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Introduction to Arithmetic Groups

Armand Borel

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Translation edited by Dave Witte Morris

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Preface to the English Translation

Fifty years after it made the transition from mimeographed lecture notes to a published book, the late Armand Borel's *Introduction aux groupes arithmétiques* is still very important for the theory of arithmetic groups. Chapter III remains the standard reference for fundamental results on reduction theory.

Before presenting a few suggestions for further reading, we first note that reduction theory is crucial for our understanding of the structure of the spaces $\mathrm{G}_{\mathbb{R}}/\Gamma$ and $\mathrm{K}\backslash\mathrm{G}_{\mathbb{R}}/\Gamma$. An example of this is given in the book's final section (§17), which describes a compactification of $\mathrm{K}\backslash\mathrm{G}_{\mathbb{R}}/\Gamma$ in the special case where the $\mathbb{Q}$-rank of G is equal to 1. This compactification is smooth, but, in higher rank, the boundary of a compactification usually has corners, rather than being a smooth manifold. For a discussion of later work that uses the "Siegel sets" of this book to construct several different compactifications of $\mathrm{K}\backslash\mathrm{G}_{\mathbb{R}}/\Gamma$, see:

[BJ] A. Borel and L. Ji, *Compactifications of symmetric and locally symmetric spaces*, Birkhäuser, Boston, MA, 2006. MR2189882

For the general theory of arithmetic groups, three books (other than *Introduction aux groupes arithmétiques*) are often listed as essential reading:

[Ma] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer, Berlin, 1991. MR1090825

[PR] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Academic Press, Boston, 1994. MR1278263

[Ra] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer, New York, 1972. MR0507234

Overviews that are more recent (but without detailed proofs) include:

[Ji] L. Ji, *Arithmetic groups and their generalizations*, American Mathematical Society, Providence, RI, 2008. MR2410298

[Mo] D. W. Morris, *Introduction to arithmetic groups*, Deductive Press, 2015. MR3307755

In this translation, the numbering of theorems, equations, and other material in the main text of the original French manuscript has been reproduced exactly (although page numbers may have changed). However, the same cannot be said of the bibliography, partly because it includes the references that Borel inserted in the main text, rather than in his bibliography. (We also note that items in the bibliography have been updated to their latest version.)

The *Math Review* of the original French manuscript observes that "the style is concise and the proofs (in later sections) are often demanding of the reader." To make the translation more approachable, numerous footnotes provide comments that are intended to be helpful. (All of the footnotes are new; the original book had no footnotes at all.) Unless marked *Translator's note*, they were added by the editor.

Typographical errors and other minor issues have been silently corrected, but significant deviations from the original French manuscript (including changes of notation) are described in footnotes. I apologize for any errors that remain (and, even more, for any new errors that I introduced!).

I would like to thank Lam Pham and the staff of the AMS Book Program for making this classic monograph accessible to a wider audience.

Dave Morris
Lethbridge, July 2019

Introduction

This book is based on the first part of a graduate course that was given at the Institut Henri Poincaré in 1964. It focuses on the so-called "reduction theory" in a real algebraic group $G_{\mathbb{R}}$, with respect to an arithmetic group Γ. The proofs of the general theorems make extensive use of the theory of linear algebraic groups. However, since the students in the course were not necessarily assumed to be acquainted with this, we first gave direct proofs of some classical cases (after all, these are where the general theory originates from), and we summarized the necessary notions and results on linear algebraic groups, as we needed them. Most of these summaries are in three sections (§7, §10, §11), which also contain some examples and proofs and thus provide, to a certain extent, an introduction to some aspects of the theory of algebraic groups.

By *reduction* in $G_{\mathbb{R}}$, with respect to Γ, we mean, roughly speaking, the search for (open or closed) subsets that meet each orbit of Γ (acting by translations on the right) in at least one point, but never in more than a finite number of points. Such sets are called fundamental sets. (Actually, we will impose precise conditions that are more restrictive (cf. 5.6, 9.6, 15.13).) Alternatively, but equivalently, the problem can be considered in the space $X = K\backslash G_{\mathbb{R}}$ of right cosets of $G_{\mathbb{R}}$ modulo a maximal compact subgroup K. When G is a classical group, we encounter, as a special case, the reduction theory of quadratic forms (and of hermitian forms).

The book has three parts. The first (§1 to §6) is mainly devoted to the reduction of quadratic forms, using methods that will be generalized in a natural way in later sections. We first consider the case where $G = \mathbf{GL}(n, \mathbb{R})$, $\Gamma = \mathbf{GL}(n, \mathbb{Z})$, and $K = \mathbf{O}(n)$, that is, where X is the space of positive-definite quadratic forms on $\mathbb{R}^n$. We show that every orbit of Γ in G meets a suitable Siegel set (1.4) and we deduce a few consequences, including Mahler's criterion for the relative compactness of a subset of the space $\mathbf{GL}(n, \mathbb{R})/\,\mathbf{GL}(n, \mathbb{Z})$ of lattices of $\mathbb{R}^n$, and the finiteness of the volume of $\mathbf{SL}(n, \mathbb{R})/\,\mathbf{SL}(n, \mathbb{Z})$. §2 translates these results into the language of quadratic forms, and establishes some links with Minkowski reduction. §4 shows that a Siegel set only meets a finite number of its translates by elements of $x\,.\,\Gamma$, for each $x \in \mathbf{GL}(n, \mathbb{Q})$, (4.6). §5 is devoted to the reduction of indefinite quadratic forms, using Hermite's method. Its key point is a finiteness property of "reduced integral" forms, which we will deduce from a more general lemma that is proved in § 6.

The second part (§7, §8, §9) is devoted to two general theorems on arithmetic groups, whose proofs require only a fairly restricted collection of results on algebraic groups (which will be recalled or proved in §7). The two main theorems are: a compactness criterion for the quotient $G_{\mathbb{R}}/\Gamma$ (§8), and a first construction of fundamental sets, which generalizes the method of Hermite. Furthermore, §8 shows that the image of an arithmetic group by an isogeny is also an arithmetic group, and §9 establishes a finiteness theorem for the number of orbits of Γ in the set of integer points of a closed orbit of G, in the space of a linear representation of G. This generalizes the finiteness of the number of classes of quadratic forms of given non-zero determinant (6.4), and also generalizes results of Jordan on the classes of homogeneous forms of degree ≥ 3, (6.5).

The third part (§10 to §17) is devoted to fundamental sets that are usually better than those of §9. Their existence is proved in two very different ways: in §13, where we rely on §9, and in §16, where we apply an extremum principle to a type of function that is studied in §14, and which generalizes, among others, the function $|cz + d|$ of the Poincaré upper half-plane. These sets are the union of a finite number of translates (by elements of $G_{\mathbb{Q}}$) of a set of a simple form, which is called a Siegel set. Finally, §17 describes, in a particular case, the space $K\backslash G_{\mathbb{R}}/\Gamma$ as the interior of a compact manifold with boundary.

The intention to make the first part self-contained, and to appeal to the theory of algebraic groups only when necessary, has led to some redundancies and inconsistencies. For example, §4 is a particular case of §15, but is not assumed in the latter, and the existence of a Bruhat decomposition is proved in §3 for $\mathbf{GL}(n, k)$, while it is assumed without proof in a much more general setting from §11 onward. Consequently, this exposition, which, *grosso modo*, follows the chronological order, and contains some quite extended "reviews," is not the most efficient possible, and reading a particular section does not necessarily require reading all of the preceding ones. Here are a few additional remarks on the interdependence of the various sections, which may serve as a "Leitfaden": §1, up to 1.11, is fundamental for the entire book, but the remainder of that section, and §2 to §5, are not used later, other than for providing concrete examples of the theory; readers who wish to reach the general theorems as quickly as possible may focus their attention on §§1, 8, 12, 14, 15, 16, if they are willing to assume a certain finiteness property whose proof here relies on §13, but which can be established more directly by using the adelic analogue of §§1 and 8 (cf. introduction to §16); finally, §6 plays a crucial role in §5 and §9, and this latter is used in §13, but nowhere else.

A first draft of the course notes (duplicated and distributed by the Institut Henri Poincaré) was written by H. Jacquet, J.-J. Sansuc and J.-P. Jouanolou. It was very useful to me, and I warmly thank the authors. I would also like to thank J. E. Humphreys, who read the manuscript, pointed out a considerable number of "misprints" and suggested some improvements in the exposition, and also A. Robert and J. Joel, for having helped me to correct the proofs.

Armand Borel
Princeton, November 1968

Notation

0.1. $\mathbb{Z}$ is the ring of integers[1] $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the fields of rational, real, and complex numbers respectively, and $\mathbb{N}$ is the set of integers that are ≥ 0. If A is a ring with unit, A^* denotes the multiplicative group of invertible elements of A. If A is a commutative ring, $\mathbf{M}(n, \mathrm{A})$ is the ring of square matrices of order n, with entries in A, $\mathbf{GL}(n, \mathrm{A})$ or $\mathbf{GL}_n(\mathrm{A})$ is the group of square matrices of order n with entries in A, whose determinant is a unit of A, and $\mathbf{SL}_n(\mathrm{A})$ or $\mathbf{SL}(n, \mathrm{A})$ is the subgroup consisting of the elements of $\mathbf{GL}_n(\mathrm{A})$ whose determinant is 1.

$\mathbf{O}(n)$ is the subgroup of $\mathbf{GL}(n, \mathbb{R})$ that leaves invariant the quadratic form $\sum_i x_i^2$ and $\mathbf{SO}(n) = \mathbf{O}(n) \cap \mathbf{SL}(n, \mathbb{R})$. If p and q are integers that are ≥ 0 and $n = p + q$, then $\mathbf{O}(p, q)$ is the subgroup of $\mathbf{GL}(n, \mathbb{R})$ that leaves invariant the quadratic form

$$(x_1^2 + \ldots + x_p^2) - (x_{p+1}^2 + \ldots + x_{p+q}^2),$$

and $\mathbf{SO}(p, q) = \mathbf{O}(p, q) \cap \mathbf{SL}(n, \mathbb{R})$.

0.2. Let G be a group and α be a homomorphism of G into $\mathbb{C}^*$. The value of α at $g \in \mathrm{G}$ will be denoted by $\alpha(g)$ or, alternatively, g^α. The latter notation implies that the pointwise product of homomorphisms of G into $\mathbb{C}^*$ will be written in additive notation[2]

0.3. Let G be a group. If $g \in \mathrm{G}$, we denote by $\mathrm{int}(g)$ the inner automorphism $x \mapsto g \,.\, x \,.\, g^{-1}$ of G. If A and H are subsets of G, then ${}^{\mathrm{A}}\mathrm{H}$ denotes the union of the sets $a \,.\, \mathrm{H} \,.\, a^{-1}$ $(a \in \mathrm{A})$.

Assume that V_i $(1 \leq i \leq n)$ are sets and that $f_i : \mathrm{V}_i \to \mathrm{G}$ are functions. The function $f : \mathrm{V}_1 \times \ldots \times \mathrm{V}_n \to \mathrm{G}$ that is defined by $(v_1, \ldots, v_n) \mapsto f_1(v_1) \cdot \ldots \cdot f_n(v_n)$ is called the product map of the f_i.

0.4. Let G be a group and G_i $(1 \leq i \leq n)$ be normal subgroups of G. We say that G is an *almost direct product* of the G_i if the product map of the natural inclusions of the G_i into G is surjective, with finite kernel[3]

0.5. A function taking values in a topological space is said to be *bounded* if its image is relatively compact.

Let X be a topological space, and let f, g be real-valued functions on X whose values are ≥ 0. We write:

$$f \prec g,$$

[1] The original French manuscript uses $\mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}, \mathbf{N}$, rather than $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{N}$.

[2] This means that if α and β are homomorphisms from G to $\mathbb{C}^*$, then the homomorphism $\alpha + \beta$ is defined by $g^{\alpha+\beta} = g^\alpha . g^\beta$.

[3] The subgroups G_i are also required to centralize each other, so the product map is a homomorphism. This requirement follows from the other assumptions if each G_i is connected, but not in general.

if there exists a constant $c > 0$ such that $f(x) \leq c\,.\,g(x)$ for all $x \in \mathrm{X}$. Also, we write $f \succ g$ if $g \prec f$, and write $f \asymp g$ if we simultaneously have $f \prec g$ and $f \succ g$. The relation $f \asymp g$ therefore means that there exist constants c, $d > 0$ such that:

$$c\,.\,f(x) \leq g(x) \leq d\,.\,f(x) \qquad (x \in \mathrm{X}).$$

If $f \prec g$ (resp. $f \succ g$, resp. $f \asymp g$), we may sometimes say that f *essentially bounds g below* (resp. *essentially bounds g above*, resp. is *comparable to g*).

0.6.[4] The normalizer of a subgroup H of a group G is denoted[5] $\mathcal{N}(\mathrm{H})$. The centralizer of H in G is denoted[6] $\mathcal{Z}(\mathrm{H})$ or $\mathcal{Z}_{\mathrm{G}}(\mathrm{H})$. In particular, $\mathcal{Z}(\mathrm{G})$ is the center of G. The transpose of a matrix g is denoted ${}^{t}g$.

[4]This paragraph was added by the editor. It is not in the original French manuscript.

[5]The original French manuscript uses N(H) (or sometimes Norm(H)) to denote the normalizer of a subgroup H, but the symbol N has another important role as a factor in the Iwasawa decomposition $\mathrm{G} = \mathrm{K}\,.\,\mathrm{A}\,.\,\mathrm{N}$.

[6]The original French manuscript uses an ordinary Z instead of $\mathcal{Z}$.

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Fifty years after it made the transition from mimeographed lecture notes to a published book, Armand Borel's *Introduction aux groupes arithmétiques* continues to be very important for the theory of arithmetic groups. In particular, Chapter III of the book remains the standard reference for fundamental results on reduction theory, which is crucial in the study of discrete subgroups of Lie groups and the corresponding homogeneous spaces.

The review of the original French version in *Mathematical Reviews* observes that "the style is concise and the proofs (in later sections) are often demanding of the reader." To make the translation more approachable, numerous footnotes provide helpful comments.

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