Contents

Chapter 1. Introduction 1

Chapter 2. General comments on references 5

Chapter 3. Examples of basic arithmetic groups 7
- 3.1. $\mathbb{Z}$ as a discrete subgroup of $\mathbb{R}$ 7
  - 3.1.1. Poisson summation formula 8
  - 3.1.2. Riemann zeta function 8
- 3.2. $\mathbb{Z}^n$ and lattices in $\mathbb{R}^n$ 9
  - 3.2.1. Lattices in $\mathbb{R}^n$ 10
  - 3.2.2. Sphere packing 11
  - 3.2.3. Poisson summation formula for lattices in $\mathbb{R}^n$ 12
  - 3.2.4. Weil-Siegel formula 13
  - 3.2.5. Voronoi formula 13
  - 3.2.6. Generalizations of the Poisson summation formula 14
- 3.3. The modular group $SL(2,\mathbb{Z})$ 15
  - 3.3.1. Fundamental domain of $SL(2,\mathbb{Z})$ in the upper half plane 15
  - 3.3.2. Reduced quadratic forms and reduction theory 16
  - 3.3.3. Generators of $SL(2,\mathbb{Z})$ 17
  - 3.3.4. Volume formula for modular curves and locally symmetric spaces 17
  - 3.3.5. Moduli space of lattices 17
  - 3.3.6. Compactifications 18
  - 3.3.7. Deformation retraction to co-compact subspaces 19
  - 3.3.8. Geodesic flow, continued fractions and symbolic dynamics 19
  - 3.3.9. Congruence subgroups 20
  - 3.3.10. Dirichlet fundamental domain 21
  - 3.3.11. Finite generation of discrete groups 22
  - 3.3.12. Arithmetic Fuchsian groups 23
  - 3.3.13. Characterization of arithmetic Fuchsian groups 23
- 3.4. Spectral theory of $\Gamma\backslash\mathbb{H}$ 24
  - 3.4.1. Weyl law 25
  - 3.4.2. Spectral decomposition 25
  - 3.4.3. Discrete spectrum and Selberg's $\frac{1}{4}$-conjecture 26
  - 3.4.4. Selberg trace formula 27

3.4.5. Generalized Weyl law 28
3.4.6. Phillips-Sarnak conjecture on cusp Mass forms and spectral degeneration 28
3.4.7. Counting lengths of geodesics–the generalized prime number theorem 29
3.4.8. Modular symbols 30
3.4.9. Selberg zeta function 31
3.4.10. Scattering geodesics and generalized Poisson relation 32
3.4.11. Modular forms and Maass forms 33
3.4.12. Automorphic representations 34
3.4.13. L-functions 35
3.4.14. Automorphic forms on adele groups 35
3.4.15. Converse theorems for L-functions 36
3.4.16. Applications of modular forms 36

Chapter 4. General arithmetic subgroups and locally symmetric spaces 37
4.1. Algebraic groups 37
4.2. Definition of arithmetic subgroups 38
4.3. Hilbert modular groups 39
4.4. Congruence subgroups and the congruence kernel 40
4.5. Arithmetic subgroups as discrete subgroups of Lie groups 41
4.6. Zariski density of arithmetic subgroups 42
4.7. Symmetric spaces 43
4.8. Non-Riemannian symmetric spaces 45
4.9. Locally symmetric spaces 45
4.10. Space forms 46
4.11. Compactness criterion for locally symmetric spaces 47
4.12. Siegel sets and fundamental sets 48
4.13. Reduction theory for arithmetic subgroups 49
4.14. Precise reduction theory for arithmetic subgroups 50
4.15. Metric properties and $\mathbb{Q}$-rank of locally symmetric spaces 51
4.16. Volume spectrum of locally symmetric spaces 52
4.17. Maximal arithmetic subgroups and automorphism groups 53
4.18. Counting of volumes of hyperbolic manifolds 54
4.19. Counting of subgroups by index 54

Chapter 5. Discrete subgroups of Lie groups and arithmeticity of lattices in Lie groups 57
5.1. Crystallographic groups and Auslander conjecture 57
5.2. Lattices in nilpotent Lie groups and Margulis Lemma 58
5.3. Lattices in solvable Lie groups 60
5.4. Lattices in semisimple Lie groups 61
5.5. Characterization of arithmetic groups 62
5.6. Non-arithmeticity of lattices in rank 1 cases 63

5.7. Arithmeticity of lattices in rank 1 cases 63
5.8. Linear discrete subgroups and Tits alternative 64
5.9. Reflection groups 65
5.10. Discrete groups related to Kac-Moody groups and algebras 65
5.11. Infinite dimensional Lie groups and discrete groups associated with them 66

Chapter 6. Different completions of $\mathbb{Q}$ and S-arithmetic groups over number fields 69
6.1. p-adic completions 69
6.2. S-integers 69
6.3. S-arithmetic subgroups 70
6.4. S-arithmetic subgroups as discrete subgroups of Lie groups 70

Chapter 7. Global fields and S-arithmetic groups over function fields 73
7.1. Function fields 73
7.2. Global fields 73
7.3. S-arithmetic subgroups over function fields 74

Chapter 8. Finiteness properties of arithmetic and S-arithmetic groups 75
8.1. Finite generation 75
8.2. Bounded generation 76
8.3. Finite presentations 77
8.4. Finiteness properties such as FP_∞ 77
8.5. Cofinite universal space for proper actions and arithmetic groups 78
8.6. Finiteness properties of arithmetic subgroups 79

Chapter 9. Symmetric spaces, Bruhat-Tits buildings and their arithmetic quotients 81
9.1. Flats in symmetric spaces and the spherical Tits building 81
9.2. Bruhat-Tits buildings 83
9.3. Action of S-arithmetic subgroups on products of symmetric spaces and buildings 83
9.4. CAT(0)-spaces and CAT(0)-groups 84
9.5. Reduction theory for S-arithmetic subgroups 85

Chapter 10. Compactifications of locally symmetric spaces 87
10.1. Why locally symmetric spaces are often noncompact 87
10.2. Compactifications of symmetric spaces 87
10.3. Limit sets of Kleinian groups and Patterson-Sullivan theory 89
10.4. Compactifications of locally symmetric spaces 91
10.5. Compactifications of Bruhat-Tits buildings 92

10.6. Compactifications for S-arithmetic groups 93
10.7. Geometry and topology of compactifications 94
10.8. Truncation of locally symmetric spaces 94

Chapter 11. Rigidity of locally symmetric spaces 97
11.1. As special Riemannian manifolds 98
11.2. Local and infinitesimal rigidity of locally symmetric spaces 98
11.3. Global (strong) rigidity of locally symmetric spaces 99
11.4. Super-rigidity of lattices 99
11.5. Quasi-isometry rigidities of lattices 100
11.6. Rank rigidity of locally symmetric spaces 102
11.7. Entropy rigidity of locally symmetric spaces and simplicial volume 102
11.8. Rigidity of Hermitian locally symmetric spaces 103
11.9. Rigidity of pseudo-Riemannian locally symmetric spaces 105
11.10. Rigidity of non-linear actions of lattices: Zimmer program 106
11.11. Rigidity in von Neumann algebras 109
11.12. Topological rigidity and the Borel conjecture 111
11.13. Methods to prove the rigidities 112
11.14. Dynamics, flows on locally symmetric spaces and number theory 113

Chapter 12. Automorphic forms and automorphic representations for general arithmetic groups 115
12.1. Automorphic forms 115
12.2. Boundary values of eigenfunctions and automorphic forms 116
12.3. Spectral decomposition 117
12.4. Weyl law 118
12.5. Counting of eigenvalues for a tower of spaces 118
12.6. Quantum chaos 119
12.7. Arthur-Selberg trace formula 120
12.8. Selberg zeta function 121
12.9. Counting of lengths of geodesics and volumes of tori 122
12.10. L-functions of automorphic representations 122
12.11. Meromorphic continuation of Eisenstein series 123
12.12. Constant term of Eisenstein series 124
12.13. Langlands program 124
12.14. Spectral theory over function fields: an example 125

Chapter 13. Cohomology of arithmetic groups 127
13.1. Cohomology groups 127
13.2. L^2- and L^p-cohomology of arithmetic groups 128
13.3. Intersection cohomology 129
13.4. Weighted cohomology 129

13.5. Continuous cohomology 130
13.6. Applications of automorphic forms to cohomology 130
13.7. Construction of cycles and relations to automorphic forms 131
13.8. Hecke trace formula on the cohomology groups 131
13.9. Euler characteristics, Gauss-Bonnet formula 131
13.10. Cohomology of S-arithmetic subgroups 133
13.11. Boundary cohomology 133

Chapter 14. K-groups of rings of integers and K-groups of group rings 135
14.1. Definitions of algebraic K-groups 135
14.2. Finite generation of $K^i(\mathbb{Z})$ 135
14.3. Relations between $K^i(\mathbb{Z})$ and cohomology of the arithmetic groups $SL(n, \mathbb{Z})$ 136
14.4. Torsion elements of $K^i(\mathbb{Z})$ 136
14.5. Applications of $K_i(\mathbb{Z}[\Gamma])$ in topology 137
14.6. Farrell-Jones conjecture, Borel conjecture and Novikov conjecture 137

Chapter 15. Locally homogeneous manifolds and period domains 139
15.1. Homogeneous manifolds as special Riemannian manifolds 139
15.2. Non-symmetric, but homogeneous spaces 140
15.3. Hodge structures, period domains and period maps 141
15.4. Homogeneous, non-Riemannian manifolds 142
15.5. Clifford–Klein forms of homogeneous spaces 143
15.6. Space forms: non-Riemannian case 144
15.7. Counting lattice points on homogeneous varieties 145

Chapter 16. Non-cofinite discrete groups, geometrically finite groups 147
16.1. Geometrically finiteness conditions 147
16.2. Applications in low dimensional topology 147
16.3. Spectral theory of geometrically finite groups 149

Chapter 17. Large scale geometry of discrete groups 151
17.1. Word metric on discrete groups and growth of groups 151
17.2. Geometric group theory and property (T) 151
17.3. Ends of groups 152
17.4. Ends of locally symmetric spaces and bottom of the spectrum 154
17.5. Asymptotic invariants 157
17.6. L^2-invariants 158
17.7. Boundaries of discrete groups 159
17.8. Asymptotic geometry of locally symmetric spaces $\Gamma\backslash X$ 161

17.9. Isoperimetric profile, Dehn functions of arithmetic subgroups 162
17.10. Trees and applications in topology 163

Chapter 18. Tree lattices 165
18.1. Structures of tree lattices 165
18.2. Arithmeticity and density of commensurability groups of tree lattices 166
18.3. Rigidity of lattices in products of trees and CAT(0) groups 167
18.4. Building lattices and applications to fake projective planes 167

Chapter 19. Hyperbolic groups 169
19.1. Basic properties of hyperbolic groups 169
19.2. Rips complex and Gromov boundary 170

Chapter 20. Mapping class groups and outer automorphism groups of free groups 173
20.1. Mapping class groups 173
20.2. Teichmüller spaces of Riemann surfaces 174
20.3. Topology of moduli spaces of Riemann surfaces and the mapping class groups 175
20.4. Compactifications of Teichmüller spaces 176
20.5. Symmetry of Teichmüller spaces 178

Chapter 21. Outer automorphism group of free groups and the outer spaces 179
21.1. Outer automorphism group of free groups 179
21.2. Outer spaces 179
21.3. Compactifications of outer spaces 180
21.4. Outer automorphism group of non-free groups 181

References 183

Index 245