Introduction

This monograph is based on ten lectures given by the second author at the CBMS sponsored conference *Hodge Theory, Complex Geometric and Representation Theory* that was held during June, 2012 at Texas Christian University, and on selected developments that have occured since then in the general areas covered by those lectures. The original material covered in the lectures and in the appendices is largely on joint work by the three authors.

This work roughly separates into two parts. One is the lectures themselves, which appear here largely as they were given at the CBMS conference and which were circulated at that time. The other part is the appendices to the later lectures. These cover material that was either related to the lecture, such as selected further background or proofs of results presented in the lectures, or new topics that are related to the lecture but have been developed since the conference. We have chosen to structure this monograph in this way because the lectures give a fairly succinct, in some places informal, account of the main subject matter. The appendices then give, in addition to some further developments, further details and proofs of several of the main results presented in the lectures.

These lectures are centered around the subjects of Hodge theory and representation theory and their relationship. A unifying theme is the geometry of homogeneous complex manifolds.

Finite dimensional representation theory enters in multiple ways, one of which is the use of Hodge representations to classify the possible realizations of a reductive, $\mathbb{Q}$-algebraic group as a Mumford-Tate group. The geometry of homogeneous complex manifolds enters through the study of Mumford-Tate domains and Hodge domains and their boundaries. It also enters through the cycle and correspondence spaces associated to Mumford-Tate domains. Running throughout is the analysis of the $G_{\mathbb{R}}$-orbit structure of flag varieties and the $G_{\mathbb{R}}$-orbit structure of the complexifications of symmetric spaces $G_{\mathbb{R}}/K$ where K contains a compact maximal torus.

Infinite dimensional representation theory and the geometry of homogeneous complex manifolds interact through the realization, due primarily to Schmid, of the Harish-Chandra modules associated to discrete series representations, especially their limits, as cohomology groups associated to homogeneous line bundles. It also enters through the work of Carayol on automorphic cohomology, which involves the Hodge theory associated to Mumford-Tate domains and to their boundary components.

Throughout these lectures we have kept the "running examples" of SL_2, $S\mathcal{U}(2,1)$, $\mathrm{Sp}(4)$ and $\mathrm{SO}(4,1)$. Many of the general results whose proofs are not given in the lectures are easily verified in the running examples. They also serve to illustrate and make concrete the general theory.

We have attempted to keep the lecture notes as accessible as possible. Both the subjects of Hodge theory and representation theory are highly developed and extensive areas of mathematics and we are only able to touch on some aspects where they are related. When more advanced concepts from another area have been used, such as local cohomology and Grothendieck duality from algebraic geometry at the end of Lecture 6, we have illustrated them through the running examples in the hope that at least the flavor of what is being done will come through.

Lectures 1 and 2 are basically elementary, assuming some standard Riemann surface theory. In this setting we will introduce essentially all of the basic concepts that appear later. Their purpose is to present up front the main ideas in the theory, both for reference and to try to give the reader a sense of what is to come. At the end of Lecture 2 we have given a more extensive summary of the topics that are covered in the later lectures and in the appendices. The reader may wish to use this as a more comprehensive introduction. Lecture 3 is essentially self-contatined, although some terminology from Lie theory and algebraic groups will be used. Lecture 4 will draw on the structure and representation theory of complex Lie algebras and their real forms. Lecture 5 will use some of the basic material about infinite dimensional representation theory and the theory of homogeneous complex manifolds. In Lectures 6 and 7 we will draw from complex function theory and, in the last part of Lecture 6, some topics from algebraic geometry. Lectures 8 and 9 will utilize the material that has gone before; they are mainly devoted to specific computations in the framework that has been established. The final Lecture 10 is devoted to issues and questions that arise from the earlier lectures.

We refer to the end of Lecture 2 for a more detailed account of the contents of the lectures and appendices.

As selected general references to the topics covered in this work we mention
- for a general theory of complex manifolds, [**Cat1**], [**Ba**], [**De**], [**GH**], [**Huy**] and [**We**];
- for Hodge theory, in addition to the above references, [**Cat2**], [**ET**], [**PS**], [**Vo1**], [**Vo2**];
- for period domains and variations of Hodge structure, in addition to the references just listed, [**CM-SP**], [**Ca**];
- for Mumford-Tate groups and domains and Hodge representation [**Mo1**], [**Mo2**], [**GGK1**] and [**Ro1**];
- for general references for Lie groups [**Kn1**] and for representation theory [**Kn2**]; specific references for topics covered in Lecture 5 are the expository papers [**Sch2**], [**Sch3**];
- for a general reference for flag varieties and flag domains [**FHW**]; [**GS1**] for an early treatment of some of the material presented below, and [**GGK2**], [**GG1**] and [**GG2**] for a more extensive discussion of some of the topics covered in this monograph;
- for a general reference for Penrose transforms [**BE**] and [**EGW**]; [**GGK2**], [**GG1**] for the material in this work;
- for mixed Hodge structures [**PS**] and [**ET**], for limiting mixed Hodge structures [**CKS1**], [**CKS2**], and [**KU**], [**KP1**] and [**KP2**] for boundary components of Mumford-Tate domains;
- for the classical theory of Shimura varieties from a Hodge-theoretic perspective [**Ke2**].

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