Preface

This book deals with a detailed analysis of the Heisenberg group and its extensions to superior steps, and their applications to Physics. The approach is using the complex and real Hamiltonian and Lagrangian formalism developed in the subRiemannian manifolds context. The exposition is enriched with chapter exercises which facilitate the reader's understanding.

An overview for the Reader: This work is a text for a course or seminars designed for graduate students interested in the most recent developments in the subRiemannian manifolds and sub-elliptic operators theory. It is useful for both pure and applied mathematicians and theoretical physicists working in the quantum mechanics area.

Scientific Outline: This book deals with the study of subRiemannian manifolds, which are manifolds with the Heisenberg principle built in. This brings the hope that Heisenberg manifolds (step 2 subRiemannain manifolds) will play a role in Quantum Mechanics in the future, similar to the role played by the Riemannian manifolds in Classical Mechanics. Some people also speculate that superior steps subRiemannian manifolds may play a similar role in Quantum Field Theory.

The subRiemannian geodesics behavior, which is very different than in the Riemannian case, plays an important role in finding heat kernels and propagators for the Schrödinger equation, as well as in finding fundamental solutions for subelliptic operators using a very ingenious geometric method involving complex modified action.

One of the novelties of this book is to introduce the complex Hamiltonian mechanics techniques and apply them in describing the fundamental solutions and heat propagators. The fact that the propagator depends on each subRiemannian distance is an expected fact in Quantum Mechanics.

Our method is quite different and new. The fact that each path has a contribution to the propagator led Feynman to introduce the path integrals. In this work we can manage without them, just working in the proper geometric framework.

For macro objects, like airplanes, planets or bullets, only the classical path is important and their motion is described by a unique geodesic in the context of Riemannian geometry. The situation for the sub-atomic particles is completely different and this book indicates the correct framework to study their motion, which is the subRiemannian geometry. The main issue which is the base of this interpretation is the fact that local and global in subRiemannian context are the same. Hence there are points arbitrary close which can be joined by infinitely many geodesics with distinct lengths and which are used in the asymptotic expansion of the heat kernel on the Heisenberg group.

The first chapter contains a detailed description of the geometry of the Heisenberg group which is the prototype of the step 2 subRiemannian manifolds. It shows that the Carnot-Carathéodory distance plays a very important role in finding the fundamental solution of the Heisenberg operator which retrieves the well-known Folland's formula.

The second chapter deals with a step 4 subRiemannian manifold and its geometry. The tools used to describe the geodesics involve complicated and explicit computations involving elliptic functions.

Chapter 3 deals with the geometry of a step $2(k+1)$ subRiemannian manifolds. As in chapter 2, the closed form solutions require the use of hypergeometric functions. We provide also a qualitative approach and we manage to prove global connectivity and geodesic completeness.

Chapter 4 is concerned with the non-isotropic multi-dimensional Heisenberg group. In this case, the group law will be

$$(x,t)\circ(x',t') = \Big(x+x', t+t'+2\sum_{j=1}^{n} a_j\left[x_{2j}x'_{2j-1} - x_{2j-1}x'_{2j}\right]\Big).$$

In the isotropic case $a_1 = a_2 = \cdots = a_n$, the results of Chapter 1 carry over with no change, except that each family of geodesics from the point $(0,t)$ to the origin is parametrized by the $(2n-1)$-sphere. However, on a general non-isotropic Heisenberg group, the analysis is more laborious. More precisely, suppose that

$$0 < a_1 \le a_2 \le \cdots \le a_p < a_{p+1} = \cdots = a_n.$$

Then every point $(0,t)$ is connected to the origin by an infinite number of geodesics. Every point (x,t) with $x'' = (x_{2p+1}, x_{2p+2}, \ldots, x_{2n}) \neq 0$ is connected to the origin by a finite number of geodesics. On the other hand, there are points (x,t), with $x \neq 0$, but $x'' = 0$, which are connected to the origin by an infinite number of geodesics. If (x,t), $x \neq 0$, is connected to the origin by an infinite number of geodesics, then the infinity of the number of geodesics connecting (x,t) to $(0,0)$ is "smaller" than the infinity of the number of geodesics connecting $(0,t)$ to $(0,0)$.

The technique of complex Hamiltonian mechanics is introduced in **chapter 5**. It consists in defining a quantum Hamiltonian and introducing a complex modified action. The latter is used in obtaining the lengths of real geodesics and in finding a geometric formula for the fundamental solution of sub-elliptic operators.

Chapter 6 is dedicated to applications in Quantum Mechanics and Electromagnetism. The main application is the fundamental solution and the Schrödinger kernel (propagator) for the Heisenberg operator by means of complex Hamiltonian formalism.

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