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CHAPTER 0

Introduction

The first noncommutative division algebra was defined by Hamilton around 150 years ago. Since then, these objects have been a source of fascination for mathematicians of many sorts. A division algebra is a very elementary object. It is just a vector space with an associative product structure where d^{-1} always exists for d nonzero. Despite this simplicity, the study of division algebras has come to involve a large collection of mathematical tools and viewpoints. For me, this contrast between simplicity and diversity is a large part of their attraction. One begins with the simplest of objects, and one finds connections to number theory, Galois theory, algebraic geometry, algebraic group theory, algebraic K theory, and more specifically to Galois cohomology, etale cohomology, and geometric invariant theory (and this list is not exhaustive). Of course, the study of division algebras has also influenced these same multiple areas of mathematics and this cross fertilization is only growing.

The goal of these lectures is to give an introduction to the theory of finite dimensional division algebras that is faithful to this diversity and depth. On the other hand, one has to keep notes like these readable, so one is limited in how far one can explore. The reader will find this tension throughout the notes.

In order not to hide the diversity of the subject, I make no pretensions that this material is self contained. I feel that to do so would inevitably distort the subject. Instead I promise to be honest. I give many references, and I hope to make clear which omitted arguments are exercises, which are elementary but very hard, and which require machinery we do not cover or even mention. On the other hand, one cannot really do a great deal in finite space. For this reason, I will, on several occasions, explore the beginnings of a topic, and then stop with what I hope are sufficient references for the interested reader to go on.

Of course, these notes reflect my individual choices and cannot hope to be anything like exhaustive. There are many many topics within this field, some of which are never mentioned at all. Others are very briefly mentioned, but not given the treatment they deserve. In making choices, I was particularly interested in touching on those subjects where I felt I have something slightly new to contribute, and where of course I feel the subject is in some sense "about division algebras."

Though traditionally a very important part of the theory of division algebras, absolutely nothing is said in these notes about involutions. This despite the fact that division algebras with involution provide a rich set of connections to the theory of algebraic groups. However, I could never hope to add anything to the magnificent "Book of Involutions" by Knus, Merkurjev, Rost, and Tignol, [**K-T**].

I also say very little about the important area of division algebras with valuations, particularly valuations which are not discrete of rank 1. These are important methods, and have yielded important achievements. Here my choice was based on the feeling I did not have very much to add. I can refer the reader to [**TW**] as a good source to begin this subject.

A final subject not included is harder for me emotionally, namely, division algebras in characteristic p. This is a subject I like very much, but is a bit tangential.

My ridiculously brief treatment of the Merkurjev-Suslin Theorem will be very noticeable to knowledgeable readers. Though I mention it and even use it, it gets nothing like the space it deserves. To be blunt, I feel I have little to add to the beautiful proof, and particular little to add that would make stronger its connection to concrete division algebras.

Nonetheless, the Merkurjev-Suslin Theorem, involutions, and other of the omitted topics illustrate a theme that I hope to emphasize myself, namely, that this is a subject with rich connections to many parts of mathematics. I hope to say enough it to make that clear, and to use other subjects to illustrate the same point.

Within topics, I will give fairly complete arguments for some results, and rather sketchy arguments for others with, I hope, adequate references to allow the reader to fill them in. Some readers may get the impression that my choices for inclusion, or not, seem random. I hope they are not. When I omit arguments, I often try say enough to give the flavor of the subject, and to also allow someone to keep reading by "taking it on faith." In making these decisions I also give weight to including an argument if I thought it a bit new, or if I thought it helped shed a slightly different light on the subject. On occasion I pursue novelty virtually for its own sake: the traditional arguments are good and enlightening, but perhaps viewing in this different way will help.

Let me now turn, instead, to saying something about what is in these lectures. In my first approach to this project, I thought the survey would involve introducing mathematical objects, one at a time, that were equivalent to division algebras or included division algebras as a special case. As the project developed, this theme could not be stuck to, but some of it remains. Often we will introduce something because it is another way of looking at division algebras, or another connection of division algebras with other objects.

Section one introduces division algebras, and gives some examples. We say a little bit here about the connection to geometry, and introduce central simple algebras. The point here is that division algebras are central simple algebras without zero divisors. We find it easier, however, to generalize even further and consider so called Azumaya algebras. These are introduced and studied in section two. This perspective makes many concepts, like specialization, much easier. Moreover, we can then view central simple algebras as Azumaya algebras at a generic point. This is also a convenient place to introduce the concept of a separable commutative algebra, which is a tool we will need.

Section three has the definition and some properties of the Brauer group, which is in the field case just a group whose elements correspond to division algebras. Of particular note here is the discussion of splitting and its relation to subfields of division algebras. Azumaya, and hence division algebras can be viewed as so called forms of matrices, and this is explained and exploited in section four. Then in section five, we use the results of four to tackle the torsion question for Brauer groups, via an approach exploiting the symmetries of the tensor power.

To construct division algebras one often has to resort to using crossed products, and this is investigated in sections six and seven. Since we need the generality of Azumaya algebra crossed products, section six is devoted to Galois extensions of commutative rings. Also in this section is some material that has never appeared in print, that is, the Galois closure of an arbitrary separable extension of commutative rings of constant rank. Here again we exploit the symmetries of tensor powers. Section seven then studies crossed products proper. This allows the construction of some examples, and also the introduction of Galois cohomology to the study of Brauer groups and hence division algebras. Beginning in this section, and then again in sections 10 and 12, we make frequent use of group and Galois cohomology. There are a number of good sources for what we need, including [**B**]; the sections in [**CF**] by Atiyah, Gruenberg, and Wall; and [**Se**].

The corestriction map has become a vital tool in the study of division algebras, and we define this construction in section eight. But we do this in a different way, one which has also never appeared in print. The perspective of the Galois closure of section six is used to define the corestriction of an Azumaya algebra over a separable S/R of constant rank. This is more general, in one way, than the usual Galois cohomology definition because Galois cohomology is not general enough for all Azumaya algebras. On the other hand, our definition does not cover the corestriction over an inseparable field extension. Also in section eight is the little we say about algebraic K theory, the Merkurjev-Suslin Theorem and its consequences.

Section nine introduces maximal orders, which are, in the sphere appropriate to us, more general than Azumaya algebras and often the only way, because of ramification, to "globalize" a division algebra. Section ten defines the ramification map, and the associated unramified Brauer group. We also touch on, far too briefly, the use of ramification in some circumstances to find splitting fields and hence to prove theorems about division algebras. Let me repeat here a point made there. Right now there are some exciting new results in so called "two dimensional" situations, and some even more exciting possibilities.

In section eleven we introduce specialization and its seeming opposite lifting. Specialization is the usual connection between so called generic objects and the objects they are generic "for". We adopt the perspective that emphasizes specialization involving rational field extensions. Lifting provides a sort of converse, telling us something about the generic objects from the lifting properties of all the objects.

In section twelve we introduce so called lattice techniques to construct Galois field extensions with the properties required in section seven. We also explain how certain generic crossed products have non zero unramified Brauer group, which will contrast them with the center of the generic division algebra and hence will be one way to view the noncrossed product result for the generic division algebra.

Section thirteen is about Brauer-Severi varieties, and their associated generic splitting fields. Here we draw a connection to a variety of isomorphisms, and also to a class of polynomials defined from matrices. The material on these polynomials, appearing in 13.19 through 13.27, is perhaps new to the literature. We use these polynomials to describe, birationally, the Brauer-Severi variety and to prove some of the known results on the connection between birational equivalences of Brauer-Severi varieties and powers in the Brauer group. We use some language from algebraic geometry throughout these notes, but in this section we use such language extensively. We refer the reader to the standard reference [**H**], but in the text we refer the reader to [**GH**] and [**Ha**] also, where the Grassmann variety, in particular, appears.

Finally, section fourteen is about the generic division algebra. We give a fairly elementary proof of its construction, and its relation to linear invariants of PGL_n and multiplicative invariants of S_n. We show how the center of the generic division algebra is "close" to rational, and see how that seems to be related to the fact that the generic division algebra itself is sometimes not a crossed product.

There cannot be a doubt that these notes owe a lot to previous treatments of this subject. I first learned about division algebras from Albert's book [**Al**], and profited from the more recent books of Draxl [**Dr**], Pierce [**Pi**], and Kersten [**Ke**]. Of course I have learned an enormous amount from Jacobson, some of which the reader can learn from his book [**J6**]. This last book might be something of an antidote to these sometimes too sketchy notes, though there are considerable differences in choice of topics treated.

Finally, the reader will immediately notice that these notes often work on the level of Azumaya algebras, for which I frequently use [**DI**] as a source supplemented by [**KO**] and [**OS**]. The first two of these valuable books are old Springer Lecture Notes and in danger of effectively disappearing from easy use. I urge interested readers to insure they have access to copies that won't disappear.

Before finally ending this introduction, let me mention some notational oddities we will observe. If $\phi : R \to S$ is a homomorphism of commutative rings, one can use ϕ to make S an R module. But for the same S there can be many ϕ's and hence many distinct modules. Thus if M is an R module, we will write the tensor product of M and S as $M \otimes_\phi S$. If R is a commutative domain, we write the field of fractions of R as $q(R)$. For any ring R, R^* will denote the group of units.