

As discussed in Section 4.4.2, Licea-Newman [**139**] (along with an improvement by Zerner [**148**]) have shown under the assumption of uniformly positive curvature for $\mathcal{B}$ that there exists a deterministic set

$$D \subset [0, 2\pi)$$

with countable complement such that, for each $\theta \in D$, the following holds a.s..

1. (Existence) There exists a collection of infinite geodesics $(\Gamma_x : x \in \mathbb{Z}^2)$ such that each Γ_x starts from x, has asymptotic direction θ, and each Γ_x and Γ_y coalesce.
2. (Uniqueness) For each x, Γ_x above is the only infinite geodesic starting from x with asymptotic direction θ.

The above theorem shows that one can take $D = [0, 2\pi)$ for existence, still under global conditions on the limit shape, but does not address uniqueness, which will be the focus of Section 5.6.

PROOF OUTLINE. This outline differs from what appears in [**68**] in that some arguments are simplified and extended to $\mathbb{Z}^d$. The idea is to work with subsequential Busemann limits in distribution. First we set $\Omega_1 = [0, \infty)^{\mathcal{E}^d}$ to be a copy of our edge-weight space with our i.i.d. joint edge-weight distribution $\mathbb{P}$ and let $\Omega_2 = \mathbb{R}^{\mathbb{Z}^d \times \mathbb{Z}^d}$ be our space for recording Busemann increments. Last, we have a space $\Omega_3 = \{0,1\}^{\vec{\mathcal{E}}^d}$, where $\vec{\mathcal{E}}^d$ is the set of oriented nearest-neighbor edges of $\mathbb{Z}^d$, which will record geodesic graphs. Put $\widetilde{\Omega} = \prod_{i=1}^3 \Omega_i$.

Let H be any supporting hyperplane for $\mathcal{B}$ at z_0 and let ρ be the vector with

$$H = \{w : w \cdot \rho = 1\}.$$

Define

$$H_\alpha = \{w : w \cdot \rho = \alpha\}$$

for $\alpha \in \mathbb{R}$. We now define Busemann increments and geodesic graphs toward H_α. For an outcome $\omega \in \Omega_1$ (written also as an edge-weight configuration (τ_e)), set

$$B_\alpha(\omega) = \big(B_\alpha(x,y) : x,y \in \mathbb{Z}^d\big) \in \Omega_2,$$

where

$$B_\alpha(x,y) = T(x, H_\alpha) - T(y, H_\alpha).$$

Furthermore define the geodesic graph configuration $\eta_\alpha(\omega)$ by

$$\eta_\alpha(\omega)((x,y)) = \begin{cases} 1 & \text{if } \{x,y\} \in \overline{\mathrm{GEO}}(z, H_\alpha) \text{ for some } z \text{ and } B_\alpha(x,y) \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Last, define $\Phi_\alpha : \Omega_1 \to \widetilde{\Omega}$ by

$$\Phi_\alpha(\omega) = (\omega, B_\alpha(\omega), \eta_\alpha(\omega))$$

and μ_α to be the push-forward of μ through Φ_α. The measure μ_α is a Borel probability measure on $\widetilde{\Omega}$.

A generic element of $\widetilde{\Omega}$ we write as

$$\widetilde{\omega} = ((\tau_e), B, \eta).$$

The coordinate B is called the reconstructed Busemann function. We would like to take α to infinity and sample a configuration $\widetilde{\omega}$ from the limit of (μ_α). Such a measure would be the distribution of Busemann increments and geodesic graphs "at infinity" in a direction related to H. However, we do not know how to show that this sequence of measures converges, so we settle for subsequential limits of averages of μ_α over α. The averaging is done to ensure that any limit is translation-invariant. That is, set

$$\mu_k^* = \frac{1}{k}\int_0^k \mu_\alpha \, \mathrm{d}\alpha$$

and let μ^* be any subsequential limit of (μ_k^*). (Tightness is not difficult to verify, due to the inequality $|B_\alpha(x,y)| \leq T(x,y)$.) Sampling from the measure μ_k^* can be thought of as representing first sampling a hyperplane H_α uniformly at random for $\alpha \in [0,k]$ and then sampling the configuration $\Phi_\alpha(\omega)$.

The idea is then to approximate the proofs of Section 5.3 to (a) find asymptotics of the function $B(0,x)$ as $|x| \to \infty$ and (b) use them to control geodesics. First, we give the mean of B, writing $\mathbb{E}_{\mu^*}$ for expectation relative to μ^*. Recall that ρ is the vector with $H_\alpha = \{w : w \cdot \rho = \alpha\}$.

Lemma 5.11. *Assume I. The mean of the reconstructed Busemann function is*

$$\mathbb{E}_{\mu^*} B(x,y) = \rho \cdot (y - x) \quad \textit{for } x,y \in \mathbb{Z}^d.$$

The idea of this lemma is that the Busemann function on average measures distances traveled perpendicular to H, but from a sort of stationary state, where the velocity is given by the time constant in direction z_0 (which is the same time constant corresponding to the hyperplane H). The reader should compare to Lemma 5.5 in the previous section, where the mean of the Busemann function is similarly given by a projection along the vertical hyperplane H used to create the function.

PROOF. The proof uses a version of the averaging trick (inspired by Gouéré [**100**]) from last section. For $k \geq 1$ we can use stationarity and the fact that $H_\alpha + x = H_{\alpha+\rho\cdot x}$ to write

$$\begin{aligned}\mathbb{E}_{\mu_k^*} B(-x,0) &= \frac{1}{k}\left[\int_0^k \mathbb{E}T(-x,H_\alpha)\ \mathrm{d}\alpha - \int_0^k \mathbb{E}T(0,H_\alpha)\ \mathrm{d}\alpha\right]\\ &= \frac{1}{k}\left[\int_0^k \mathbb{E}T(0,H_{\alpha+\rho\cdot x})\ \mathrm{d}\alpha - \int_0^k \mathbb{E}T(0,H_\alpha)\ \mathrm{d}\alpha\right]\\ &= \frac{1}{k}\left[\int_k^{k+\rho\cdot x} \mathbb{E}T(0,H_\alpha)\ \mathrm{d}\alpha - \int_0^{\rho\cdot x} \mathbb{E}T(0,H_\alpha)\ \mathrm{d}\alpha\right].\end{aligned}$$

Choosing k_j so that $\mu_{k_j}^* \to \mu^*$, one can show using translation invariance and the bound

$$\mathbb{E}_{\mu_{k_j}^*} B(-x,0)^2 \leq \mathbb{E}T(-x,0)^2$$

that the left side $\mathbb{E}_{\mu_k^*} B(-x,0)$ converges to $\mathbb{E}_{\mu^*} B(0,x)$. Noting that the last term vanishes in the limit when divided by k, we can change variables in the first term to obtain the formula

$$\mathbb{E}_{\mu^*} B(0,x) = \lim_{j\to\infty} \int_0^{\rho\cdot x} \frac{\mathbb{E}T(0,H_{\alpha+k_j})}{k_j}\ \mathrm{d}\alpha.$$

One can show using the shape theorem (see for example [**68**, Lemma 3.6]) that the integrand converges to 1, so by the dominated convergence theorem, we obtain a limit of $\rho \cdot x$. By translation invariance, then, of μ^*, one has

$$\mathbb{E}_{\mu^*} B(x,y) = \mathbb{E}_{\mu^*} B(0,y-x) = \rho\cdot(y-x).$$

□

As usual, we can prove asymptotics for the reconstructed Busemann function in the form of a shape theorem. However, due to the extra randomness introduced through α, we may have lost ergodicity for μ^* under translations, and therefore obtain a random "shape."

LEMMA 5.12 (Shape theorem for B). *Assume I. There exists a random vector $\varrho \in \mathbb{R}^2$ such that for any $\epsilon > 0$,*

$$\mu^*\left(|B(0,x) - x\cdot\varrho| > \epsilon\|x\|_1 \text{ for infinitely many } x \in \mathbb{Z}^d\right) = 0.$$

The vector ϱ satisfies the following conditions:

(1) *μ^*-a.s., the hyperplane*

$$H_\varrho := \{w \in \mathbb{R}^d : w\cdot\varrho = 1\}$$

is a supporting hyperplane for $\mathcal{B}$ at z_0.

(2) *The mean of ϱ under μ^* is ρ.*

The proof of the above result establishes first radial limits of the form $\lim_{n\to\infty} B(0,nx)/n$ and patches them together using shape theorem arguments. A key tool is that the vector ϱ is invariant under translating the edge-weights, Busemann increments, and geodesic graphs. The mean of ϱ follows directly from the Lemma 5.11, whereas the fact that H_ϱ is a supporting line for $\mathcal{B}$ follows from the statements (a) $\mu^*(x\cdot\varrho\le 1)=1$ for each $x\in\mathcal{B}$ and (b) $\mu^*(z_0\cdot\varrho=1)=1$. The claim (a) is shown by noting that for $x\in\mathcal{B}$, one has

$$x\cdot\varrho=\lim_{n\to\infty}\frac{B(0,nx)}{n}\le\lim_{n\to\infty}\frac{T(0,nx)}{n}=\mu(x)\le 1.$$

In particular,

$$\mu^*(\rho\cdot z_0\le 1)=1. \tag{5.7}$$

On the other hand, using the mean of ϱ, one has

$$\mathbb{E}_{\mu^*}(\varrho\cdot z_0)=\rho\cdot z_0=1.$$

Combining this with (5.7) gives $\varrho\cdot z_0=1$ with μ^*-probability one.

At this point, we may identify the asymptotics of the reconstructed Busemann function if $\partial\mathcal{B}$ is differentiable at z_0. In this case, H is the unique supporting hyperplane H_{z_0} for $\mathcal{B}$ at z_0, so this must be H_ϱ. In other words, one has

COROLLARY 5.13. *Assume I. If $\partial\mathcal{B}$ is differentiable at z_0, then*

$$\mu^*(\varrho=\rho)=1.$$

In the presence of differentiability, the asymptotics of the Busemann function are given exactly by projection onto the hyperplane H used to create the function. This result gives us a major piece needed to establish directional properties of geodesic rays sampled from μ^*. To complete the proofs of the directional results, we simply need to argue as in the proof of Theorem 5.7, combining asymptotics of B with the shape theorem for T.

To do this, we consider the configuration $\eta\in\Omega_3$ sampled from μ^* and build a directed graph from it. Specifically, we set $\mathbb{G}=\mathbb{G}(\eta)$ to be the directed graph with vertex set equal to $\mathbb{Z}^d$ and with edge set equal to $\{(x,y):\eta((x,y))=1\}$. Simple properties of geodesics carry over to the weak limit. For example: with μ^*-probability one, for all $x,y\in\mathbb{Z}^d$,

- Each directed path in $\mathbb{G}$ is a geodesic and from each x there is an infinite self-avoiding directed path.
- If $x\to y$ in $\mathbb{G}$ (there is a directed path from x to y in $\mathbb{G}$), then $B(x,y)=T(x,y)$.
- Under assumption II, viewed as an undirected graph, $\mathbb{G}$ has no circuits.
- Each vertex x has out-degree at least 1 in $\mathbb{G}$. Under assumption II, the out-degree is exactly 1, and thus emanating from each x is a unique infinite directed path Γ_x.

Given these properties, the proof of the next result is analogous to that of Theorem 5.7, so we omit it.

THEOREM 5.14 (Damron-Hanson [**68**], Theorem 5.3). *Assume I. With μ^*-probability 1, each directed infinite path starting from 0 in $\mathbb{G}$ is asymptotically directed in $S_\varrho=\partial\mathcal{B}\cap H_\varrho$.*

In the case that $\partial\mathcal{B}$ is differentiable at z_0, the hyperplane H_ϱ is μ^*-a.s. equal to H_{z_0} and therefore $S_\varrho=S_{z_0}$. Therefore Theorem 5.14 implies a version of Theorem 5.9 on the space $\widetilde{\Omega}$. To deduce the result on the original space, one applies a pull-back argument using the regular conditional measure of μ^* given the edge-weights.

For Theorem 5.10, item 1 is a restatement of Theorem 5.9 using translation invariance of μ^*. The proof of the coalescence statement follows the argument of Licea-Newman outlined in Theorem 4.28. The main difference is that one must be careful to construct barrier events on the original space Ω_1 and port them over to $\widetilde{\Omega}$, and this creates a considerable headache. We refer the reader to [**68**, Section 6]. As for absence of infinite backward paths, we have

THEOREM 5.15 (Damron-Hanson [**68**], Theorem 6.9). *Assume I and II in dimension $d = 2$. With μ^*-probability 1, the set $\{y : 0 \in \Gamma_y\}$ is finite.*

PROOF. The idea is a modification of that of Burton-Keane [**46**], from their proof of uniqueness of the infinite occupied cluster in Bernoulli percolation. For $x \in \mathbb{Z}^2$, define the event A_x that in $\mathbb{G}$, x has an infinite forward path and two disjoint infinite backward paths. On A_x, the vertex x is, in the language of [**46**], an "encounter point." A counting argument (see [**46**, p. 504]) shows that in the box $B(n) = [-n, n]^2$, one has deterministically

$$|\{x \in B(n) : A_x \text{ occurs}\}| \leq Cn,$$

as the number of points for which A_x occurs is bounded above by the number of points on the boundary of $B(n)$. On the other hand, if we show that $\mathbb{P}(A_x) > 0$, then translation invariance of μ^* gives

$$\mathbb{E}_{\mu^*}|\{x \in B(n) : A_x \text{ occurs}\}| = |B(n)|\mu^*(A_x) = (2n+1)^2\mu^*(A_x),$$

and this would be a contradiction for large n.

Assume then that with positive probability, the set $\{y : 0 \in \Gamma_y\}$ is infinite; we will show that this implies $\mu^*(A_x) > 0$. By directedness of paths in $\mathbb{G}$, we may find a line L intersecting $\mathbb{Z}^2$ (which we will for illustration take to be the e_2-axis) such that each Γ_y eventually stays on one fixed side of L, say $\{z : z \cdot e_1 > 0\}$. Choose y on the other side $\{z : z \cdot e_1 < 0\}$ of L and follow Γ_y on the event that $\{z : y \in \Gamma_z\}$ is infinite until its last intersection with L. Such a point has an infinite forward path in $\mathbb{G}$ that does not intersect L except at its initial point, and has an infinite backward path. By the ergodic theorem, we can then find $x_1, x_2 \in L$ such that with positive μ^*-probability, each of x_1 and x_2 has an infinite forward path that does not touch L except at the initial point and has an infinite backward path. Because all forward paths in $\mathbb{G}$ coalesce, we may choose x to be the coalescence point of Γ_{x_1} and Γ_{x_2} and so A_x occurs. Therefore there is an x such that $\mu^*(A_x) > 0$. □

Combining the above results with an argument to pull back to Ω_1 ends the proof of Theorem 5.10. □

REMARK 5.16. Here we explain the relation between the above method and Kingman's approach. The analogue of Kingman's variables defined in (2.20) and (2.21) would be, respectively (using the notation from (5.3))

$$f_k = \frac{1}{k}\sum_{j=1}^{k}(T(0, H_j) - T(e_1, H_j)) = \frac{1}{k}\sum_{j=1}^{k} B_j(0, e_1)$$

and the replacement for

$$f_k + Tf_k + \ldots + T^{n-1}f_k$$

would be $\frac{1}{k}\sum_{j=1}^{k}\left[B_j(0,e_1)+\cdots+B_{j+n-1}((n-1)e_1,ne_1)\right]$, but it is more natural for us to choose

$$\frac{1}{k}\sum_{j=1}^{k} B_j(0,ne_1).$$

Note that f_k is similar to the Busemann field coordinate B under the averaged measure μ_k^*, but the averaging is done one level lower (at the level of random variables rather than measures). To make this precise, suppose that f_k converges a.s. to some random variable f and that μ_k^* converges to a measure μ. Then one would have

$$f = \mathbb{E}_\mu\left[B(0,e_1) \mid (\tau_e)\right],$$

where we have conditioned on the edge-weight configuration (τ_e). In other words, f would be an average of the reconstructed Busemann function $B(0,e_1)$ over the additional randomness we introduced into μ (the uniformly random hyperplane H_α).

In fact, exactly this approach is taken in Liggett's proof of his improved subadditive ergodic theorem [**142**]. In that result, Kingman's assumptions are weakened, and the question of building a decomposition into an additive and nonnegative subadditive process in Theorem 2.30 is handled using weak limits, instead of employing weak-$*$ compactness to find a limit point for the sequence (f_k) in $(L^1)^{**}$, as Kingman did. Precisely, for a subadditive ergodic sequence $(X_{m,n})$, Liggett defined an independent uniform $\{1,\ldots,n\}$ random variable U_n (similar to the index of our uniformly random hyperplane H_α) and set

$$Y_i^{(n)} = X_{0,i+U_n} - X_{0,i+U_n-1}.$$

The distribution of $Y_i^{(n)}$ is similar to that of our $B(0,e_1)$ under μ_n^*, but is reflected. Then he set $(Y_1,Y_2,\ldots)$ to be any subsequential limit in distribution of the sequences $(Y_1^{(n)},Y_2^{(n)},\ldots)$ and used the distributional monotonicity

$$(Y_1, Y_1+Y_2, Y_1+Y_2+Y_3\ldots) \le_{st} (X_{0,1}, X_{0,2}, X_{0,3}\ldots)$$

in place of Kingman's $f \le X_{0,1}$.

One could ask whether the approach used in this section could work by only considering quantities at the level of the random variable f_k rather than at the level of μ_k^*. In other words, instead of averaging the distributions of the quantities

$$(T(x,H_\alpha)-T(y,H_\alpha))_{x,y}$$

over α, could we simply average the random variables as

$$\left(\frac{1}{n}\int_0^n (T(x,H_\alpha)-T(y,H_\alpha))\,\mathrm{d}\alpha\right)_{x,y},$$

which would be more similar to Kingman's original approach? This may not be possible in general translation-ergodic environments. Suppose that geodesics from 0 to H_n and e_1 to H_n have exactly two pairs of subsequential limiting geodesics $\Gamma_0^{(1)},\Gamma_0^{(2)}$ and $\Gamma_{e_1}^{(1)},\Gamma_{e_1}^{(2)}$. Then if $\Gamma_0^{(i)}$ and $\Gamma_{e_1}^{(i)}$ coalesce for each fixed i, one would expect the Busemann functions constructed from these limiting geodesics to be different. However it is still possible that the limit f of f_k exists, assuming the subsequences on which we took limits are regular enough. Then f is simply an average of these two Busemann functions corresponding to the different geodesics and for this reason, it will no longer necessarily have property (a) from Lemma 5.2.

That is, if only one of the two geodesics passes over the edge from 0 to e_1, then it need not be true that $f = \tau_{\{0,e_1\}}$. This property is essential for deriving directional properties of geodesics from their Busemann functions.

5.5. Busemann functions along boundaries in $\mathbb{Z}^2$

In one case, point-to-point Busemann limits similar to (5.4) can be shown to exist deterministically, and this implies that limits exist a.s. for certain sequences of finite geodesics. We explain in this section the results of Auffinger-Damron-Hanson [**20**], where these statements are proved when Busemann limits are taken to points on boundaries of subsets in $\mathbb{Z}^2$. The existence of a boundary will allow for a "paths-crossing" trick due to Alm-Wierman [**13**] (first introduced in [**11**] in a special case), and existence of Busemann limits follows from this.

For simplicity, the results below will be stated for the half plane with vertices

$$V_H = \{(x,y) \in \mathbb{Z}^2 : y \geq 0\}$$

and edges

$$E_H = \{\{x,y\} : x,y \in V_H, |x-y| = 1\},$$

although we will remark about extending them to general subsets. For $x, y \in V_H$, let $T_H(x,y)$ be the minimum passage time among all paths with all vertices in V_H from x to y.

THEOREM 5.17 (Auffinger-Damron-Hanson [**20**], Theorem 1.1). *Let $x_n = ne_1$ and let (τ_e) be any edge-weight configuration in $[0,\infty)^{E_H}$. For all $x, y \in V_H$, the Busemann limit to x_n exists:*

$$B_H(x,y) = \lim_{n\to\infty} \left[T_H(x,x_n) - T_H(y,x_n)\right]. \tag{5.8}$$

As usual, existence of Busemann limits gives us quite a bit of control on geodesics. Using Theorem 5.17, we can prove existence of limiting geodesic graphs. Formally, we represent geodesic graphs as elements of a directed graph space, as in Section 5.4. Let $\vec{E}_H$ be the set of directed edges of V_H

$$\vec{E}_H = \{(x,y) : x,y \in V_H, |x-y| = 1\}$$

and write η for an arbitrary element of $\{0,1\}^{\vec{E}_H}$. Build the graph $\mathbb{G} = \mathbb{G}(\eta)$ as before: the vertices are all the vertices of V_H and an edge $e \in \vec{E}_H$ is present in the graph if and only if $\eta(e) = 1$. For the sequence of vertices $(x_n) = (ne_1)$, we let $\eta_n(e) = 1$ if $e = (x,y)$ is in a geodesic from some point to x_n and

$$T_H(x,x_n) \geq T_H(y,x_n);$$

we then set $\mathbb{G}_n = \mathbb{G}(\eta_n)$. The graphs $\mathbb{G}_n$ converge to a graph $\mathbb{G} = \mathbb{G}(\eta)$ if for each $e \in \vec{E}_H$, one has $\eta_n(e) \to \eta(e)$.

THEOREM 5.18 (Auffinger-Damron-Hanson [**20**], Theorem 1.3). *Let $\mathbb{P}$ be a probability measure on $[0,\infty)^{E_H}$ such that*

$$\mathbb{P}(\exists \text{ geodesic between } x,y \text{ for all } x,y \in V_H) = 1. \tag{5.9}$$

Then with a.s., $(\mathbb{G}_n)$ converges to a graph $\mathbb{G}$. Each directed path in $\mathbb{G}$ is a geodesic.

REMARK 5.19. The above two theorems are valid in the following more general setting. Let V be any subset of $\mathbb{Z}^2$ that is connected and infinite and has infinite connected complement. (The reader can think of slit planes, sectors, etc.) There is a unique doubly infinite path in the dual lattice $\mathbb{Z}^2 + (1/2, 1/2)$ which separates V and V^c. Enumerate the vertices in V that are adjacent to this dual path as

$$\dots, x_{-1}, x_0, x_1, \dots.$$

If we denote passage times between vertices x, y in V as $T_V(x, y)$, the minimum passage time among all paths with all vertices in V connecting x and y, then the Busemann limits

$$B_V(x, y) = \lim_{n\to\infty} [T_V(x, x_n) - T_V(y, x_n)]$$

exist for all edge-weight configurations in $[0, \infty)^{E_V}$, and limiting geodesic graphs exist a.s. for any measure on $[0, \infty)^{E_V}$ that has the geodesic property (5.9). Here, E_V is the set of nearest-neighbor edges with both endpoints in V.

In the case of the half-plane, one can say more about the structure of the limiting graph $\mathbb{G}$. We will only state the theorem (with a restricted class of edge-weight distributions), and refer to [**20**] for the complete proof.

THEOREM 5.20 (Auffinger-Damron-Hanson [**20**], Theorem 1.5). *Let $\mathbb{P}$ be an i.i.d. product measure on $[0, \infty)^{E_H}$ with continuous marginals. The limiting geodesic graph $\mathbb{G}$ from Theorem* 5.18 *satisfies the following a.s..*

(1) *Each vertex in V_H has out-degree 1. Therefore from each $x \in V_H$ emanates a unique infinite directed path Γ_x.*
(2) *Viewed as an undirected graph, $\mathbb{G}$ has no circuits.*
(3) *(Coalescence) For all $x, y \in V_H$, Γ_x and Γ_y coalesce. That is, their edge symmetric difference is finite.*
(4) *(Finiteness of backward paths) For each $x \in V_H$, the backward cluster $\{y \in V_H : x \in \Gamma_y\}$ is finite.*

PROOF. We give the ideas of the proofs of Theorems 5.17 and 5.18. For Theorem 5.17, existence of Busemann limits as in (5.8) are proved as a consequence of the "paths-crossing" trick of Alm and Wierman [**13**]. We begin by showing the limit for $x = m_1 e_1, y = m_2 e_1$ with $m_1 < m_2 \in \mathbb{Z}$. For simplicity, let us assume that for each $w, z \in V_H$, there is a geodesic between w and z. If this is not the case, then the geodesics can be replaced by paths that have passage time within ϵ of the infimum.

Let $n_2 > n_1 > m_2$ and let σ_1 be a geodesic from x to $n_1 e_1$, with σ_2 a geodesic from y to $n_2 e_1$. Note that by planarity, the paths σ_1 and σ_2 must share a vertex z. Define the path $\hat\sigma_1$ by traversing σ_1 from x to z and then σ_2 from z to $n_2 e_1$. Define $\hat\sigma_2$ by traversing σ_2 from y to z and then σ_1 from z to $n_1 e_1$. Then (see Figure 13),

$$\begin{aligned} T_H(x, n_1 e_1) + T_H(y, n_2 e_1) = T_H(\sigma_1) + T_H(\sigma_2) &= T_H(\hat\sigma_1) + T_H(\hat\sigma_2) \\ &\geq T_H(x, n_2 e_1) + T_H(y, n_1 e_1). \end{aligned}$$

Rearranging this inequality, we obtain

$$T_H(x, n_1 e_1) - T_H(y, n_1 e_1) \geq T_H(x, n_2 e_1) - T_H(y, n_2 e_1).$$

Therefore the sequence in (5.8) is monotone (and bounded by subadditivity) and the limit exists.

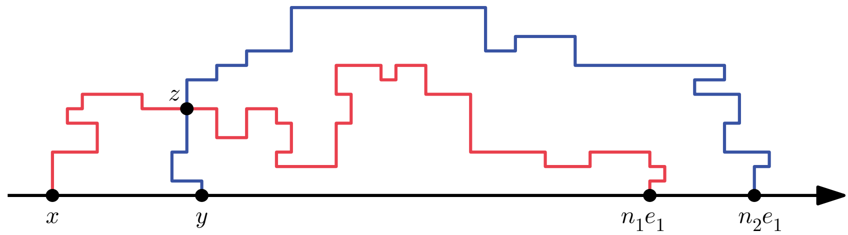

FIGURE 13. "Paths-crossing" trick: the path σ_1 from x to n_1e_1 and the path σ_2 from y to n_2e_1 must intersect at a point z.

To generalize this argument to points off the axis, we take $x \in V_H$ to be of the form (x_1, x_2) with $x_2 > 0$ and $x' \in V_H$. Let B be the box $[-k, k] \times [0, k]$ for some k large enough so that B contains both x and x'. The set $V' := V_H \setminus B$ is connected and infinite with complement that is also connected and infinite. So we define for $y, z \in V'$ the restricted passage time $T'(y, z)$ to be the minimum passage time of all paths from y to z which have only vertices in V'. One may then repeat the above "paths-crossing" trick to see that if y, z are in $\partial B \cap V'$ (they are each in V' but have a $\mathbb{Z}^2$-neighbor which is in B), then the limit

$$B'(y, z) = \lim_{n\to\infty} [T'(y, x_n) - T'(z, x_n)]$$

exists. Furthermore, for all such y, z, the sequence defining $B'(y, z)$ is monotone.

Now the idea is to use existence of Busemann limits $B'(y, z)$ for $y, z \in \partial B \cap V'$ to prove existence for x and x'. The crucial point is that for large n,

$$T_H(x, x_n) = \min\{T_H(x, y) + T'(y, x_n) : y \in \partial B \cap V'\}.$$

For our point x, another fixed point $z \in \partial B \cap V'$, and $y \in \partial B \cap V'$ variable, we set

$$\psi_n(y) = T_H(x, y) + T'(y, x_n) - T'(z, x_n).$$

We find then that $\psi_n(y)$ has a finite limit $\psi(y) = T_H(x, y) + B'(y, z)$ for each y, and furthermore

$$\lim_{n\to\infty} [T_H(x, x_n) - T'(z, x_n)] = \min\{\psi(y) : y \in \partial B \cap V'\}.$$

Reversing the roles of x and x', we find that $\lim_{n\to\infty} [T_H(x, x_n) - T_H(x', x_n)]$ exists.

To prove existence of limiting geodesic graphs, we take $e = (x, y) \in \vec{E}_H$ and again let B be a box of the form $[-k, k] \times [0, k]$ which is large enough to contain x and y. If N is large enough so that $x_n \notin B$ for all $n \geq N$, then if we again put $V' = V_H \setminus B$ one has for all $y \in \partial B \cap V'$,

$$y \text{ is on a geodesic from } x \text{ to } x_n \Leftrightarrow \psi_n(y) = \min\{\psi_n(u) : u \in \partial B \cap V'\},$$

where $\psi_n(y)$ was defined as above as $T_H(x, y) + T'(y, x_n) - T'(z, x_n)$ for the fixed $z \in \partial B \cap V'$. However, one can show using monotonicity of the differences $T'(y, x_n) - T'(z, x_n)$ in n for $y \in \partial B \cap V'$ that the set of minimizers $\mathfrak{m}_n$ of ψ_n is eventually constant for n large. Thus, the set of points in $\partial B \cap V'$ which are in geodesics from x to x_n is eventually constant in n.

The above work readily implies that $(\mathbb{G}_n)$ converges. Indeed, one can check that for our $e = (x, y)$, one has $\eta_n(e) = 1$ if and only if e is in a geodesic from x to a vertex of $\mathfrak{m}_n$. Since $\mathfrak{m}_n$ is eventually constant, the value of $\eta_n(e)$ must also be eventually constant. Since e was arbitrary, this completes the proof. □

5.6. Nonexistence of bigeodesics in fixed directions

Recently, Busemann functions have been used in general last-passage percolation (LPP) models by Georgiou, Rassoul-Agha, and Seppäläinen to prove analogues (and improved versions) of the directional results from [**68**] in FPP. This is a big advance because most work on LPP has assumed exponential or geometric weights, where random matrix and queueing theory analysis can be used.

A main tool in the general two-dimensional LPP models is the directedness of paths, which allows one to use various forms of the "paths-crossing" argument of Alm and Wierman. Applied to Busemann functions, this gives a certain monotonicity of directional Busemann functions. In FPP, paths are not directed, and this creates a fundamental obstacle to obtaining similar improvements.

In this section, we describe recent work of Damron-Hanson [**69**] which shows that the LPP results are also valid in FPP. These theorems address the "uniqueness" issue as described below Theorem 5.10, and consequently rule out bigeodesics in fixed directions. In that paper, the following theorem is proved for general edge-weight distributions considered in the first work of Damron-Hanson [**68**], but we will focus again on i.i.d. weights, assuming I and II from Section 5.4.

We will again make some assumptions on the limiting shape: let $\theta \in [0, 2\pi)$ and let v_θ be the point on $\partial\mathcal{B}$ in direction θ. When v_θ is a point of differentiability of $\partial\mathcal{B}$, let L_θ be the unique tangent line and let S_θ be the sector of angles of contact of L_θ with $\partial\mathcal{B}$. Let θ_1 and θ_2 be the endpoints of S_θ. We say that a sequence (x_n) of distinct points of $\mathbb{Z}^2$ is directed in S_θ if all limit points of the sequence of arguments $(\arg x_n)$ are contained in S_θ. As before, we say an infinite geodesic with vertices (x_n) is directed in S_θ if the sequence (x_n) is.

THEOREM 5.21 (Damron-Hanson [**69**], Theorem 1). *Assume I, II, and that $\partial\mathcal{B}$ is differentiable at θ, θ_1, and θ_2. The following hold a.s..*

(1) *(Uniqueness) For each $x \in \mathbb{Z}^2$, there is exactly one infinite geodesic Γ_x that is directed in S_θ such that for any (random) sequence (x_n) of distinct points directed in S_θ,*
$$\Gamma_x = \lim_{n\to\infty} \mathrm{Geo}(x, x_n).$$
(2) *(Coalescence) For each $x, y \in \mathbb{Z}^2$, the geodesics Γ_x and Γ_y coalesce.*
(3) *(Finiteness of backward paths) There are no bigeodesics with one end directed in S_θ.*

Notice that this result shows Licea-Newman's set D (see Theorem 4.26) of directions to be equal to $[0, 2\pi)$. Furthermore, their result rules out bigeodesics with both ends directed in D, whereas the above theorem only requires directedness of one end (in a sector). The reader should also consult the recent preprint of Ahlberg and Hoffman [**5**] for more work on these questions.

From this result we obtain the following progress on the bigeodesic conjecture.

COROLLARY 5.22. *Assume I, II and that $\partial\mathcal{B}$ is differentiable. Then for each θ,*
$$\mathbb{P}(\textit{there is a bigeodesic with one end in direction } \theta) = 0.$$

REMARK 5.23. Since geodesic lines with fixed directions cannot exist, one should ask if infinite geodesics are even required to have directions. One can show using planarity and the results of [**68**] that if $\partial\mathcal{B}$ is differentiable and I and II hold, then the following statements are true a.s.:

(1) for all θ, there is an infinite geodesic starting from 0 directed in S_θ and
(2) every infinite geodesic is directed in S_θ for some θ.

As a consequence of Theorem 5.21 and arguments like those presented elsewhere in this section (for instance in Section 5.3), one has the following result.

THEOREM 5.24 (Damron-Hanson [**69**], Theorem 2). *Assume I, II, and that $\partial\mathcal{B}$ is differentiable at θ, θ_1, and θ_2. A.s., for each $x, y \in \mathbb{Z}^2$ and (random) sequence (x_n) of distinct points directed in S_θ, the limit*

$$B(x,y) = \lim_n [T(x,x_n) - T(y,x_n)]$$

exists. Furthermore, letting ρ be the unique vector in $\mathbb{R}^2$ such that $\{r \in \mathbb{R}^2 : r\cdot\rho = 1\}$ is the tangent line to $\mathcal{B}$ in direction θ, one has:

(1) $\mathbb{E}B(0,x) = \rho \cdot x$ *for all* $x \in \mathbb{Z}^2$.
(2) *For each $\epsilon > 0$, the set of $x \in \mathbb{Z}^2$ such that $|B(0,x) - \rho \cdot x| > \epsilon \|x\|_1$ is finite.*

REMARK 5.25. The conclusions of Theorems 5.21 and 5.24 are also valid under a weaker condition on $\partial\mathcal{B}$. Namely one must only assume that θ_i is a limit of extreme points of $\mathcal{B}$ for each $i = 1, 2$.

In these theorems, one may ask the degree to which the differentiability assumption is necessary. The main obstacle to removing differentiability is the result of Häggstrom-Meester (see Remark 2.25 and Theorem 7.6), which we recall states that given any compact, convex subset $\mathcal{C}$ of $\mathbb{R}^d$ that is symmetric ($x \in \mathcal{C} \Rightarrow -x \in \mathcal{C}$), there exists a translation-ergodic distribution of edge-weights whose limit shape is $\mathcal{C}$. In particular, there are models of FPP whose limit shapes are polygons. In these situations, it is reasonable to believe that one can construct models in which the only infinite geodesics are ones which do not have asymptotic directions — they wander across the sectors corresponding to the sides of $\mathcal{B}$. This leads one to consider directedness in sectors. However, it is reasonable to expect existence of some translation-ergodic edge-weight distributions which have polygonal limit shapes, and for which there are infinite geodesics directed toward the corners. In this case, uniqueness of infinite geodesics in sectors corresponding to sides abutting such corners cannot hold. These situations are prevented by making a differentiability assumption on $\partial\mathcal{B}$.

REMARK 5.26. The above results give progress toward solving the "BKS midpoint problem" introduced in [**26**], and which was listed earlier as Question 3.2.1. The question is: is it true that the probability that $(n/2)e_1$ is in a geodesic from 0 to ne_1 goes to zero at $n \to \infty$? By a translation, if the answer is no, then

$$\mathbb{P}\left(0 \in \overline{\text{GEO}}(-ne_1, ne_1) \text{ for infinitely many } n\right) > 0.$$

On this event, 0 is in a bigeodesic, and under the differentiability assumptions of Theorem 5.21, one of its ends is directed in the sector corresponding to $\theta = 0$. This is impossible by Theorem 5.21. Therefore under the differentiability assumption of Theorem 5.21 for $\theta = 0$, the answer to the midpoint problem is yes. For more recent work on this problem, the reader is directed to Ahlberg and Hoffman [**5**].

SKETCH OF PROOF OF THEOREM 5.21. The proof we give is more similar to the proof in last-passage percolation [**97**] because, for ease of exposition, we will omit the many applications of the Jordan curve theorem needed to deal with the fact that paths in FPP are not necessarily oriented.

For simplicity, we will take $\theta = \pi/2$ and

$$\pi > \theta_1 \geq \theta \geq \theta_2 > 0.$$

We will first reduce to problem to a the corresponding one on the upper half-plane, where it is easy to put an ordering on infinite geodesics. We will skip this step, although it is the most work, because it does not involve Busemann functions, but just topological arguments. However we mention that this reduction is done using the fact that every path that is directed in S_θ must have at most finitely many intersections with

$$L_0 = \{x : x \cdot e_2 = 0\}.$$

So consider the upper half-plane with vertices V_H and nearest-neighbor edges E_H, and define passage times $T_H(x, y)$ for vertices $x, y \in V_H$ by considering only paths in the upper half-plane. We will content ourselves with proving the theorem for geodesics constructed in the upper half-plane using T_H.

Let $\mathcal{G}_H(x)$ be the collection of infinite geodesics starting from x in the upper half-plane which are directed in S_θ. Because all these geodesics are in the upper half-plane, for each $x \in L_0$, it is possible to define a leftmost (counterclockwisemost) infinite geodesic in $\mathcal{G}_H(x)$, written as $\Gamma^L_{x,H}$. Similarly there is a rightmost infinite geodesic written as $\Gamma^R_{x,H}$. Using our differentiability assumptions, one can show (see [**69**, Proposition 4.7]) that $\Gamma^L_{x,H}$ and $\Gamma^R_{x,H}$ themselves are directed in S_θ and the first lies counterclockwise to the second. The main step is to argue that that

$$\Gamma^L_{0,H} = \Gamma^R_{0,H} \quad \text{a.s..} \tag{5.10}$$

Once we show (5.10), then part 1 of Theorem 5.21 follows from the following lemma, whose proof uses the assumption that $\partial\mathcal{B}$ is differentiable at the endpoints θ_1, θ_2.

LEMMA 5.27. *A.s., the following holds. If (x_n) is any (random) sequence of points directed in S_θ, and Γ is a subsequential limit of upper half-plane geodesics from 0 to x_n, then Γ is directed in S_θ.*

The idea of the proof of this lemma is to use trapping. Since $\partial\mathcal{B}$ is differentiable at θ_1, one can find infinitely many extreme points of $\mathcal{B}$ which converge to v_{θ_1} from outside of S_θ. One then derives an upper half-plane version of the existence of infinite geodesics directed in sectors from [**68**] (see Theorem 5.9), and uses this to construct infinitely many geodesics which are directed in distinct sectors whose endpoints converge to v_{θ_1} from outside S_θ. The subsequential limit Γ cannot intersect any of these geodesics more than once, due to uniqueness of passage times, and so it must be directed "on or to the right" of θ_1. A similar argument works for θ_2 and traps Γ in S_θ.

If we assume (5.10) and use Lemma 5.27, we find that any subsequential limit Γ of half-plane geodesics from 0 to a (random) sequence (x_n) directed in S_θ must itself be directed in S_θ, and must then lie between $\Gamma^L_{0,H}$ and $\Gamma^R_{0,H}$. This would mean all three geodesics $\Gamma, \Gamma^L_{0,H}$, and $\Gamma^R_{0,H}$ are equal and would prove item 1 of Theorem 5.21.

Now we argue for (5.10). Similarly to the definition of leftmost and rightmost geodesics in the upper half-plane, one can with considerably more effort (see [**69**, Section 4]) define leftmost and rightmost infinite geodesics from $x \in \mathbb{Z}^2$ in the full-plane; we write these as Γ_x^L and Γ_x^R. One can show (see [**69**, Proposition 4.10]) that for $x \in L_0$, with positive probability,

$$\Gamma_{x,H}^L = \Gamma_x^L,$$

and similarly that with positive probability,

$$\Gamma_{x,H}^R = \Gamma_x^R.$$

This is argued using the fact that any path that is directed in S and starts on L_0 has a last intersection with L_0. One can then use a Licea-Newman style argument (as in Theorem 4.28) to show that each Γ_x^L and Γ_y^L coalesce for $x, y \in \mathbb{Z}^2$ (and similarly for rightmost geodesics) and then deduce this same statement for $\Gamma_{x,H}^L$ and $\Gamma_{y,H}^L$ for $x, y \in V_H$ (and similarly for rightmost upper half-plane geodesics). This allows one to define Busemann functions for $x, y \in V_H$ and $* = L$ or R:

$$B_H^*(x,y) = \lim_{n\to\infty} \left[T_H(x, x_n) - T_H(y, x_n)\right],$$

where (x_n) is the sequence of vertices on $\Gamma_{0,H}^*$. By coalescence, the definition would be the same if we replace $\Gamma_{0,H}^*$ by $\Gamma_{z,H}^*$ for any $z \in V_H$, and this fact implies a horizontal translation covariance similar to that in (5.5) for these Busemann functions. Indeed, if θ is a horizontal integer translation, then

$$B_H^*(\theta(x), \theta(y))(\theta(\omega)) = B_H^*(x,y)(\omega).$$

A similar definition is given for full-plane geodesics and in this case, one may define the Busemann functions on all of $\mathbb{Z}^2$, not just V_H, and they are translation covariant relative to all integer translations.

By the ergodic theorem (like in (5.1)), if we set

$$\Delta_H(x,y) = B_H^L(x,y) - B_H^R(x,y)$$

(and similarly for $\Delta(x,y)$ as the difference of full-plane Busemann functions), then there is a c such that

$$\frac{1}{n}\Delta_H(0, ne_1) \to c \text{ and } \frac{1}{n}\Delta_H(0, -ne_1) \to -c \quad \text{a.s..}$$

Furthermore, one can use equality of full-plane and half-plane geodesics (with positive probability) to show that these limits exist for Δ (the full-plane difference) and are equal to c and $-c$, respectively:

$$\frac{1}{n}\Delta(0, ne_1) \to c \text{ and } \frac{1}{n}\Delta(0, -ne_1) \to -c \quad \text{a.s..}$$

Now (5.10) follows from the next two lemmas.

LEMMA 5.28. *Under our differentiability assumption on $\partial\mathcal{B}$, $c = 0$.*

PROOF. For $* = L, R$ the full-plane function $x \mapsto \mathbb{E}B^*(0,x)$ is linear, and so as usual there is a vector ρ^* such that

$$\mathbb{E}B^*(x,y) = \rho^* \cdot (y - x) \quad \text{for } x, y \in \mathbb{Z}^2.$$

Once again (as in Lemma 5.6), one may upgrade this to a sort of shape theorem: for each $\epsilon > 0$,

$$\mathbb{P}\left(|B^*(0,x) - \rho^* \cdot x| > \epsilon \|x\|_1 \text{ for infinitely many } x \in \mathbb{Z}^2\right) = 0.$$

To prove the lemma, we will show that for for $* = L, R$, the line

$$L^* = \{r \in \mathbb{R}^2 : r \cdot \rho^* = 1\}$$

is a supporting line for the limit shape $\mathcal{B}$ at some direction (and therefore any direction) of the sector S_θ. By our differentiability assumptions, there is only one such line, and so we can conclude

$$\rho^L = \rho^R.$$

In other words, this will show that

$$\mathbb{E}B^*(0, e_1) = \rho^L \cdot e_1$$

for both $* = L, R$, and the ergodic theorem will complete the proof.

The proof of the supporting property of L^* is similar to that of Theorem 5.7. Label the last intersections of Γ_0^* with

$$L_0, L_1, L_2, \dots$$

as

$$x_0^*, x_1^*, x_2^*, \dots.$$

(Here, $L_k = L_0 + ke_2$.) Then choose a subsequence $(x_{n_k}^*)$ such that $x_{n_k}^*/\mu(x_{n_k})$ converges to some vector $z_0 \in \partial\mathcal{B}$ (whose angle necessarily lies in S_θ). By the standard shape theorem,

$$B^*(0, x_n^*)/\mu(x_n) = T(0, x_n^*)/\mu(x_n) \to 1 \quad \text{as } n \to \infty.$$

On the other hand, this converges on the subsequence (n_k) to $\rho^* \cdot z_0$, so

$$\rho^* \cdot z_0 = 1.$$

In other words, z_0 is both a point of L^* and of $\partial\mathcal{B}$. Furthermore, as usual,

$$\rho^* \cdot z \leq 1$$

for all $z \in \mathcal{B}$, using the fact that Busemann functions are bounded by the passage time:

$$\rho^* \cdot z = \lim_{n\to\infty} \frac{B^*(0, nz)}{n} \leq \lim_{n\to\infty} \frac{T(0, nz)}{n} = \mu(z) \leq 1.$$

This means that L^* is a supporting line for $\mathcal{B}$ at z_0, which is a point with angle in S_θ. This completes the proof. □

Lemma 5.29. *Almost surely,* $\Delta_H(0, e_1) \leq 0$. *If* $\Gamma_{0,H}^L \neq \Gamma_{0,H}^R$ *with positive probability, then* $c < 0$.

Proof. If $\Gamma_{0,H}^L \neq \Gamma_{0,H}^R$, then by coalescence, the same must be true for geodesics starting from any point on L_0. Since the edge-weights have a continuous distribution, one then deduces that

$$\Delta(0, e_1) \neq 0.$$

So if we can show that

$$\Delta(0, e_1) \leq 0$$

a.s., one then has

$$\mathbb{E}\Delta(0, e_1) < 0,$$

and the ergodic theorem will prove the lemma.

To show the a.s. inequality, we use the "paths-crossing" trick. We claim that $\Gamma^R_{0,H}$ and $\Gamma^L_{e_1,H}$ must share a vertex z. Indeed, if

$$\Gamma^L_{0,H} = \Gamma^R_{0,H},$$

then this holds by coalescence. Otherwise, if follows from planarity and the fact that $\Gamma^R_{0,H}$ coalesces with $\Gamma^R_{e_1,H}$, which is "to the right" of $\Gamma^L_{e_1,H}$. Let y be a vertex on $\Gamma^L_{e_1,H}$ beyond z and let y' be a vertex on $\Gamma^R_{0,H}$ beyond z. Then

$$\begin{aligned} T_H(0,y') + T_H(e_1,y) &= T_H(0,z) + T_H(z,y') + T_H(e_1,z) + T_H(z,y) \\ &\geq T_H(0,y) + T_H(e_1,y'). \end{aligned}$$

Rearranging and taking $y, y' \to \infty$ along their respective geodesics, we obtain $B^L_H(0,e_1) \leq B^R_H(0,e_1)$. □

Given (5.10) from the previous two lemmas, we can return to the proof of Theorem 5.21. We already argued for item 1 below (5.10). Coalescence of leftmost and rightmost geodesics implies item 2 of the theorem. For the last item, we use the argument of Theorem 5.15. This argument applies to general cases in which from each point x, there is an infinite geodesic Γ_x such that (a) for $x, y \in \mathbb{Z}^2$, Γ_x and Γ_y coalesce, and (b) there is a line L such that a.s., each Γ_x intersects this line finitely often. In such cases, the conclusion is that the tree formed by the union of the geodesics $(\Gamma_x : x \in \mathbb{Z}^2)$ cannot have infinite backward paths. In other words, a.s., the set $\{y : 0 \in \Gamma_y\}$ is finite. If there were a bigeodesic with one end directed in S_θ, then it would have to be contained in this tree of geodesics. However this tree cannot have an infinite backward path, and this would be a contradiction, so we conclude that a.s. there are no bigeodesics with one end directed in S_θ.

□