CHAPTER 1

THE CONCEPT OF AN ANALYTIC FUNCTION

The theory of functions is concerned with complex-valued functions of a complex variable. Our study is confined to those functions which are *differentiable* in a sense which will be made precise later on; such functions are known as *analytic functions*. In order to create a basis for the theory, we begin by introducing the complex numbers in a manner which will lead us naturally to their interpretation as vectors in the plane.

§1. THE COMPLEX NUMBERS

1.1. Two-dimensional Vector Spaces

We begin by stating the axioms for a two-dimensional vector space over the real numbers.

Let there be given a set R, whose elements $a, b, \ldots, x, y, \ldots$ shall be called *points* or *vectors*, satisfying the following conditions.

I. To every two elements $a, b \in R$ there corresponds an element $c \in R$, known as their sum and written $c = a + b$, obeying the following rules:

I.1. $a + b = b + a$ (the commutative law).

I.2. $a + (b + c) = (a + b) + c$ (the associative law).

I.3. There is a zero in R, denoted $x = 0$, with the property that $a + 0 = a$ for every $a \in R$.

I.4. The equation $a + x = b$ has one and only one solution, $x = b - a \in R$.

II. To every vector a and every real number λ there corresponds a vector $b = \lambda a \in R$, known as their product, and obeying the following rules.

II.1. $\lambda(\mu a) = (\lambda\mu)a$ (λ, μ real numbers).

II.2. $(\lambda + \mu)a = \lambda a + \mu a$, $\lambda(a + b) = \lambda a + \lambda b$ (the distributive law).

II.3. $1 \cdot a = a$.

II.4. The product λa vanishes if and only if $\lambda = 0$, or $a = 0$, or both $\lambda = 0$ and $a = 0$.†

II.5. *The axiom of dimension:* there exist two vectors a and b in R which are linearly independent, that is, for which the equation

† We shall use the symbol 0 for the number zero as well as for the vector zero without, we trust, provoking any confusion.

$\lambda a + \mu b = 0$ has only the solution $\lambda = \mu = 0$, but every three vectors a, b, c in R are linearly dependent, that is, the equation $\lambda a + \mu b + \nu c = 0$ always has a solution such that at least one of the three numbers λ, μ, ν does not vanish.

This axiom asserts that the dimension of the vector space R is equal to *two*. In the resulting "affine plane" every vector x admits a representation in terms of its coordinates in a two-dimensional reference system. Such a system is given by a *basis* for R, that is, by two linearly independent vectors e_1, $e_2 \in R$. From II.5 it follows that every vector $x \in R$ has two numbers ξ_1 and ξ_2 associated with it (its coordinates in this reference system) such that

$$x = \xi_1 e_1 + \xi_2 e_2.$$

1.2. Plane Euclidean Geometry

Axioms I and II define a two-dimensional vector space whose geometry is the geometry of the affine plane. It becomes a Euclidean geometry once we introduce a (Euclidean) measure of length and angle. We can arrive at such a measure by defining, for any two vectors x and y in R, a scalar product (x, y) with the following properties.

III.1. (x, y) is a real, symmetric function of its arguments x and y: $(x, y) = (y, x)$.

III.2. (x, y) is linear in each argument.†

III.3. (x, y) is positive definite, that is, $(x, x) \geqq 0$, and equality holds only for $x = 0$.

The *length*, *norm*, or *modulus* $|x|$ of a vector x is defined by

$$|x| = +\sqrt{(x, x)}.$$

It is easily proved (Exercises 1 and 2)‡ that the following inequalities hold:

1) *Schwarz's inequality* $(x, y)^2 \leqq |x|^2 \, |y|^2$;

2) *The triangle inequality* $|x + y| \leqq |x| + |y|$.

The angle $[x, y]$ between two vectors x, y $(\neq 0)$ is defined by

$$\cos\,[x, y] = \frac{(x, y)}{|x| \, |y|}.$$

Two vectors are therefore orthogonal if $(x, y) = 0$.

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† A function $f(x)$ is said to be linear, if $f(\lambda x) = \lambda f(x)$ and $f(x_1 + x_2) = f(x_1) + f(x_2)$. The linearity of the scalar product (x, y) asserts, therefore, that this product obeys the distributive law.

‡ Unless there are indications to the contrary, the numbers will always refer to the exercises at the end of the chapter.

If e_1, e_2 is a basis for R and if the vectors x, y have the representations

$$x = \xi_1 e_1 + \xi_2 e_2, \qquad y = \eta_1 e_1 + \eta_2 e_2,$$

in terms of this basis, then

$$(x, y) = (\xi_1 e_1 + \xi_2 e_2, \eta_1 e_1 + \eta_2 e_2) = \sum_{i,k=1}^{2} g_{ik} \xi_i \eta_k,$$

where the g_{ik} denote the real constants

$$g_{ik} = (e_i, e_k) \qquad (g_{12} = g_{21}).$$

The square of the norm of x is the quadratic form

$$|x|^2 = (x, x) = \sum_{i,k} g_{ik} \xi_i \xi_k = g_{11} \xi_1^2 + 2g_{12} \xi_1 \xi_2 + g_{22} \xi_2^2.$$

It reduces to the Pythagorean form

$$|x|^2 = \xi_1^2 + \xi_2^2$$

if and only if the coordinate system is orthonormal; that is,

$$(e_1, e_2) = 0, \qquad |e_1| = |e_2| = 1$$

(the Cartesian coordinate system).

1.3. Extension of the Set *R* to a Vector Algebra

In what follows, we shall not introduce a metric into the plane R for the time being, so that we shall be dealing with an affine geometry on R defined by the postulates in groups I and II. The problem before us is to see whether it is possible to extend I and II so as to give R the structure of a field (or algebra), and, if this is possible, to discover in how many different ways it can be done.

The vector space R becomes an algebra once we are able to define, for any two elements $x, y \in R$, a "product"

$$z = xy \in R$$

which satisfies the following axioms.

IV.1. The product is commutative: $xy = yx$.

IV.2. The product is bilinear, that is, linear in each factor.

IV.3. The product is associative: $x(yz) = (xy)z$.

IV.4. The product xy vanishes, $xy = 0$, if and only if at least one factor vanishes.

1.4.

Our task, then, is *to find all bilinear forms* $xy \in R$ *which satisfy these axioms IV*. In order to arrive at the general solution to this problem, we shall assume, at first, that we already have a product xy defined on R in accordance with axioms IV and see what this tells us.

If we fix the vector $y \neq 0$ in the product $z = xy$, we obtain a linear transformation in x which maps the plane R into itself. This mapping is one-to-one, for if $z_1 = x_1 y$, $z_2 = x_2 y$, then

$$z_1 - z_2 = (x_1 - x_2)y.$$

Since y was assumed to be different from 0, $z_1 - z_2$ will vanish if and only if $x_1 - x_2 = 0$. Different vectors x therefore have (for each fixed $y \neq 0$) different image vectors $z = xy$.

On the other hand, the range of the mapping $z = xy$ is the whole plane R. For, if x_1 and x_2 are two vectors in R, and λ_1 and λ_2 are two arbitrary real numbers, then

$$(\lambda_1 x_1 + \lambda_2 x_2)y = \lambda_1 x_1 y + \lambda_2 x_2 y = \lambda_1 z_1 + \lambda_2 z_2,$$

where $z_1 = x_1 y$, $z_2 = x_2 y$. From this we see that the image vectors z_1, z_2 are linearly independent if and only if the vectors x_1, x_2 are linearly independent. Hence, if x_1, x_2 is a basis for R, then z_1, z_2 will also be a basis. If the vector x has the coordinates λ_1, λ_2 in the system (x_1, x_2), then its image vector has the same coordinates in the system (z_1, z_2), for $z = xy = \lambda_1 z_1 + \lambda_2 z_2$. Hence, the set of image vectors $z = xy$ covers the plane R exactly once if x runs through all values in R (for y fixed).

Thus, for any given vector $y \neq 0$, there is precisely one vector x which makes the product xy take a prescribed value z; that element is the "quotient" $x = z/y$.

1.5. Definition of the Unit Vector *e*

If, in particular, we take $z = y(\neq 0)$, then there is a definite vector $e = e_y \in R$ having the property that $e_y y = y$. We shall show that e_y is *independent* of the choice of y. Let y_1 and y_2 be two non-zero vectors. If $e_1 y_1 = y_1$, $e_2 y_2 = y_2$, then

$$e_2 y_2 = y_2 = y_1 \frac{y_2}{y_1} = (e_1 y_1) \frac{y_2}{y_1},$$

and this last expression is, by axiom IV.3 (the associative law), equal to $e_1(y_1 y_2/y_1) = e_1 y_2$. Hence, $e_2 y_2 = e_1 y_2$, or $(e_1 - e_2)y_2 = 0$, from which it follows that $e_1 = e_2$, since $y_2 \neq 0$.

The element $e\ (\neq 0)$ defined uniquely by the equation

$$ey = ye = y \tag{1.1}$$

is called the *unit vector*, or *unit*, in R.

1.6. Definition of the Vector *i*

Let a be an arbitrary vector in R and consider the equation

$$x^2 = a.$$

If this equation has a solution $x = x_1$, $x_1^2 = a$, then, for every vector $x \in R$, we have

$$x^2 - a = x^2 - x_1^2 = (x - x_1)(x + x_1),$$

so that the equation $x^2 - a = 0$ has, in addition to $x = x_1$, one further solution $x = -x_1$.

Let us choose $a = -e$ and solve the equation

$$x^2 + e = 0. \tag{1.2}$$

The existence of a solution will be shown in an exercise (Exercise 3). We denote the solutions by $x = \pm i$ ($\neq 0$). The vector i is linearly independent of the vector e, for, if $i = \lambda e$ (λ real), then we would have $-e = i^2 = (\lambda e)^2 = \lambda^2 e^2 = \lambda^2 e$, or $(1 + \lambda^2)e = 0$, which is impossible, since both $1 + \lambda^2 \neq 0$ and $e \neq 0$.

The vectors $x = e$ and $x = i$ span the entire plane R. An arbitrary vector $x \in R$ has the coordinate representation

$$x = \xi e + \eta i.$$

This representation has been found under the assumption that there is a product, defined for pairs of vectors $x_1, x_2 \in R$, which satisfies the axioms IV. If x_1 and x_2 are written in terms of coordinates,

$$x_1 = \xi_1 e + \eta_1 i, \qquad x_2 = \xi_2 e + \eta_2 i,$$

it follows from IV and the definition of the basis vectors e and i via (1.1) and (1.2) that the product $x_1 x_2$ must have the form

$$\begin{aligned} x_1 x_2 &= (\xi_1 e + \eta_1 i)(\xi_2 e + \eta_2 i) \\ &= (\xi_1 \xi_2 - \eta_1 \eta_2)e + (\xi_1 \eta_2 + \eta_1 \xi_2)i. \end{aligned} \tag{1.3}$$

The quotient x_1/x_2 ($x_2 \neq 0$) is defined to be that vector $x = \xi e + \eta i$ which, when multiplied by the vector $x_2 = \xi_2 e + \eta_2 i$, yields the vector $x_1 = \xi_1 e + \eta_1 i$. With the aid of (1.3), we can obtain the coordinates ξ, η of x from the equations

$$\xi_2 \xi - \eta_2 \eta = \xi_1, \qquad \eta_2 \xi + \xi_2 \eta = \eta_1.$$

Therefore the quotient x_1/x_2 is given by the expression

$$\frac{x_1}{x_2} = \frac{\xi_1 e + \eta_1 i}{\xi_2 e + \eta_2 i} = \frac{\xi_1 \xi_2 + \eta_1 \eta_2}{\xi_2^2 + \eta_2^2} e + \frac{\eta_1 \xi_2 - \xi_1 \eta_2}{\xi_2^2 + \eta_2^2} i. \tag{1.4}$$

1.7. The Solution of the Extension Problem

We now turn all this around and choose any two linearly independent vectors in R, label them e and i, and *define* the product $x_1 x_2$ of two vectors $x_j = \xi_j e + \eta_j i$ ($j = 1, 2$) by means of Eq. (1.3). We shall then have $ex = xe = x$ and $i^2 + e = 0$, and all the axioms IV will be satisfied. The verification of axioms IV.1–3 we leave to the reader. To prove IV.4, we observe that the equation $x_1 x_2 = 0$ is equivalent, by (1.3), to the coordinate equations

$$\xi_1\xi_2 - \eta_1\eta_2 = 0, \qquad \xi_1\eta_2 + \eta_1\xi_2 = 0.$$

Squaring and then adding, we obtain

$$(\xi_1^2 + \eta_1^2)(\xi_2^2 + \eta_2^2) = 0.$$

Consequently, $\xi_1 = \eta_1 = 0$ or $\xi_2 = \eta_2 = 0$, that is, $x_1 = 0$ or $x_2 = 0$ (or $x_1 = x_2 = 0$), as required by axiom IV.4.

We have, therefore, completely solved the problem before us:

If the vectors e and i are any two arbitrarily chosen linearly independent vectors, then (1.3) *furnishes a definition for the product of two vectors in R which makes R into a field* (or algebra, that is, a vector space which satisfies the axioms IV), *and this definition of the product is the only one that is compatible with all the axioms.*

1.8. Notation for Complex Numbers. Absolute Value and Argument

Having made R into a field in which every vector, or *complex number*, can be written as $\xi_1 e + \xi_2 i$, we want to say something about notation. Vectors ξe (ξ real) along the e-axis we shall denote, for brevity, by ξ alone, by dropping the e. In view of the property $xe = ex = x$ which defines the unit, this can hardly lead to confusion. Furthermore, in keeping with a long-standing custom we shall denote the coordinates of a complex number $z = \xi_1 e + \xi_2 i = \xi_1 + \xi_2 i$ by $\xi_1 = x$, $\xi_2 = y$, and write

$$z = x + iy.$$

The real number x is called the *real part* of z, and the real number y is called the *imaginary part* of z. These terms can be abbreviated to

$$x = \operatorname{Re} z, \qquad y = \operatorname{Im} z.$$

We now introduce a Euclidean metric into the "complex plane" R by defining the scalar product (z_1, z_2) of two complex numbers $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ as

$$(z_1, z_2) = x_1x_2 + y_1y_2.$$

This means that the basis vectors e and i are orthogonal to one another, and that their lengths are one: $|e| = |i| = 1$.

The *modulus* or *absolute value* of a complex number $z = x + iy$ is then given by

$$|z| = +\sqrt{(z, z)} = +\sqrt{x^2 + y^2}.$$

If we go over to polar coordinates, we get

$$z = r(\cos\phi + i\sin\phi),$$

where $r = |z|$, $\phi = \arctan y/x$. The quantity ϕ is called the *argument* of z:

$$\arg z = \phi = \arctan y/x.$$

As long as $z \neq 0$, ϕ is defined up to a multiple of 2π (we say "modulo 2π," and write "mod 2π").

In this notation, the product of two complex numbers

$$z_k = r_k(\cos \phi_k + i \sin \phi_k) \ (k = 1, 2)$$

is

$$z_1 z_2 = r_1 r_2\{\cos (\phi_1 + \phi_2) + i \sin (\phi_1 + \phi_2)\}.$$

From this it follows that

The absolute value of the product of two complex numbers is equal to the product of their absolute values, while the argument of the product is equal to the sum (mod 2π) *of the arguments of the factors:*

$$|z_1 z_2| = |z_1|\,|z_2|, \qquad \arg (z_1 z_2) = \arg z_1 + \arg z_2 \ (\mathrm{mod}\ 2\pi).$$

The latter rule presupposes that the factors are different from zero, since the argument of the number $z = 0$ is indeterminate.

From the product rule it follows that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \qquad \arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2 \ (\mathrm{mod}\ 2\pi).$$

If all n factors, $z = r(\cos \phi + i \sin \phi)$, of a product are equal we obtain

$$[r(\cos \phi + i \sin \phi)]^n = r^n(\cos n\phi + i \sin n\phi).$$

This yields as a special case, for $r = 1$, *de Moivre's formula*

$$(\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi. \tag{1.5}$$

The numbers $x + iy$ and $x - iy$ are said to be *complex conjugates*. The complex conjugate of the complex number z is denoted by $\bar{z}$; obviously,

$$z\bar{z} = |z|^2; \qquad \mathrm{Re}\, z = \frac{1}{2}(z + \bar{z}), \qquad \mathrm{Im}\, z = \frac{1}{2i}(z - \bar{z}).$$

Geometrically speaking, the addition of complex numbers corresponds to vector addition (according to the parallelogram rule). The difference $z_1 - z_2$ corresponds to a vector whose initial point is at z_2 and whose end-point is at z_1. The modulus of the difference $|z_1 - z_2|$ gives the distance between the points z_1 and z_2.

Since the complex numbers form an algebra (axioms I–IV), the rational operations of arithmetic (addition, subtraction, multiplication, and division) obey the same rules as in the real case. Over and beyond this, the defining equation $i^2 = -1$ must be taken into account.

§2. POINT SETS IN THE COMPLEX PLANE

1.9. Convergent Sequences

A sequence of complex numbers

$$z_1, z_2, \ldots, z_n, \ldots \tag{1.6}$$

tends to a limit,

$$\lim_{n\to\infty} z_n = z, \tag{1.7}$$

if, to any arbitrarily prescribed number $\epsilon > 0$, *a number* $n_\epsilon > 0$ *can be found such that*

$$|z_n - z| < \epsilon \qquad \textit{for} \qquad n \geqq n_\epsilon. \tag{1.8}$$

The condition (1.8) says, geometrically, that all the points z_n ($n \geqq n_\epsilon$) lie in a circle about z with radius ϵ.

Let $z = x + iy$, $z_n = x_n + iy_n$.

Then the conditions

$$\lim_{n\to\infty} x_n = x \qquad \textit{and} \qquad \lim_{n\to\infty} y_n = y \tag{1.9}$$

are necessary and sufficient for (1.7) *to hold.*

The necessity of the condition (1.9) follows immediately from the inequalities

$$|x_n - x| \leqq |z_n - z|, \qquad |y_n - y| \leqq |z_n - z|.$$

Conversely, if (1.9) is fulfilled, then there exists a number N with the property that

$$|x_n - x| < \frac{\epsilon}{2} \qquad \text{and} \qquad |y_n - y| < \frac{\epsilon}{2} \qquad \text{for} \qquad n \geqq N.$$

Consequently, for all $n \geqq N$ we have

$$|z_n - z| = \sqrt{(x_n - x)^2 + (y_n - y)^2} < \frac{\epsilon}{\sqrt{2}} < \epsilon,$$

which shows that the condition (1.9) is also sufficient. The following theorem is also easy to prove (Exercise 16):

If $z \neq 0$, *the conditions*

$$\lim_{n\to\infty} |z_n| = |z| \qquad \text{and} \qquad \lim_{n\to\infty} \arg z_n = \arg z \ (\operatorname{mod} 2\pi)$$

are necessary and sufficient for the validity of (1.7).

When the sequence z_n ($n = 1, 2, \ldots$) is such that

$$\lim_{n\to\infty} |z_n| = \infty,$$

we say that the sequence tends to ∞ and write simply

$$\lim_{n\to\infty} z_n = \infty.$$

This limit, ∞, is taken as a point, the *point at infinity*, of the complex plane. The plane, completed by the single point at infinity, is called the *extended*, or *closed*, plane.† In many questions, the point $z = \infty$ has an equal status with the finite points of the plane (cf. Section 3.13).

1.10. The Topology of the Complex Plane

The set of points z belonging to the interior of a disk of radius r with center at the point $z_0 = x_0 + iy_0 \neq \infty$:

$$K_r: \qquad |z - z_0| < r$$

is called a *circular neighborhood* of z_0. A circular neighborhood of the point $z = \infty$ will be taken to mean the set of points which lie outside some circle of radius r about the origin: $|z| > r$.

A set of points $\{z\}$ in the extended plane $|z| \leqq \infty$ is said to be *open* if each of its points is the center of some circular neighborhood which belongs entirely to the set.

An open set of points $\{z\}$ in the extended plane $|z| \leqq \infty$ forms a *domain* if it is possible to join any two points in $\{z\}$ by a polygonal path which lies entirely in $\{z\}$. (This condition makes the open set *connected.*)

Any domain containing the point z_0 is called a *neighborhood* of z_0.

A point a is called a *cluster point* (or sometimes, a *limit point* or *accumulation point*) of a set of complex numbers $\{z\}$ if every circular neighborhood of a contains at least one point $z \neq a$ of $\{z\}$. From this it follows that every neighborhood of a cluster point a must contain infinitely many points of the set.

If a set contains all of its cluster points, the set is said to be *closed.*

The set of points $|z| < \infty$ is open. The extended plane $|z| \leqq \infty$ is both open and closed (Exercise 18). Open sets and closed sets are important particular classes of sets, but an arbitrary set of points is, in general, neither open nor closed.

A closed set which cannot be split into two disjoint closed subsets is called a *continuum.*

A set of points $\{z\}$ is said to be *compact* if any infinite subset of it has a cluster point belonging to $\{z\}$. (A compact set is therefore closed.) The closed plane $|z| \leqq \infty$ is compact.

The set of points in the plane which do not belong to a given set $\{z\}$ forms what is called the *complement* of $\{z\}$. The complement of an open set is closed, and the complement of a closed set is open (Exercise 20).

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† The finite plane can also be extended in other ways. For example, in projective geometry, there is the so-called line at infinity with its infinitely many, infinitely distant points.

Let G be a domain. If G does not contain every point of $|z| \leqq \infty$, then the points of the complement of G fall into two classes:

a) *Boundary points of G.* These do not belong to G, but are cluster points of G. The set of boundary points forms the *boundary* of G.

b) *Exterior points of G.* These are points which belong neither to G nor to the boundary of G. This set can be empty.

If Γ is the boundary of the domain G, then the union $G \cup \Gamma$ (that is, the set of all points which belong either to G or Γ) is a closed set (Exercise 21). It is called the *closure* of the domain G.

The union of a domain and its boundary is also called a *closed domain.*

§3. FUNCTIONS OF A COMPLEX VARIABLE

1.11. Definition of a Function. Continuity

Functions of a complex variable are defined in the same manner as functions of a real variable:

If to every value z in a domain G there corresponds a definite complex value w, then the mapping $f: z \rightarrow w$ is said to be a function defined in the domain G.

The number $w = f(z)$ is called the value of the function at the point z.

In what follows we shall consider first only those functions which assume *finite* values in a *finite* domain, that is, a domain belonging to the finite plane $|z| < \infty$.

The real and imaginary parts, u and v, of the function $f(z)$ are real functions of the real variables x and y $(z = x + iy)$:

$$u = u(x, y), \qquad v = v(x, y).$$

Conversely, any two such functions always define one complex function $f(z) = u + iv$ of $z = x + iy$.

Continuity is defined in the same way as in the real case:

A function $w = f(z)$ is continuous at the point z if, to every positive number ϵ, there corresponds a positive number ρ_ϵ, such that

$$|f(z + \Delta z) - f(z)| < \epsilon \qquad \textit{whenever} \qquad |\Delta z| < \rho_\epsilon.$$

Geometrically speaking, the continuity of a function $w = f(z)$ at $z = z_0$ means this: to an arbitrarily small disk K_w centered at $w_0 = f(z_0)$ there corresponds a disk K_z about z_0 with the property that $w = f(z)$ lies in K_w whenever z lies in K_z.

The limit of a function is defined in the same way as the limit of a sequence in Section 1.9. Everything that was said there about the limit of a sequence of complex numbers applies here as well. Combining the definitions of continuity and of a limit we can say that a function $f(z)$ is continuous at a point z if

$$\lim_{\Delta z \to 0} f(z + \Delta z) = f(z).$$

The real and imaginary parts of a continuous function are obviously continuous functions themselves, and conversely. The same holds for the absolute value and the argument of a continuous function, provided that the function does not vanish $\big(f(z) \neq 0\big)$.

1.12. Differentiable Functions

The derivative of a function of a complex variable is defined in the same way as in the real case.

Let $f(z)$ be a function defined in a neighborhood of the point z. If the difference quotient

$$\frac{\Delta f}{\Delta z} = \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

tends to a finite limit A whenever Δz tends to zero, then the number A is called the derivative of f at the point z and is denoted by

$$f'(z) = A = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}. \tag{1.10}$$

The difference $\epsilon = \Delta f / \Delta z - A$ therefore tends to zero with Δz. Hence, Δf has the expansion

$$\Delta f = A \Delta z + \Delta z \, \epsilon = A \Delta z + \Delta z (\Delta z), \tag{1.10$'$}$$

where (Δz) denotes a number which tends to zero as $\Delta z \to 0$.

If, on the other hand, there exists a constant A such that (1.10)′ is satisfied, then $\Delta f / \Delta z = A + (\Delta z)$. Accordingly, the difference quotient $\Delta f / \Delta z$ tends to the limit A as $\Delta z \to 0$. Hence, A is the derivative $f'(z)$ of the function $f(z)$.

Equations (1.10) and (1.10)′ are therefore equivalent.

The expansion (1.10)′ expresses the fact that the function f is *differentiable* at the point z. The first term is the *differential*

$$df = f'(z) \Delta z$$

corresponding to the increment Δz. The total increment Δf of the function f is obtained by adding to the differential df the remainder term $\Delta z(\Delta z)$.

If $f'(z) \neq 0$, this remainder term is negligible in comparison with df as $\Delta z \to 0$. Therefore, for small values of $|\Delta z|$, the differential df is a good approximation to the increment Δf: the ratio of

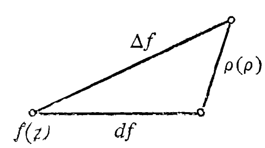

Figure 1

the length of the vector $\rho(\rho)$ (where $\rho = |\Delta z|$) to $|df|$ tends to zero with Δz.

If, in particular, we choose $f(z) \equiv z$, then $dz = \Delta z$. Accordingly, we can write $df = f'(z)\, dz$. The derivative f' is therefore the quotient† of two differentials df and dz:

$$\frac{df}{dz} = f'(z).$$

(This notation is due to Leibniz.)

1.13. The Cauchy-Riemann Differential Equations

We shall now investigate what form the equations that define the derivative (1.10, 1.10′) assume when we separate the variables into their real and imaginary parts:

$$z = x + iy, \qquad f(z) = u(x, y) + i\, v(x, y).$$

Accordingly, $\Delta z = \Delta x + i\Delta y$ and

$$u(x + \Delta x, y + \Delta y) - u(x, y) = \Delta u,$$
$$v(x + \Delta x, y + \Delta y) - v(x, y) = \Delta v.$$

We also write

$$A = \alpha + i\beta, \qquad \Delta z(\Delta z) = \rho\big((\rho)_1 + i(\rho)_2\big),$$

where $(\rho)_1$ and $(\rho)_2$ denote real numbers which tend to zero with

$$\rho = |\Delta z| = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

If the function f is differentiable, then, by (1.10)′,

$$\Delta u + i\Delta v = (\alpha + i\beta)(\Delta x + i\Delta y) + \rho\big((\rho)_1 + i(\rho)_2\big).$$

Comparison of the real and imaginary parts yields

$$\Delta u = \alpha\Delta x - \beta\Delta y + \rho(\rho)_1, \qquad \Delta v = \beta\Delta x + \alpha\Delta y + \rho(\rho)_2.$$

These formulas say that $u(x, y)$ and $v(x, y)$, considered as functions of the real variables x and y, are differentiable at the point (x, y). If we set $\Delta y = 0$ (so that now $\rho = |\Delta x|$) and then divide by Δx, the first equation yields

$$\frac{\Delta u}{\Delta x} = \alpha + \frac{|\Delta x|}{\Delta x}(\Delta x) \to \alpha \qquad \text{as} \qquad \Delta x \to 0.$$

The function $u(x, y)$ therefore possesses the partial derivative $u_x = \alpha$.

† The differentials $dz = \Delta z$ and $df = f'\, dz$ are *finite* (and not "infinitesimal") complex quantities. On the other hand, as we have already seen, the differential df approximates Δf better and better, the smaller the modulus of $\Delta z = dz$ is.

In a similar way, we can prove that $u_y = -\beta$, $v_x = \beta$ and $v_y = \alpha$. We thus have the following result.

If the function $f(z) = u + iv$ is differentiable, then the functions $u(x, y)$ and $v(x, y)$ are also differentiable. The partial derivatives are related by the equations

$$u_x = v_y, \qquad u_y = -v_x. \tag{1.11}$$

These partial differential equations, which relate the real and imaginary parts of a differentiable function, are known as the *Cauchy-Riemann differential equations.*

The converse of this result is also true.

If $u(x, y)$ and $v(x, y)$ are differentiable functions of x and y, and if their partial derivatives u_x, u_y, v_x, v_y satisfy the Cauchy-Riemann differential equations, then the complex function

$$f(z) = u(x, y) + iv(x, y)$$

is differentiable with respect to the variable $z = x + iy$. Its derivative, $f'(z)$, is

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

Proof. The assumption that u and v are differentiable at the point (x, y) means that they have the expansions

$$\Delta u = u_x\Delta x + u_y\Delta y + \rho(\rho)_1, \qquad \Delta v = v_x\Delta x + v_y\Delta y + \rho(\rho)_2. \tag{1.12}$$

Here, the quantities $(\rho)_1$ and $(\rho)_2$ vanish when $\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ tends to 0. If we multiply the second equation in (1.12) by i and add it to the first equation, we get

$$\Delta f = \Delta u + i\Delta v = (u_x + iv_x)\Delta x + (u_y + iv_y)\Delta y + \rho(\rho),$$

where $(\rho) \equiv (\rho)_1 + i(\rho)_2 \to 0$ as $\rho \to 0$. Since, by hypothesis, the Cauchy-Riemann equations (1.11) are satisfied, we can write

$$\begin{aligned}\Delta f &= (u_x + iv_x)\Delta x + (-v_x + iu_x)\Delta y + \rho(\rho)\\ &= (u_x + iv_x)(\Delta x + i\Delta y) + \rho(\rho).\end{aligned}$$

But, according to (1.10)′, this means that $f(z)$ is differentiable and has the derivative $f'(z) = u_x + iv_x = v_y - iu_y$.

Remark. If a function $u(x, y)$ of two real variables is differentiable at a point (x, y):

$$\Delta u = \alpha\Delta x + \beta\Delta y + \rho(\rho) \quad ((\rho) \to 0 \quad \text{as} \quad \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2} \to 0),$$

then, as we pointed out above, it possesses partial derivatives $u_x = \alpha$ and $u_y = \beta$ at this point. We know from the theory of real functions of several variables that, conversely, the mere existence of the derivatives u_x and u_y at the point (x, y) in no way implies the differentiability of the function $u(x, y)$ at this point. If, however, we make the additional hypothesis that the partial derivatives u_x and u_y also exist in a neighborhood of the point (x, y) and are *continuous* at this point, then one can conclude that u is differentiable at the point. (We use the mean-value theorem; cf. Exercises 27 and 28.)

1.14. The Definition of an Analytic Function

If $f(z)$ is defined in a finite domain G, and is differentiable in z at each point of G, then $f(z)$ is said to be an analytic function in G.

A function is said to be *analytic at a point* if it is analytic in a neighborhood of the point.

The definition of an analytic function in the domain G says that at every point of G this function possesses a finite derivative

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}.$$

This definition of an analytic function was given by the founder of the theory of complex functions, Augustin Cauchy (1789–1857). From what we have just discussed, it is equivalent to the following definition, given by Bernhard Riemann (1826–1866).

A function $f(z) = u(x, y) + i\,v(x, y)$ is analytic in a domain G if the functions $u(x, y)$ and $v(x, y)$ are differentiable throughout G and the Cauchy-Riemann differential equations

$$u_x = v_y, \qquad u_y = -v_x$$

are satisfied.

Later on, we shall show that still more follows from these assumptions, namely, that an analytic function is *continuously* differentiable (that is, it possesses a continuous derivative), and the same holds for its real and imaginary parts. In their original definition of an analytic function, the founders of complex function theory, Cauchy and Riemann, required the continuity of the derivatives. That this property already follows from the differentiability assumption was proved only at a later date (Édouard Goursat, 1900).

We shall see, in fact, that the existence of *all* the higher derivatives follows from the analyticity of a function. Despite its apparent simplicity, then, the definition of an analytic function constitutes an enormous restriction compared with the general definition of a complex function.

The real and imaginary parts u and v of an analytic function have continuous partial derivatives of all orders, as we shall prove later on. From the Cauchy-Riemann differential equations (1.11) it follows that u and v satisfy Laplace's equation

$$\Delta U \equiv \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

(cf. Exercise 32). Functions of this sort are called *harmonic* functions.

Two harmonic functions which are related by the Cauchy-Riemann equations are said to be *harmonic conjugates*. The real part and the imaginary part of an analytic function are therefore conjugate harmonic functions.

1.15. The Rules of Differentiation

The derivative of a complex function has been defined in precisely the same way as the corresponding notion in the case of a function of a single real variable. The definition is based upon the rational operations of arithmetic and the notion of limit, both of which remain unchanged when we go over from the real domain to the complex domain. Accordingly, all the rules of differential calculus (for example, for the derivative of a sum, or a product, or a quotient) remain intact and can be applied, without further ado, to complex functions.

The same also holds for the *composition* of functions. Suppose that the function $w = w(z)$ is analytic in the domain G_z of the z-plane and that its values lie in a domain G_w of the w-plane. If, further, $\zeta = \zeta(w)$ is an analytic function of w in the domain G_w, then the composite function $z \to w \to \zeta$

$$\zeta = \zeta(w(z)) \equiv f(z)$$

is analytic in G_z.

To prove this, it is only necessary to show that f is a differentiable function of z. We leave this to the reader as an exercise (Exercise 29). At the same time, we note that the chain rule for the derivative of a composite function holds in the complex domain:

$$\frac{d\zeta}{dz} = f'(z) = \frac{d\zeta}{dw}\frac{dw}{dz}.$$

1.16. Conformal Mapping by Analytic Functions

Suppose that the function $w = w(z)$ is continuous in a neighborhood of the point z_0 and is differentiable at z_0. Suppose, further, that $w'(z_0) \neq 0$.

If the point z moves along the ray $\arg(z - z_0) = \phi$ (= const.), then its image $w(z) = w(z_0) + \Delta w$ moves along a well-defined curve γ_ϕ which starts at $w_0 = w(z_0)$ and is such that

$$\frac{\Delta w}{\Delta z} = w'(z_0) + (\Delta z) = w'(z_0)[1 + \epsilon(\Delta z)],$$

where $\epsilon(\Delta z) = (\Delta z)/w'(z_0) \to 0$ as $\Delta z \to 0$. Therefore, as $\Delta z \to 0$, we have

$$\frac{|\Delta w|}{|\Delta z|} \to |w'(z_0)|, \qquad \arg\frac{\Delta w}{\Delta z} = \arg \Delta w - \phi \to \arg w'(z_0). \tag{1.13}$$

From the second formula in (1.13) it follows that γ_ϕ possesses a tangent at the point w_0 which makes an angle

$$\psi = \phi + \arg w'(z_0)$$

with the real axis.

If we take two rays, $\arg (z - z_0) = \phi_\nu$ $(\nu = 1, 2)$, then the angle between the tangents to their image curves at the point w_0 turns out to be

$$\psi_2 - \psi_1 = \phi_2 - \phi_1.$$

Under the mapping from the z-plane into the w-plane this angle therefore remains fixed. If the angle of inclination of the ray in the z-plane increases, then the angle of inclination of the tangent to the image curve also increases: the orientation is preserved.

From the first formula in (1.13), we conclude that the ratio of corresponding distances $|\Delta w|$ and $|\Delta z|$ tends to the same limit $|w'(z_0)|$ independently of the direction chosen for the vector Δz.

Geometrically speaking, this means that the mapping of the z-plane into the w-plane resembles a similarity mapping in a neighborhood centered at z_0, and the resemblance becomes stronger as the size of the neighborhood shrinks.

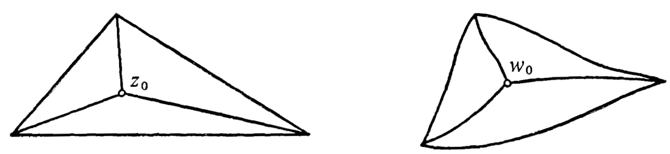

Figure 2

On the basis of these considerations we make the following definition:

If the function $w = w(z)$ has a non-vanishing derivative at the point z_0, then the mapping $z \to w$ is called *conformal* at the point z_0.

This definition is equivalent to the existence of the following (finite) limits:

$$\lim_{\Delta z \to 0} \arg \frac{\Delta w}{\Delta z}, \tag{a}$$

$$\lim_{\Delta z \to 0} \frac{|\Delta w|}{|\Delta z|} \neq 0, \tag{b}$$

where $\Delta w = w(z_0 + \Delta z) - w(z_0)$.

1.17.

In order to enlarge upon the foregoing considerations, we shall separate the complex function $w(z)$ into its real and imaginary parts. We shall assume that the real and imaginary parts of the function

$$w = w(z) = u(x, y) + i\, v(x, y)$$

are differentiable at the point z_0, and that the functional determinant is positive,

$$\frac{\partial(u, v)}{\partial(x, y)} = u_x v_y - u_y v_x > 0 \tag{1.14}$$

at the point z_0.

If, in addition, one of the two limits (a) and (b) mentioned in Section 1.16 exists, then $w(z)$ is differentiable at the point z_0 and $w'(z_0) \neq 0$.

To prove this, we shall first assume the existence of the limit (a). If we write $\Delta z = dz = dx + i dy \neq 0$ and $\Delta w = \Delta u + i\Delta v$, then

$$\lim_{\Delta z \to 0} \arg \frac{\Delta w}{\Delta z} = \operatorname{arc\,tan} \frac{dv\, dx - du\, dy}{du\, dx + dv\, dy},$$

since, by (1.14), du and dv do not both vanish. This limit and therefore also the expression

$$\frac{dv\, dx - du\, dy}{du\, dx + dv\, dy} = \frac{v_x\, dx^2 + (v_y - u_x)\, dx\, dy - u_y\, dy^2}{u_x\, dx^2 + (u_y + v_x)\, dx\, dy + v_y\, dy^2}$$

is independent of dx and dy. From this it follows that the coefficients of the quadratic forms occurring in the numerator and denominator are proportional,

$$v_x = \lambda u_x, \qquad u_y = -\lambda v_y, \qquad v_y - u_x = \lambda(u_y + v_x),$$

where λ depends only upon z_0. Consequently,

$$(1 + \lambda^2)(v_y - u_x) = 0$$

and so

$$u_x = v_y, \qquad u_y = -v_x.$$

The Cauchy-Riemann equations therefore hold, and the existence of $w'(z_0) \neq 0$ follows from (1.14) and Section 1.13.

We shall now prove the same result from the existence of the limit (b) instead of (a). (The hypotheses made at the beginning of this section are still assumed to be in force.)

The expression

$$\lim_{\Delta z \to 0} \frac{|\Delta w|^2}{|\Delta z|^2} = \frac{du^2 + dv^2}{dx^2 + dy^2}$$
$$= \frac{(u_x^2 + v_x^2)\, dx^2 + 2(u_x u_y + v_x v_y)\, dx\, dy + (u_y^2 + v_y^2)\, dy^2}{dx^2 + dy^2} \; (\neq 0)$$

is independent of dx and dy. From this it follows, as in the first case, that

$$u_x u_y + v_x v_y = 0, \qquad u_x^2 + v_x^2 = u_y^2 + v_y^2 > 0, \tag{1.15}$$

and, consequently,

$$\frac{u_x}{v_y} = \frac{-v_x}{u_y} = \mu \qquad \text{or} \qquad u_x = \mu v_y, \qquad -v_x = \mu u_y.$$

From the second equation in (1.15), we obtain $\mu = \pm 1$. Because of assumption (1.14), $\mu = 1$, and the Cauchy-Riemann equations hold. From this we conclude, as above, that the derivative $w'(z_0) \neq 0$ exists.

Remark. According to the foregoing proof, it follows from the existence of the limit (b) that $\mu = +1$ or $\mu = -1$, depending upon whether the functional determinant $u_x v_y - u_y v_x$ is positive or negative. In the latter case, the mapping $z \to w$ is indirectly conformal at the point z_0 (Exercise 39).

EXERCISES ON CHAPTER 1

1. Prove Schwarz's inequality $(x, y)^2 \leqq |x|^2 \, |y|^2$ (Section 1.2). Under what conditions does equality hold?

Hint. $|\lambda x + \mu y|^2 = (\lambda x + \mu y, \lambda x + \mu y) = \lambda^2(x, x) + 2\lambda\mu(x, y) + \mu^2(y, y)$ is positive definite with respect to λ and μ (or it vanishes identically).

2. Prove the triangle inequality (Section 1.2):

$$\big||z_1| - |z_2|\big| \leqq |z_1 + z_2| \leqq |z_1| + |z_2|.$$

Under what conditions does equality hold? Generalize the inequality on the right to n complex numbers.

3. Prove, with the aid of axioms IV, Section 1.3, that the equation

$$x^2 + e = 0 \tag{1.2}$$

possesses a solution. Here e is that vector for which $ex = xe = x$ holds for all values of x.

Hint. Take as a basis for the x-plane the vector e and a vector f, linearly independent of e. Let

$$x = \alpha_1 e + \alpha_2 f, \qquad y = \beta_1 e + \beta_2 f, \qquad f^2 = \gamma_1 e + \gamma_2 f$$

be the coordinate representation of two arbitrary vectors x and y and of the vector f^2. Then

$$\begin{aligned} xy &= \alpha_1\beta_1 e + (\alpha_1\beta_2 + \alpha_2\beta_1)f + \alpha_2\beta_2 f^2 \\ &= (\alpha_1\beta_1 + \gamma_1\alpha_2\beta_2)e + (\alpha_1\beta_2 + \alpha_2\beta_1 + \gamma_2\alpha_2\beta_2)f. \end{aligned}$$

Since $xy = 0$ only if $x = 0$ or $y = 0$, the system of equations

$$\alpha_1\beta_1 + \gamma_1\alpha_2\beta_2 = 0, \qquad \alpha_2\beta_1 + (\alpha_1 + \gamma_2\alpha_2)\beta_2 = 0$$

has only the solutions $\alpha_1 = \alpha_2 = 0$ or $\beta_1 = \beta_2 = 0$. Such will be the case if and only if the determinant of the coefficients of β_1 and β_2,

$$\alpha_1(\alpha_1 + \gamma_2\alpha_2) - \gamma_1\alpha_2^2 = \alpha_1^2 + \gamma_2\alpha_1\alpha_2 - \gamma_1\alpha_2^2 \neq 0$$

for all $\alpha_1^2 + \alpha_2^2 > 0$. This quadratic form in α_1 and α_2 is therefore *definite*, and the discriminant of the form is positive:

$$-(4\gamma_1 + \gamma_2^2) > 0.$$

Now solve Eq. (1.2), setting $x = \alpha_1 e + \alpha_2 f$. The equation

$$x^2 + e = (\alpha_1^2 + \gamma_1\alpha_2^2 + 1)\, e + (2\alpha_1\alpha_2 + \gamma_2\alpha_2^2) f = 0$$

is equivalent to the system of equations

$$\alpha_1^2 + \gamma_1\alpha_2^2 + 1 = 0, \qquad \alpha_2(2\alpha_1 + \gamma_2\alpha_2) = 0,$$

which possesses two pairs of real roots:

$$\alpha_1 = \pm\frac{\gamma_2}{\sqrt{-(4\gamma_1 + \gamma_2^2)}}, \qquad \alpha_2 = \mp\frac{2}{\sqrt{-(4\gamma_1 + \gamma_2^2)}}.$$

4. Show that the product x_1x_2 defined in Section 1.7 satisfies the axioms IV. 1–3.

5. What is the geometric interpretation of the multiplication of complex numbers?

6. Find the real and imaginary parts of the quotient of the complex numbers $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ by multiplying top and bottom by the complex conjugate $\bar{z}_2$ of z_2.

7. Prove that the determinant of the real and imaginary parts of the complex numbers a and b is $(\bar{a}b - a\bar{b})/2i$.

8. The value of a rational expression goes over into its complex conjugate when every complex number occurring in the expression is replaced by its complex conjugate. (Show this.) Deduce from this the theorem:

If the coefficients of the algebraic equation $a_0z^n + a_1z^{n-1} + \cdots + a_n = 0$ *are real, then the complex roots of the equation are conjugate in pairs.*

9. Prove that ($z = x + iy$)

$$\frac{|x| + |y|}{\sqrt{2}} \leqq |z| \leqq |x| + |y|.$$

When does equality hold?

10. Prove that

$$\left|\frac{a - b}{1 - \bar{a}b}\right| = 1,$$

if a and b are complex numbers with $|a| = 1$ or $|b| = 1$.

11. Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1$$

if $|a| < 1$ and $|b| < 1$.

12. Prove that

$$\left|\frac{az+b}{\bar{b}z+\bar{a}}\right| = 1$$

for $|z| = 1$.

13. Derive the formulas

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\,(z_1\bar{z}_2),$$
$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}\,(z_1\bar{z}_2).$$

The second one includes the Law of Cosines.

14. Separate the following expressions into real and imaginary parts (z is complex):

$$iz^3, \quad \frac{1}{z-i}, \quad \frac{z-i}{z+i}, \quad \frac{1}{z^2}.$$

15. Explain the geometrical significance of the absolute value and the argument of the function $(z - i)/(z + i)$ and investigate how they vary as z varies.

16. Prove the following theorem:

$$\lim_{n\to\infty} z_n = z \neq 0$$

holds if and only if the conditions

$$\lim_{n\to\infty} |z_n| = |z| \qquad \text{and} \qquad \lim_{n\to\infty} \arg z_n = \arg z \pmod{2\pi}$$

are fulfilled.

17. Prove the *Bolzano-Weierstrass Theorem: Every bounded infinite set of points has a cluster point.*

18. Show that the set of points $|z| \leqq \infty$ is both open and closed.

19. Give an example of a set of points in the plane which is neither open nor closed.

20. Show that in the complex plane (a) the complement of an open set is closed; (b) the complement of a closed set is open.

21. Prove that (a) the boundary of a domain is a closed set; (b) the union $G \cup \Gamma$ of a domain G with its boundary Γ is a closed set.

22. Prove that the union $A \cup B$ of two closed sets A and B is closed.

23. Prove that the intersection $A \cap B$ of two closed sets A and B (that is, the set of points common to A and B) is closed.

24. Prove that (a) the union $A \cup B$ and (b) the intersection $A \cap B$ of two open sets A and B are open.

25. Prove that the union of arbitrarily many open sets is open.

26. Prove the *Heine-Borel Theorem: If a bounded, closed set A is covered by a set C of open disks (that is, A is contained in the union of the disks in C), then there is a finite number of disks in C which cover A.*

27. Prove that if a real function $u(x, y)$ of two real variables possesses partial derivatives which are continuous at a point, then the function is differentiable at this point.

28. Show by means of the example $u = x^2y/(x^2 + y^2)$ that the continuity of a function and the existence of partial derivatives is not sufficient for the differentiability of the function.

29. Prove the chain rule for the differentiation of a composite function.

30. Verify that the real and imaginary parts of the functions of z given in Exercise 14 satisfy the Cauchy-Riemann equations.

31. By use of polar coordinates, investigate the real and imaginary parts of z^n and $1/z^n$, with particular rerefence to their sign.

32. Prove that the real and imaginary parts of an analytic function $w(z) = u(x, y) + i\, v(x, y)$ satisfy Laplace's equation

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0,$$

provided u and v possess continuous partial derivatives of the second order.

33. Prove that, in a disk on which the derivative of a complex function vanishes identically, the function is constant.

34. Show that every polynomial of the first degree $a + bx + cy$ with real coefficients is the real part of an analytic function of z $(= x + iy)$, and construct this function.

35. What are the most general polynomials, with real coefficients, of second degree

$$U(x, y) = a + bx + cy + dx^2 + exy + fy^2,$$
$$V(x, y) = a' + b'x + c'y + d'x^2 + e'xy + f'y^2,$$

for which $U(x, y) + i\, V(x, y)$ is an analytic function of $z = x + iy$? Show that the function in question is a polynomial of second degree in z.

36. What is the most general polynomial, with real coefficients, of the form

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

which is the real part of an analytic function? Construct this function.

37. Show that the form of the Cauchy-Riemann equations and of Laplace's equation is preserved when the coordinate system x, y is replaced by another rectangular coordinate system whose axes are in the same position relative to one another as were the original coordinate axes.

38. Show that in polar coordinates the Cauchy-Riemann equations have the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \phi}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \phi}$$

and Laplace's equation has the form

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} = 0.$$

39. The real and imaginary parts of a function $w(z) = u(x, y) + i\,v(x, y)$ are differentiable at a certain point z_0 and $\partial(u, v)/\partial(x, y) = u_x v_y - u_y v_x < 0$. Prove that if the limit (b) in Section 1.16 exists, the mapping $z \to \bar{w} = u - iv$ is conformal at the point z_0 and the given mapping $z \to w$ is indirectly conformal (conformal with the sense of the angles reversed).

40. Suppose that the function $w(z)$ is analytic in a domain G which is symmetric with respect to the real axis. Show that $f(z) = \overline{w(\bar{z})}$ is then an analytic function of z in G.