CHAPTER 1

Introduction

The wave packet analysis of this book originated in the celebrated work of L. Carleson [**5**] on almost everywhere convergence of Fourier series of L^2 functions. Wave packet analysis is not named in Carleson's paper, but appears as an ad hoc means of solving the difficult problem of a.e. convergence of Fourier series. In the decade following Carleson's paper, several authors picked up the ideas and refined the analysis. P. Billard [**1**] proved the analogue for Walsh Fourier series, R. Hunt [**15**] extended the result to L^p with $1 < p < 2$, P. Sjölin [**37**] gave an extension to higher dimensions, and C. Fefferman [**9**] gave an influential new proof of Carleson's theorem based on a decomposition of the relevant Carleson operator. Several books have been devoted to Carleson's theorem.

Despite the wide recognition of Carleson's theorem and manifold research activity, the profound ideas of wave packet analysis have for quite some time only been applied to problems very closely related to the original problem motivating the work of Carleson. Consequently, interest in Carleson's ideas had become less pronounced by the nineteen nineties, when M. Lacey [**16**] and Lacey and C. Thiele in a series of papers [**20**],[**21**],[**22**],[**23**] revived interest in wave packet analysis as a tool to proof bounds on the bilinear Hilbert transform. The bilinear Hilbert transform had been known to A. Calderon in the sixties, who encountered it in his investigations of the socalled first Calderon commutator. Calderon succeeded in estimating all commutators appearing in the multilinear expansion of the Cauchy integral on Lipshitz curves [**3**], [**4**] without the use of the bilinear Hilbert transform. The problem of proving L^p bounds for the bilinear Hilbert transform remained as a separate challenge to the analysis community and was not seriously advanced until the above mentioned work.

For a very brief moment in history it may have appeared as if Carleson's theorem and bounds on the bilinear Hilbert transform would form new small equivalence class of problems without applications to a wider field of Mathematics. But subsequent developments have shown that at least this equivalence class is rather large with many interesting ramifications. Moreover, questions arising from different fields have been approached and solutions been produced, the most striking example being the estimates by Lacey and Li [**19**] on the Hilbert transform along vectorfields, a problem with an independent history. Also, connections with ergodic theory [**17**] and scattering theory [**29**], [**30**], [**34**] have been studied and may hopefully bear further fruit in the future.

The main part of these lecture notes provides a snapshot of wave packet analysis in the mid nineteen nineties, at which time Carleson's theorem and bounds on the bilinear Hilbert transform were known, albeit in a language that has developed since. It certainly would be desireable to have a comprehensive discussion of the work on wave packet analysis since, but such a project has turned out to be beyond the scope of the current lecture notes. Only in the last lecture we summarize some but not all of the more recent results, just enough to deliver evidence to the claim that the research has not stopped with the bounds on the bilinear Hilbert transform.

One can readily point to a common feature of both Carleson's operator and the bilinear Hilbert transform that necessitates wave packet analysis, and that is modulation symmetries of these objects. Modulation of a function is the same as a translation of its Fourier transform and can be expressed in spatial coordinates as multiplication with a pure harmonic function:

$$M_\xi f(x) = e^{2\pi i \xi x} f(x)$$

Carleson's operator is defined as

$$Cf(x) = \sup_\xi \left| p.v. \int f(x-t) e^{2\pi i \xi t} \frac{dt}{t} \right|$$

and has the one parameter modulation symmetry

$$C(M_\eta f) = Cf \quad .$$

The bilinear Hilbert transform with parameter $\alpha \neq 0, -1$ can be written

$$B(f,g)(x) = p.v. \int f(x-t) g(x+\alpha t) \frac{dt}{t}$$

and has the one parameter modulation symmetry

$$B(M_{\alpha\eta} f, M_\eta g) = M_{(\alpha+1)\eta} B(f,g) \quad .$$

The lack of a comparable linear singular integral operator with a one parameter family of modulation symmetries may have delayed the development of wave packet analysis.

The classical theory of singular integral operators such as convolution with $p.v. \frac{1}{t}$ all has built in a special role of zero frequency through cancellation conditions or mean zero conditions. All these techniques cannot directly be applied when the problem forbids a special role of zero frequency. Wave packet analysis is then the natural generalization of singular integral theory to the modulation invariant setting. As such it generalizes wavelet theory, which is a modern framework in which much of classical singular integral theory can be formulated. Of course much weaker forms of modulation invariant settings than actual symmetries as pointed out above will suffice to necessitate wave packet analysis. The symbolic formalism of J. Gilbert and A. Nahmod [**10**],[**11**],[**12**], introduces invariant conditions on operators, where the operators themselves need not be invariant. In the work of C. Muscalu [**27**] and L. Grafakos and X. Li [**14**] symmetries are reduced to some infinitesimal modulation symmetries, and so on.

In the first three lectures after the introduction we discuss aspects of the classical part of singular integral theory. Lecture 2 introduces bump functions and introduces basic localization properties. These bump functions are essentially the wavelets of classical singular integral theory, but without any strict conditions on orthogonality. Lecture 3 introduces some interpolation theory developed in [**28**] that is particulary adapt to multilinear operators. Lecture 4 discusses paraproducts, which are the basic multilinear operators in classical singular integral theory. In Lecture 5 we discuss wave packets, which are generalizations of the bump functions in Lecture 2 with additional modulation parameter. We prove again basic localization results and results which allow to structure the set of all wave packets.

In Lecture 6 we prove bounds on the bilinear Hilbert transform and closely related multilinear forms and operators, and in Lecture 7 we prove Carleson's theorem. Lecture 8 is devoted to the Walsh model, a discrete model of wave packet analysis. This lecture is worthwile reading simultaneously or even prior to Lectures 5 to Lecture 7. Finally, Lecture 9, summarizes without proofs a few further developments of the theory.