LECTURE 1

Introduction

When a group G acts on a manifold M, one would like to understand the relation between:

- algebraic properties of the group G,
- the topology of M,
- the G-invariant geometric structures on M, and
- dynamical properties of the action (such as dense orbits, invariant measures, etc.).

If we assume that G is a connected Lie group, then the structure theory (A1.3) tells us there are two main cases to consider:

- *solvable*
  Solvable groups are usually studied by starting with $\mathbb{R}^n$ and proceeding by induction.
- *semisimple*
  There is a classification that provides a list of the semisimple groups ($\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SO}(p,q)$, etc.), so a case-by-case analysis is possible. Alternatively, the groups can sometimes be treated via other categorizations, e.g., by real rank.

The emphasis in these lectures is on the semisimple case.

(1.1) **Assumption.** *In this lecture, G always denotes a connected, noncompact, semisimple Lie group.*

1A. Discrete versions of G

A connected Lie group may have discrete subgroups that approximate it. There are two notions of this that play very important roles in these lectures:

- *lattice*
  This is a discrete subgroup Γ of G such that G/Γ is compact (or, more generally, such that G/Γ has finite volume).
- *arithmetic subgroup*
  Suppose G is a (closed) subgroup of $\mathrm{GL}(n,\mathbb{R})$, and let $\Gamma = G \cap \mathrm{GL}(n,\mathbb{Z})$ be the set of "integer points" of G. If Γ is Zariski dense in G (or, equivalently, if $G \cap \mathrm{GL}(n,\mathbb{Q})$ is dense in G), then we say Γ is an *arithmetic* subgroup of G.

Discrete versions inherit important algebraic properties of G:

(1.2) **Example.** Suppose Γ is a discrete version of G (that is, Γ is either a lattice or an arithmetic subgroup).

1) Because Assumption 1.1 tells us that G is semisimple, we know G has no solvable, normal subgroups (or, more precisely, none that are connected and nontrivial). One can show that Γ also has no solvable, normal subgroups (or, more precisely, none that are infinite, if G has finite center). This follows from the Borel Density Theorem (A6.1), which tells us that Γ is "Zariski dense" in (a large subgroup of) G.
2) If G is simple, then, by definition, G has no normal subgroups (or, more precisely, none that are connected, nontrivial, and proper). If we furthermore assume $\mathbb{R}\text{-rank}(G) \geq 2$ (i.e., G has a subgroup of dimension ≥ 2 that is diagonalizable over $\mathbb{R}$) and the center of G is finite, then theorems of G. A. Margulis link Γ and G more tightly:
   (a) Γ has no normal subgroups (or, more precisely, none that are infinite and of infinite index), and
   (b) roughly speaking, every lattice in G is arithmetic.
   (See Theorems A7.3 and A7.9 for precise statements.)

(1.3) *Remark.* Both of the conclusions of Example 1.2(2) can fail if we eliminate the assumption that $\mathbb{R}\text{-rank}(G) \geq 2$. For example, in $\mathrm{SL}(2, \mathbb{R})$, the free group F_2 is a lattice, and some other lattices have homomorphisms onto free groups.

(1.4) **Example** (Linear actions). Let $M = V$, where V is a finite-dimensional vector space (over $\mathbb{R}$ or $\mathbb{C}$). A linear action of G on M (that is, a homomorphism $G \to \mathrm{GL}(V)$) is known as a (finite-dimensional) *representation* of G. These representations are classified by the well-known theory of *highest weights*, so we can think of the linear actions of G as being known.

If $\mathbb{R}\text{-rank}(G) \geq 2$ (and G is simple), then the linear actions of Γ are closely related to the linear actions of G. Namely, the spirit, but not exactly the statement, of the Margulis Superrigidity Theorem (A7.4) is that if $\varphi\colon \Gamma \to \mathrm{GL}(V)$, then either

- φ extends to a representation of G, or
- the image of φ is contained in a compact subgroup of $\mathrm{GL}(V)$.

In other words, any representation of Γ extends to a representation of G, modulo compact groups. This is another illustration of the close connection between G and its discrete versions.

(1.5) *Remark.* An alternative formulation (A7.7) of the Margulis Superrigidity Theorem says that (in spirit) each representation of Γ either extends to G or extends to G after composing with some field automorphism of $\mathbb{C}$.

1B. Nonlinear actions

These lectures explore some of what is known about the nonlinear actions of G and its discrete versions. (As was mentioned in Example 1.4, the theory of highest weights provides a largely satisfactory theory of the linear actions.) The connections between G and Γ are of particular interest.

We begin the discussion with two basic examples of G-actions.

(1.6) **Example.** G acts on G/Γ, where Γ is any lattice. More generally, if $G \hookrightarrow H$, and Λ is a lattice in H, then G acts by translations on H/Λ.

(1.7) **Example.** The linear action of $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{R}^n$ factors through to a (nonlinear) action of $\mathrm{SL}(n,\mathbb{R})$ on the projective space $\mathbb{R}P^{n-1}$. More generally, $\mathrm{SL}(n,\mathbb{R})$ acts on Grassmannians, and other flag varieties.

Generalizing further, any semisimple Lie group G has transitive actions on some projective varieties. They are of the form G/Q, where Q is a "parabolic" subgroup of G. (We remark that there are only finitely many of these actions, up to isomorphism, because there are only finitely many parabolic subgroups of G, up to conjugacy.)

(1.8) *Remark.* In the case of $\mathrm{SL}(n,\mathbb{R})$, any parabolic subgroup Q is block upper-triangular:

$$Q = \begin{bmatrix} \boxed{*} & & * \\ & \boxed{*} & \\ 0 & & \boxed{*} \end{bmatrix}$$

A natural question that guides research in the area is:

$$\text{\textit{To what extent is every action on a compact manifold (or perhaps a more general space) built out of the above two basic examples?}} \tag{1.9}$$

Unfortunately, there are methods to construct actions that seem to be much less amenable to classification. The two known methods are *Induction* and *Blowing up*; we will briefly describe each of them.

1B(a). Induction. Suppose some subgroup H of G acts on a space Y. Then

1) H acts on $G \times Y$ by
$$h \cdot (g, y) = (gh^{-1}, hy),$$
and
2) G acts on the quotient $X = (G \times Y)/H$ by
$$a \cdot [(g, y)] = [(ag, y)].$$

The action of G on X is said to be *induced* from the action of H on Y.

(1.10) *Remark.*

1) Ignoring the second coordinate yields a G-equivariant map $X \to G/H$, so we see that X is a fiber bundle over G/H with fiber Y. Note that the fiber over $[e] = eH$ is H-invariant, and the action of H on this fiber is isomorphic to the action of H on Y.
2) Conversely, if G acts on X, and there is a G-equivariant map from X to G/H, then X is G-equivariantly isomorphic to an action induced from H. Namely, if we let Y be the fiber of X over $[e]$, then Y is H-invariant, and the map $(g, y) \mapsto g \cdot y$ factors through to a G-equivariant bijection $(G \times Y)/H \to X$.

(1.11) **Example.** Let H be the stabilizer in $\mathrm{SL}(n,\mathbb{R})$ of a point in $\mathbb{R}P^{n-1}$, so

$$H = \left\{ h = \begin{bmatrix} \lambda & * \; \cdots \; * \\ 0 & \\ \vdots & \boxed{\quad A \quad} \\ 0 & \end{bmatrix} \;\middle|\; \det A = \lambda^{-1} \right\}$$

Then $h \mapsto \log|\lambda|$ is a homomorphism from H onto $\mathbb{R}$, so every $\mathbb{R}$-action yields an action of H; hence, by induction, an action of $\mathrm{SL}(n, \mathbb{R})$.

The upshot is that every vector field on any compact manifold M yields an $\mathrm{SL}(n, \mathbb{R})$-action on a compact manifold M'. Thus, all the complications that arise for $\mathbb{R}$-actions also arise for $\mathrm{SL}(n, \mathbb{R})$-actions.

It is hopeless to classify all $\mathbb{R}$-actions on compact manifolds, so Example 1.11 puts a damper on the hope for a complete classification of actions of simple Lie groups. The following example discourages the belief in a classification theorem even more.

(1.12) **Example.** Let Γ be a lattice in $\mathrm{PSL}(2, \mathbb{R})$, such that Γ has a homomorphism onto a nonabelian free group F. By the same argument as in Example 1.11, we see that every action of F on a compact manifold yields an action of $\mathrm{PSL}(2, \mathbb{R})$ on a manifold (and the manifold is compact if $\mathrm{PSL}(2, \mathbb{R})/\Gamma$ is compact).

In $\mathrm{SL}(n, \mathbb{R})$, let

$$Q = \begin{bmatrix} \boxed{\begin{matrix} * & * \\ * & * \end{matrix}} & \begin{matrix} * & \cdots & * \\ * & \cdots & * \end{matrix} \\ \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & \boxed{\quad * \quad} \end{bmatrix},$$

where the box in the top left corner is 2×2, so Q has a homomorphism onto $\mathrm{PSL}(2, \mathbb{R})$. Thus,

$$F\text{-action} \to \mathrm{PSL}(2, \mathbb{R})\text{-action} \to Q\text{-action} \to \mathrm{SL}(n, \mathbb{R})\text{-action}.$$

So the free-group problem arises for every $\mathrm{SL}(n, \mathbb{R})$, not just $\mathrm{SL}(2, \mathbb{R})$.

1B(b). Blowing up (Katok-Lewis, Benveniste)**.** Let Γ be a finite-index subgroup of $\mathrm{SL}(n, \mathbb{Z})$, so Γ acts on $\mathbb{T}^n$ by automorphisms, and assume the action has two distinct fixed points x and y.

1) First, *blow up* at these two fixed points. That is, letting $T_p(\mathbb{T}^n)$ be the tangent space to $\mathbb{T}^n$ at p,
   - replace x with $\{\text{rays in } T_x(\mathbb{T}^n)\}$, and
   - replace y with $\{\text{rays in } T_y(\mathbb{T}^n)\}$.

   Thus, x and y have each been replaced with a sphere.
2) Now, glue the two spheres together.

The upshot is that the union of two fixed points can be replaced by a projective space with the usual action of $\mathrm{SL}(n, \mathbb{Z})$ on $\mathbb{R}P^{n-1}$.

(1.13) *Remark.*

1) Blowing up results in a different manifold; the fundamental group is different. (This will be discussed in Example 4.18 below.)
2) The construction can embed the action on a projective space (which is not volume preserving) into a volume-preserving action.
3) The change in the manifold is on a submanifold of positive codimension.

The above construction is due to A. Katok and J. Lewis. It was generalized by J. Benveniste along the following lines.

(1.14) **Example.**

1) Embed G in a larger group H, let $M = H/\Gamma$, for some lattice Λ in H.
2) Assume G is contained in a subgroup L of H, such that L has a closed orbit on H/Γ.

For some pairs of closed L-orbits, we can blow up transversally (that is, take the set of rays in a subspace transverse to the L-orbit), and then glue to make a new action. Varying the gluing results in actions that have nontrivial perturbations.

1C. Open questions

The examples in §1B(a) and §1B(b) suggest there is a limit to what can be done, so we present a few problems that are likely to be approachable. Some will be discussed in later lectures.

1) Are there actions of Γ on low-dimensional manifolds?

   Low-dimensional can either mean low in an absolute sense, as in dimension ≤ 3, or it can mean low relative to Γ, which is often taken to be less than

   $$(\text{lowest dimension of a representation of } G) - 1$$

   For example, if $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, then "low-dimensional" means either ≤ 3 or $< n - 1$.
2) When are actions locally rigid?

   Not all actions are locally rigid. For example, if the action is induced from a vector field, then it may be possible to perturb the vector field. (Note that perturbations may be non-linear actions, even if the original action is linear.)
3) Suppose G preserves a geometric structure on M, defined by a structure group H. Then what is the relation between G and H?

   Sometimes, assuming that G preserves a suitable geometric structure eliminates the examples constructed above.
4) Does every action have either:
   - an invariant geometric structure of rigid type (at least, on an open set), or
   - an equivariant quotient G-manifold with such a structure?
5) If G acts on M, what can be said about the fundamental group $\pi_1(M)$? What about other aspects of the topology of M?
6) What are the consequences of assuming there is a G-invariant volume form on M?

Approaches to these questions must bear in mind the constructions of §1B(a) and §1B(b) that provide counterexamples to many naive conjectures.

Comments

Proofs of the fundamental theorems of G. A. Margulis mentioned in Examples 1.2 and 1.4 can be found in [**9, 10, 11**].

The blowing-up construction of §1B(b) is due to A. Katok and J. Lewis [**7**]. Example 1.14 appeared in the unpublished Ph. D. thesis of E. J. Benveniste [**1**].

See [**4**] for details of a stronger result. A few additional examples created by a somewhat different method of gluing appear in [**6**, §4].

Some literature on the open questions in Section 1C:

1) Actions on manifolds of low dimension ≤ 2 are discussed in Lecture 2 (and the comments at the end).
2) See [**3**] for a recent survey of the many results on local rigidity of group actions.
3) Actions with an invariant geometric structure are discussed in Lectures 3, 5 and 6.
4) The examples of Katok-Lewis and Benveniste do not have a rigid geometric structure that is invariant [**2**]. On the other hand, the notion of "almost-rigid structure" is introduced in [**2**], and all known volume-preserving, smooth actions of higher-rank simple Lie groups have an invariant structure of that type.
5) Results on the fundamental group of M are discussed in Lectures 4, 5 and 7.
6) Many of the results discussed in these lectures apply only to actions that are volume preserving (or, at least, have an invariant probability measure). Only Lecture 9 is specifically devoted to actions that are *not* volume preserving.

See the survey of D. Fisher [**5**] (and other papers in the same volume) and the ICM talk of F. Labourie [**8**] for a different view of several of the topics that will be discussed in these lectures.

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