CHAPTER 0

The Principle of the Fermionic Projector – A New Mathematical Model of Space-Time

The mathematical model of space-time has evolved in history. In Newtonian mechanics, space is described by a Euclidean vector space. In special relativity, space and time were combined to Minkowski space, a vector space endowed with a scalar product of signature $(+ - - -)$. In general relativity, the vector space structure of space-time was given up on the large scale and was replaced by that of a Lorentzian manifold. The first hint that the notions of space and time should be modified also on the microscopic scale was obtained by Planck, who observed that the gravitational constant, Planck's constant and the speed of light give rise to a quantity of the dimension of length,

$$l_P \;=\; \sqrt{\frac{\hbar\, \kappa}{c^3}} \;\approx\; 1.6 \cdot 10^{-35}\, \mathrm{m}\;,$$

and he conjectured that for distances as tiny as this so-called Planck length, the conventional laws of physics should no longer hold, and yet unknown physical effects might become significant. Later, this picture was confirmed by quantum field theory. Namely, due to the ultraviolet divergences, perturbative QFT is well-defined only after regularization, and the regularization is then removed using the renormalization procedure. While renormalization ensures that the observable quantities do not depend on the regularization, the theoretical justification for the renormalization program lies in the assumption that the continuum theory should be valid only down to some microscopic length scale, and it seems natural to associate this length scale to the Planck length.

Today most physicists agree that in order to make progress in fundamental physics, one should go beyond the continuum field theory and try to get a better understanding of the microscopic structure of space-time. However, giving up the usual space-time continuum leads to serious problems, and this is one reason why there is no consensus on what the correct mathematical model for "Planck scale physics" should be. Let us illustrate the difficulties by briefly discussing a few of the many approaches. The simplest and maybe most natural approach is to assume that on the Planck scale space-time is no longer a continuum but becomes in some way "discrete." This idea is for example used in lattice gauge theories, where space-time is modeled by a four-dimensional lattice (see Figure 0.1(a)). Using the specific structures of a lattice like the nearest-neighbor relation and the lattice spacing d, one can set up a quantum field theory which is ultraviolet finite [**Ro**]. Lattice gauge theories are very useful for numerical simulations [**K**]. However, they are not fully satisfying from a conceptual point of view because a space-time lattice is not consistent with the equivalence principle of general relativity. Namely, if one considers the lattice in the reference frame of an accelerated observer (denoted in

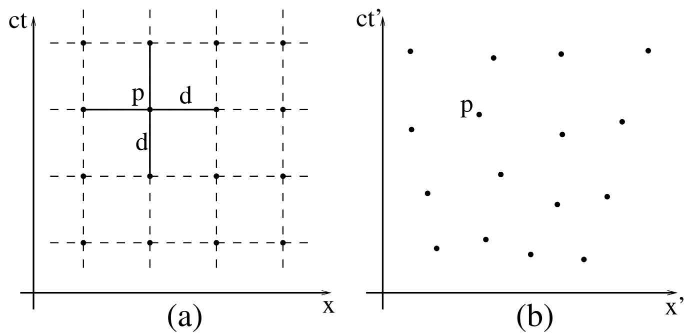

FIGURE 0.1. A lattice regularization for two different observers.

in Figure 0.1(b) by (t', x')), the lattice points are no longer in a regular configuration. Thus the structure of a lattice is not invariant under general coordinate transformations and hence is not compatible with the equivalence principle.

An alternative approach is to hold on to a space-time continuum, but to work with objects which are spread out in space-time, thereby avoiding the ultraviolet problems which occur for very small distances. The most prominent example in this direction is string theory, where physics is on a fundamental level described by so-called strings, which are extended in space-time and are therefore ultraviolet finite. The basic problem with such theories is that they are formulated using the structures of an underlying continuum space-time (like the vector space structure, the topology or even a metric), although all observable quantities (like the Lorentz metric, particles, fields, etc.) are to be derived from the non-localized objects, without referring to the underlying space-time continuum. Therefore, the structures of the underlying "continuum background" may seem artificial, and serious conceptual problems arise when these background structures are not compatible with basic physical principles (for example, a background vector space is not compatible with the equivalence principle). For short, one says that the theory is not background-free (for a more detailed discussion see [**Ba**] and the references therein).

Thus one difficulty in finding a promising model for Planck scale physics is that it should be background-free and should respect the basic physical principles (like the equivalence principle, the local gauge principle, etc.). There are approaches which actually meet these requirements. One is Connes' noncommutative geometry. As pointed out by Grothendieck, there is a one-to-one correspondence between the points of a manifold and the prime ideals of the (commutative) algebra of functions on this manifold. Thus the geometry of a manifold can be recovered from an underlying algebraic structure, and this makes it possible to extend the notions of space and time by considering more general, noncommutative algebras (see [**Co, CC**] for details and physical applications). The other approach is quantum gravity as pursued by Ashtekar and his school [**ARS, Th**]. The hope is that the ultraviolet divergences of QFT should disappear as soon as gravity, quantized in a non-perturbative way, is included.

Ultimately, a model for space-time on the Planck scale must be verified or falsified by physical experiments. Unfortunately, experiments on the Planck scale would require such enormously high energies that they are at present out of reach.

Indirect experiments seem possible in principle [**ARA**] but have so far not been carried out. In my opinion, one should not hope for important new experimental input in the near future, but one should try to make due with the experimental data which is now available. Indeed, this situation is not as hopeless as it might appear at first sight. Namely, in present physical models like the standard model, a lot of information from experiments is built in empirically, like the masses of the elementary particles, the gauge groups, the coupling constants, etc. Therefore, one can get a connection to experiments simply by trying to reproduce this well known empirical data. If successful, this procedure could give strong physical evidence for a model. For example, even if based on ad-hoc assumptions on the microscopic structure of space-time (which cannot be verified directly), a model would be very convincing and physically interesting if it gave the correct value for the fine structure constant and explained e.g. why the strong gauge group is $SU(3)$ or why neutrinos do not couple to the electromagnetic field. Thus the goal of a mathematical model for space-time on the Planck scale is to give a more fundamental explanation for the structures and empirical parameters in the standard model. To our opinion, only such concrete results can justify the model. Clearly, it is far from obvious how a promising model should look, or even in which mathematical framework it should be formulated. But at least for a mathematician, this makes the problem only more interesting, and it seems a challenging program to search for such models and to develop the mathematical methods needed for their analysis.

Our point of view that the mathematical model needs justification by known experimental data is not just a requirement which the model should fulfill at the very end, but it also gives a few hints on how one should proceed in order to find a promising model. First of all, one can expect concrete results only if one makes specific assumptions. Therefore, generalizing the notion of a Lorentzian manifold does not seem to be sufficient, but one should make a concrete ansatz for the microscopic structure of space-time (as it is done e.g. in string theory and lattice gauge theories). Furthermore, in order to make it easier to get a connection to well-established theories like classical field theory and quantum mechanics, it seems a good idea to take these theories as the starting point and to try to work as closely as possible with the objects used in these theories. Namely, if one drops important structures of the classical theories and/or introduces too many new structures ad hoc, it might become very difficult if not impossible to obtain a relation to observable data.

In our model of space-time we have tried to follow the above considerations. Our starting point is relativistic quantum mechanics and classical field theory. We assume that space-time is discrete on the Planck scale. But our notion of "discrete space-time" is much more general than a four-dimensional lattice; in particular, we do not assume any discrete symmetries in space-time, we keep the local gauge freedom, and we also extend the diffeomorphism invariance of a manifold in such a way that the equivalence principle is respected in discrete space-time. Furthermore, our model is background-free. In contrast to string theory, we do not introduce any new objects, but hold on to the structures already present in classical Dirac theory. We build in our physical ideas simply by prescribing which of these structures we consider as being fundamental, and then carry over these structures to discrete space-time. In the resulting mathematical framework, it is impossible to formulate the conventional physical equations, and thus we propose instead new equations of different type, referred to as the equations of discrete space-time. In a certain limiting case, the so-called continuum limit, we get a connection to the conventional formulation of physics in a space-time continuum. We point out that, in contrast to the Ashtekar program, we do not work with second quantized fields. But our concept is that the equations of discrete space-time should also account for the physical effects of quantized fields if one goes beyond the continuum limit.

More specifically, we describe the physical system by the *fermionic projector* $P(x,y)$, which can be regarded as the projector on all occupied fermionic states of the system, including the states of the Dirac sea. After carrying over the fermionic projector to discrete space-time, we can set up variational principles like our "model variational principle"

$$\sum_{x,y \in M} \mathcal{L}[P(x,y)\, P(y,x)] \;\to\; \min\,,$$

where the "Lagrangian" $\mathcal{L}$ is given by

$$\mathcal{L}[A] \;=\; |A^2| - \mu |A|^2\,,$$

with μ a Lagrangian multiplier. Here $|A|$ is the so-called spectral weight defined as the sum of the absolute values of the eigenvalues of the matrix A (or, in case that A is not diagonalizable, of the zeros of its characteristic polynomial). We study the above variational principle for a fermionic projector which in the vacuum is the direct sum of seven identical massive sectors and one massless left-handed sector, each of which is composed of three Dirac seas. Analyzing the continuum limit for an interaction via general chiral and (pseudo)scalar potentials, we find that the sectors spontaneously form pairs, which are referred to as blocks. The resulting so-called effective interaction can be described by chiral potentials corresponding to the effective gauge group

$$SU(2) \otimes SU(3) \otimes U(1)^3\,.$$

This model has striking similarity to the standard model if the block containing the left-handed sector is identified with the leptons and the three other blocks with the quarks. Namely, the effective gauge fields have the following properties.

- The $SU(3)$ corresponds to an unbroken gauge symmetry. The $SU(3)$ gauge fields couple to the quarks exactly as the strong gauge fields in the standard model.
- The $SU(2)$ potentials are left-handed and couple to the leptons and quarks exactly as the weak gauge potentials in the standard model. Similar to the CKM mixing in the standard model, the off-diagonal components of these potentials must involve a non-trivial mixing of the generations. The $SU(2)$ gauge symmetry is spontaneously broken.
- The $U(1)$ of electrodynamics can be identified with an Abelian subgroup of the effective gauge group.

The effective gauge group is larger than the gauge group of the standard model, but this is not inconsistent because a more detailed analysis of our variational principle should give further constraints for the Abelian gauge potentials. Moreover, there are the following differences to the standard model, which we derive mathematically without working out their physical implications.

- The $SU(2)$ gauge field tensor F must be simple in the sense that $F = \Lambda\, s$ for a real 2-form Λ and an $su(2)$-valued function s.

- In the lepton block, the off-diagonal $SU(2)$ gauge potentials are associated with a new type of potential, called nil potential, which couples to the right-handed component.

These results give a strong indication that the principle of the fermionic projector is of physical significance. Nevertheless, the goal of this book is not to work out our model variational principle in all details. Our intention is to develop the general concepts and methods from the basics, making them easily accessible to the reader who might be interested in applying them to other equations of discrete space-time or to related problems.

These notes are organized as follows. In order to make the presentation self-contained, Chapter 1 gives a brief account of the mathematical and physical preliminaries. Chapter 2 introduces the fermionic projector in the continuum and provides the mathematical methods needed for its detailed analysis. In Chapter 3 we go beyond the continuum description and introduce our mathematical model for space-time on the Planck scale. In Chapter 4 we develop a mathematical formalism suitable for the analysis of the continuum limit. In Chapter 5 we present and discuss different equations of discrete space-time in the vacuum, and we choose the most promising equations as our "model equations". In the last Chapters 6-8 we analyze interacting systems in the continuum limit. The appendices contain additional material and will be referred to from the main chapters.