Preface

In one guise or another, many mathematicians are familiar with certain arithmetic groups, such as $\mathbb{Z}$ and $\mathrm{SL}(n, \mathbb{Z})$. But their relatives, for example, S-arithmetic groups such as $\mathrm{SL}\left(n, \mathbb{Z}\left[\frac{1}{p_1}, \ldots, \frac{1}{p_m}\right]\right)$, where $p_1, \ldots, p_m$ are prime numbers, and their analogues over function fields such as $\mathrm{SL}(n, \mathbb{F}_p[t])$, where $\mathbb{F}_p$ is a finite field and t is a variable, may not be so well-known. The purpose of this expository book is to explain through some brief and informal comments what these groups are, why they are important to study, and how they can be understood and applied to many fields such as analysis, geometry, topology, number theory, representation theory and algebraic geometry.

We try to emphasize the point of view that it is the group action on good spaces that makes the group interesting and understandable, and that it is also the group action which makes the spaces invloved more interesting and useful. In fact, problems naturally arise and are solved by such pairs of groups and spaces. For example, though symmetric spaces are important and interesting in themselves, their quotients by arithmetic groups, which are locally symmetric spaces of finite areas and sometimes called arithmetic locally symmetric spaces, are much richer in structures and have more applications. One obvious reason is that symmetric spaces of noncompact type have trivial topology, but cohomology groups of locally symmetric spaces are usually nontrivial and carry a lot of valuable information both for the arithmetic groups and other related objects, in particular when the locally symmetric spaces can be interpreted as moduli spaces in algebraic geometry and number theory. This point of view can be seen clearly in comparison between the Poincaré upper half plane $\mathbb{H}$ and its arithmetic quotients such as modular curves and Shimura curves.

Though Riemannian symmetric spaces and locally symmetric spaces are probably the most natural spaces associated with arithmetic subgroups of Lie groups, we also try to emphasize that it is natural and important to consider other related spaces such as pseudo-Riemannian symmetric and locally symmetric spaces, buildings (both spherical and Euclidean type) and Teichmüller spaces, which can feed back to symmetric and locally symmetric spaces, besides their own interests. In fact, for S-arithmetic groups such as $SL\left(n, \mathbb{Z}\left[\frac{1}{p_1}, \ldots, \frac{1}{p_m}\right]\right)$, both symmetric spaces and buildings are needed simultaneously in order to understand them.

We also discuss some related groups such as tree lattices, building lattices, CAT(0)-groups, mapping class groups, and outer automorphism groups of free groups, and their related spaces such as Teichmüller spaces and moduli spaces of Riemann surfaces, outer spaces, by emphasizing their similarities to arithmetic groups and symmetric spaces, and also to S-arithmetic groups and Bruhat-Tits buildings. By putting all these related groups and spaces together at one place, results from one class can motivate and suggest results for another class.

Hopefully the reader will be convinced of and appreciate more the importance and ubiquity of arithmetic groups in mathematics, and consequently, learn some non-obvious connections between different subjects through them, by the snapshots on motivations, informal descriptions of basic objects and applications, and the many references included in this book, which might also serve as a partial guide to the huge literature on arithmetic groups and related topics.