Contents

Introduction

1. The Subject Matter

The main theme of this book is the study of uniformly continuous and, in particular, Lipschitz functions between Banach spaces. Such functions appear in many contexts, and their study involves a rich interplay between topology, geometry of Banach spaces, geometric measure theory, classical analysis, probability and even combinatorics. The study of uniformly continuous and Lipschitz maps is naturally connected to the classification of Banach spaces and their subsets (especially balls) with respect to uniform homeomorphisms and bi-Lipschitz maps. It also leads to the development of notions and tools which are of interest in the applications of nonlinear functional analysis to other branches of mathematics.

To put the theory of uniform classification in context, we quote two well-known results. The first, due to Mazur and Ulam [**407**], is that a surjective isometry between two Banach spaces which takes 0 to 0 is necessarily linear. The second (and much deeper) result, due to Kadec [**303**], is that all separable infinite-dimensional Banach spaces are mutually homeomorphic. (And, as shown by Anderson [**16**], they are homeomorphic to the product space $\mathbb{R}^{\aleph_0}$.) The first result means that the linear structure of a Banach space is completely determined by its structure as a metric space, while the second says that its structure as a topological space, which is very simple, contains no information whatsoever on its linear structure.

The uniform classification lies between these two extremes. Even when the isometry assumption is slightly weakened, and instead of isometries we consider mappings which "almost" preserve distances, we already encounter difficult and very interesting questions (see Chapters 14 and 15, and Theorem 6.14). The situation becomes even more involved (and interesting) when we move further away from isometries and consider bi-Lipschitz maps and uniform homeomorphisms.

In addition to uniformly continuous maps, we also discuss some other topics in the nonlinear theory which are closely related to questions in the geometry of Banach spaces.

We treat several topics in the context of finite-dimensional Banach spaces. However most of the discussion concerns infinite-dimensional spaces. In many classical topics, once one leaves the locally compact setting, one immediately encounters the type of problems which are the main subject of this book.

We shall now outline the highlights of the various chapters.

In Chapter 1 we consider uniformly continuous (and Lipschitz) retractions, extensions and selections. We prove that some important metric spaces are Lipschitz retracts of any metric space containing them. These include the spaces of continuous functions on compact metric spaces (with the usual sup norm) and the space of all closed, bounded and convex subsets of a given convex set in a Banach space (equipped with the Hausdorff metric). We then study the question of extending uniformly continuous functions, defined on a subset of a Banach space E and taking values in a Banach space F, to the whole space E. This question is of special interest when E or F are Hilbert spaces. In the last section we study selections of set-valued maps. The problems of retraction and of extension can both be formulated as selection problems, and our main interest here is in obtaining uniformly continuous selections whose moduli of continuity are as good as possible.

In Chapter 2 we consider several independent topics related to the material of Chapter 1. We mention just three of these topics. The first is the question of approximating a general uniformly continuous function by functions with a prescribed modulus of continuity. In particular we obtain the exact exponent α so that every uniformly continuous function from the unit ball of L_p into L_q can be approximated arbitrarily well by functions of Hölder class α. Another subject is the nearest point map onto convex sets in a uniformly convex and uniformly smooth Banach space. We obtain precise estimates on the modulus of continuity of this map. These estimates yield, in particular, that a space whose moduli of convexity and smoothness are of the best possible order must be isomorphic to a Hilbert space. The third topic is the Steiner point map, a classical method to select a distinguished point in a bounded convex subset in $\mathbb{R}^n$. We show, among other results, that its modulus of continuity on the set of closed bounded convex subsets in $\mathbb{R}^n$ (equipped with the Euclidean norm) is the smallest possible among all such selection maps.

Chapter 3 is devoted to fixed point theory. Besides the basic standard results of Banach and Schauder, we also consider Lipschitz and nonexpansive maps of closed convex sets in Banach spaces. The result in this chapter which is most closely related to the rest of the book is that if C is a closed convex noncompact set in a Banach space, then there is a Lipschitz map f from C into itself with no approximate fixed point, i.e., $\inf\{\|f(x) - x\| : x \in C\} > 0$. In particular it follows that if E is an infinite-dimensional Banach space, then there is a Lipschitz retraction from the unit ball of E onto its unit sphere. We also discuss in some detail the rather delicate problems of the existence of fixed points and of the nonlinear ergodic theorem for nonexpansive maps.

One of the main tools of nonlinear functional analysis is the "linearization" of maps. A natural way to do so is to use derivatives. In the first section of Chapter 4 we present the two most natural concepts of differentiability, the Gâteaux and the Fréchet derivatives. We also present the basic calculus results that carry over with no difficulty from the finite-dimensional setting to infinite dimensions. The study of the existence of derivatives in infinite-dimensional spaces is much more involved than in finite-dimensional ones, and the main part of the chapter is devoted to the question of differentiation of convex real-valued functions. This subject is quite well understood today. There is even a complete characterization of sets in separable Banach spaces that can be the sets of points of nondifferentiability (in the Gâteaux sense) of some convex continuous function.

In Chapter 5 we study Banach spaces which have the Radon-Nikodým property (RNP). This class of spaces appears naturally in a variety of contexts and has many equivalent characterizations involving, for example, vector-valued measures, vector-valued martingales, or the extremal structure of closed bounded convex sets. The characterization most relevant to the subject of this book is that spaces with RNP are exactly those spaces E for which every Lipschitz function from $\mathbb{R}$ to E is differentiable almost everywhere.

In Chapter 6 we turn to Gâteaux differentiability of Lipschitz functions. To study this topic we need, in addition to the concept of RNP spaces, also the notion of "almost everywhere" in infinite-dimensional Banach spaces. In spite of the fact that infinite-dimensional Banach spaces do not carry a natural measure, one can still define in a meaningful way the notion of negligible sets. Actually, this can be done in several nonequivalent ways. We give some examples which illustrate these notions and also exhibit their delicate nature. With these tools (RNP and negligible sets) at hand, we proceed to prove the existence almost everywhere of Gâteaux derivatives of Lipschitz functions between certain Banach spaces. This generalizes in a very useful way the classical theorem of Rademacher on the differentiability of Lipschitz functions between finite-dimensional spaces. In a separate section we show that the "almost everywhere" results do not hold for Fréchet differentiation even for real-valued convex and continuous functions on a separable Hilbert space. The chapter ends with a section which summarizes and gives an overview of the various classes of negligible sets which appear in this volume.

In Chapter 7 we apply the Gâteaux differentiability results of Chapter 6 in order to linearize Lipschitz retractions and bi-Lipschitz maps. This procedure reduces in many important cases the study of Lipschitz embeddings and retractions to the linear theory. It follows from the discussion in this chapter that some concrete separable spaces which are not linearly isomorphic to each other also fail to be Lipschitz equivalent. On the other hand, we show that every separable Banach space is Lipschitz equivalent to a subset of c_0. This demonstrates the decisive role of the existence of derivatives in some of the results. We also present an example of two nonisomorphic spaces which are Lipschitz equivalent. (This example is nonseparable. No separable example of this type is known.) We end this chapter with examples of Lipschitz equivalences of l_2 onto itself with some surprising properties.

In Chapter 8 we turn to uniform embeddings; i.e., uniformly continuous maps whose inverses are uniformly continuous on their domain of definition. In this chapter we study uniform embeddings into Hilbert space and give a complete linear characterization of the normed (even quasi-normed) spaces that embed uniformly into Hilbert space. They are exactly the spaces which are linearly isomorphic to a subspace of $L_0(\mu)$, the space of all the measurable functions on some measure space with the topology of convergence in measure.

In Chapter 9 we characterize the Banach lattices whose unit spheres are uniformly homeomorphic to the unit sphere of Hilbert space. Examples (related to the Krivine-Maurey stability theory) show that for spaces which are not lattices the situation is more complicated. By using various methods (starting with the classical explicit map of Mazur, and proceeding via lattice factorization, complex interpolation and the Pełczyński decomposition method), we prove that the unit spheres of a very large class of spaces are uniformly homeomorphic to the unit sphere of a Hilbert space. We also show that the uniform classification of unit balls of Banach spaces is essentially the same as the classification of their spheres.

In Chapter 10 we study uniform homeomorphism between Banach spaces. It turns out that for several classes of Banach spaces, the structure of the space as a uniform space determines its linear structure. Moreover, for all Banach spaces the uniform structure already determines the finite-dimensional linear subspaces of the space (up to an isomorphism constant independent of the dimension of the subspace). Comparing these results with those of Chapter 9, one sees that there is a big difference between the uniform classification of spheres and that of spaces. We also give some examples that put the results of this chapter into perspective. For example, there are separable uniformly convex spaces which are uniformly homeomorphic but are not linearly isomorphic. There is even a separable uniformly convex space E with the following property: Any Banach space which is uniformly homeomorphic to E is either isomorphic to E or to $E \oplus H$ (where H is an infinite-dimensional Hilbert space), and E and $E \oplus H$ are uniformly homeomorphic but not linearly isomorphic. Another topic discussed in this chapter is the structure of discrete nets in Banach spaces. We show that knowing a discrete net in a Banach space up to Lipschitz equivalence gives much information on the space (and sometimes even determines it uniquely up to linear isomorphism). Infinite-dimensional Banach spaces determine their nets up to Lipschitz equivalence, but this is no longer true for finite-dimensional spaces (except, of course, when the dimension is 1).

In Chapter 11 we study quotient maps, a notion dual to that of embedding in the context of the linear theory. The direct analog, namely, surjective Lipschitz or uniformly continuous maps, turns out to be insufficient from our point of view. Indeed, for each pair of separable infinite-dimensional Banach spaces E and F there is a Lipschitz (and C^1) surjective map from E onto F. We introduce another notion of uniform (or Lipschitz) quotient map and study its properties. With this new notion we get, for example, that a uniform quotient space of a Hilbert space must be isomorphic to a Hilbert space. In finite dimensions the notion of Lipschitz quotient maps is related to the theory of quasi-regular maps.

In Chapter 12 we show that a real-valued uniformly continuous function on the unit sphere of a finite-dimensional Banach space is essentially constant on large sections of the sphere. More precisely, for every n, modulus of continuity φ, and $\varepsilon > 0$, there is a $k = k(n, \varepsilon, \varphi)$ (with $\lim_{n\to\infty} k(n, \varepsilon, \varphi) = \infty$ for fixed ε and φ), such that if f is a uniformly continuous real-valued function on the unit sphere of an n-dimensional Banach space E, and if the modulus of continuity of f is bounded by φ, then there is a k-dimensional subspace F of E such that the oscillation of f on the unit sphere of F is smaller than ε. We actually prove two results of this type. The first gives concrete estimates of k (which are sometimes asymptotically sharp), but does not show explicitly how F is located in E. The second gives much smaller k (and we do not even attempt to get any explicit estimate), but produces a subspace F which is nicely located in E: If E has a basis, then F is spanned by a "block basis". The heart of the first approach is the theorem of Dvoretzky on the existence of almost spherical sections in high-dimensional convex bodies. The second approach also reduces essentially to a result of a similar nature: For every k and ε there is an n such that every n-dimensional Banach space with a basis contains a block basis of length k which is ε-close to the unit vectors of l_p^k for some $1 \le p \le \infty$.

In Chapter 13 we consider the analogous question in an infinite-dimensional setup. We consider a uniformly continuous real-valued function f on the unit sphere of an infinite-dimensional Banach space E and ask whether there is an infinite-dimensional subspace F of E such that the restriction of f to the unit sphere of F has a small oscillation (prescribed in advance). The results of the previous chapter ensure the existence of finite-dimensional subspaces F of arbitrarily high dimension, but it turns out that in general there does not have to be an infinite-dimensional subspace F with the required property. The main example here is that of a distorted norm on a Hilbert space; i.e., an equivalent norm on a separable Hilbert space H (which we consider as a Lipschitz function on the unit sphere on

H) for which there is a $\delta > 0$ such that the oscillation of the new norm on the unit sphere of any infinite-dimensional subspace F of H is at least δ. The construction and the proof are quite involved, and the norm is not constructed directly on a Hilbert space. The first step is the construction of a rather complicated Banach space. Results on the structure of this Banach space are then transferred to Hilbert space via an appropriate nonlinear map. We also show in this chapter that every uniformly continuous function from the unit sphere of c_0 to $\mathbb{R}$ has arbitrarily small oscillation when restricted to a suitable infinite-dimensional subspace of c_0. It turns out that c_0 is essentially the only Banach space with this property.

Chapter 14 considers maps which are close to isometries. After reviewing the main results concerning isometries, we present the theory developed by F. John in the framework of the theory of elasticity. Consider a map f defined on a bounded open set Ω which is locally almost an isometry; then one asks whether f is injective on all of Ω, or even close to an actual isometry there. Some of these questions can be answered for general Banach spaces, while others have a nice answer only in Hilbert space (actually only in finite-dimensional Euclidean spaces). In finite-dimensional spaces, the maps we consider are differentiable almost everywhere, and it is natural to study their derivative. It is in the framework of this study that BMO functions were introduced for the first time.

In Chapter 15 we also consider the approximation of maps which are close to isometries by isometries, but an essential feature here is that the maps are globally defined and surjective. This issue is now well understood, and we present two quite precise results concerning it. This study is closely related to that of functions which are almost linear in the sense that $\|f(x+y) - f(x) - f(y)\| = o(\|x\| + \|y\|)$ as $\|x\|, \|y\| \to \infty$.

In Chapter 16 we study functions which are close to being linear in a weaker sense, i.e., $\|f(x+y) - f(x) - f(y)\| = O(\|x\| + \|y\|)$. It turns out that these functions (the so-called quasi-linear functions) come up naturally in the study of "twisted sums" of Banach spaces. (A space G is a twisted sum of E and F if $G \supset E$ and G/E is F.) The natural setting for this study is not of Banach spaces, but that of quasi-Banach spaces. Quasi-linear functions are used in particular in order to produce nontrivial twisted sums of Hilbert space with itself. (The existence of such a twisted sum is far from obvious.) The properties of twisted sums of Hilbert space with itself are studied in some detail. They give examples of Banach spaces which are interesting from many different points of view such as the study of unconditional structure or in the framework of complex Banach spaces, where they are used to produce two complex Banach spaces which are linearly isometric as real spaces but fail to be isomorphic as complex spaces.

In the final chapter, 17, we study questions related to Hilbert's fifth problem (concerning Lie groups) in the setting of groups modeled on an infinite-dimensional Banach space. We first examine whether there exist nontrivial group structures on a Banach space. It is shown that under some metric and other natural conditions, the only way to put a commutative group structure on a Banach space is to transfer to it the addition operations from another Banach space via a uniform homeomorphism. We also consider the analogue of Hilbert's problem concerning local groups. We study how far one can go in the construction of a Lie structure without changing the underlying model space. It is clear that without change of the model space one cannot solve the problem completely, and it seems to be unknown whether one can solve the infinite-dimensional analogue of Hilbert's problem if a change in the model space is allowed. In all these considerations it is essential that one assumes that the group operation is uniformly continuous, and thus the subject is intimately connected to the uniform classification of Banach spaces.

A more detailed outline of the material in chapters 7-12 is given in the survey [**362**] by Lindenstrauss. Note however that some of the open problems mentioned in [**362**] have already been solved; see, e.g., Examples 7.17 and 11.24.

Volume 2 of this book will treat topics which complement those considered here. The three main topics in Volume 2 will be

(i) Fréchet differentiability of Lipschitz functions between Banach spaces.
(ii) Lipschitz maps from the discrete point of view.
(iii) Connections with analytic functions.

Let us briefly describe these topics:

(i) In this volume we present existence theorems for Gâteaux differentiability of Lipschitz maps between Banach spaces. Fréchet differentiability is considered only for convex (real-valued) functions. The question of existence of Fréchet derivatives of Lipschitz functions between Banach spaces is much harder. Presently only one general result (due to Preiss [**471**]) is known in this direction: Every real-valued Lipschitz map defined on a space E with a separable dual is Fréchet differentiable on a dense set. Easy examples show that a Lipschitz map from ℓ_2 into itself may be nowhere Fréchet differentiable. This also holds for Lipschitz maps on other infinite-dimensional spaces. In Volume 2 we shall present the above-mentioned result on real-valued functions and also discuss several weakened notions of Fréchet differentiability. In terms of these weakened versions several positive results are available for Lipschitz maps between pairs of Banach spaces E and F (when both may be infinite-dimensional). The discussion involves also a new type of exceptional sets and a notion related to density points in infinite dimensions.

(ii) It is proved in Chapter 10 of this volume that uniformly homeomorphic Banach spaces have the same finite-dimensional subspaces. This means that all Banach space notions which involve only finitely many vectors (i.e., the entire so-called "local theory of Banach spaces"; see e.g. Appendix G) are invariant under uniform homeomorphism. It follows that at least in principle, these notions can be defined using only distances and without any use of vector space operations. We can even go one step further and use these notions in their metric version and apply them to general metric spaces. Naturally, there are major problems in such a program. Finding explicit metric formulations for linear notions is not always easy, and the metric formulation can sometimes be done in several different ways. Nevertheless, there is an emerging theory in this spirit for general (and in particular for finite) metric spaces.

We shall present this theory in Volume 2. It involves, e.g., such questions as the possibility of embedding metric spaces with small distortion into Banach spaces (and in particular Hilbert spaces); metric notions corresponding to type, cotype and superreflexivity; extension of Lipschitz maps defined on finite metric spaces and taking values in a Banach space, etc. There is even a metric version of Dvoretzky's theorem. The main tools in this study are of probabilistic and combinatorial nature. Some connections have already been found between this direction and theoretical computer science. Discrete problems also enter, to a very limited extent, into some discussions of the present volume (e.g., in the study of Banach groups in Chapter 17).

(iii) As one example of the connection of analytic functions to the subject matter of this book we mention the notion of Analytic Radon-Nikodým Property (ARNP). A Banach space E has the ARNP if every bounded E-valued analytic function on the open unit disc has radial limits a.e. on the unit circle. There is a theory of this class of spaces which is analogous to the theory of RNP spaces presented in Chapter 5 of this volume. The notions of convexity and convex functions that appear in the RNP theory are replaced in this theory by notions of complex convexity such as harmonic and plurisubharmonic functions. Also, general martingales are replaced by "analytic" martingales.

Another topic that falls under this heading and will be treated in Volume 2 is the biholomorphic classification of some subsets in complex Banach spaces. For example, it turns out that when the unit balls of two complex Banach spaces are biholomorphically equivalent, then the spaces are necessarily linearly isometric.

Besides these three major directions, Volume 2 will also contain some other topics. We mention just one such additional topic. A nice and useful result (e.g. in infinite-dimensional manifold theory) is that the general linear group of l_2 is contractible. This will be proved in Volume 2, and we shall also consider which other Banach spaces can replace l_2 in the statement above.

After reviewing the contents of the two volumes, it should be clear that large parts of what is usually called nonlinear functional analysis are not treated at all in this book. In particular we do not cover the one aspect of nonlinear functional analysis which is treated most often, namely, the theory of nonlinear differential equations. We also do not treat here the related subjects of infinite-dimensional manifold theory, degree theory, index theories, bifurcations results, etc. There are, of course, many books which treat this material. The books by Krasnoselskii [**336**], Vainberg [**567**], Schwartz [**526**], Nirenberg [**438**], Browder [**96**], Berger [**55**], Zeidler [**593**], Deimling [**139**], and Ambrosetti and Prodi [**12**] comprise just a sample of the books in which the general theory is presented. See also the collections of expository articles [**393, 394**] edited by Matzeu and Vignoli.

We also do not treat here questions related to purely topological homeomorphism, i.e., the theory which is usually called infinite-dimensional topology. Although several arguments in this theory use the geometry of Banach spaces, the final results are, as we have already mentioned above, not connected to the linear structure of the spaces. We refer the interested reader to the books by Bessaga and Pełczyński [**62**] and by van Mill [**576**], where this subject is covered in detail.

2. The Book. Conventions, Prerequisites, Etc.

We tried to write the book so that it will be possible to read each chapter independently of the other chapters. Thus each chapter gives a rather complete picture of one aspect of the subject, and cross-references between the chapters are few and are usually restricted to quoting a theorem or a definition.

Each chapter ends with a Notes and Remarks section in which we refer to the literature, present additional comments and results in the same direction, and often also mention some open problems.

The end of a proof is marked with a $\square$ or a $\diamondsuit$. The $\square$ is used when the proof comes directly after the statement of the result. If an auxiliary result or a discussion separates the proof of a result from its statement, we use the $\diamondsuit$ symbol.

The book ends with a detailed subject index. We point out, however, that the list of theorems in the subject index is *not* the list of the main results proved in the book. Rather, it is a list of theorems which have commonly used names attached to them, and most of them are only quoted here without a proof. The bibliography serves also as an enhanced author index; every item contains a reference to the page numbers in the text where the item is quoted.

The prerequisites for reading this book are a knowledge of material usually covered in the first year of graduate studies (in real analysis, functional analysis and also some complex analysis). Some further more specialized topics, which are used in a few places in the book, are presented in the various appendices at the end. In these appendices we present the necessary definitions and basic results. Some of these results are proved, but in most cases we refer to books and to journal articles where proofs can be found. We emphasize that the appendices are not a prerequisite for the main body of the book. They should be consulted only at the places in the text where a specific reference is made.

Two general remarks concerning the Banach spaces which appear in this book: For the large majority of the results, it does not matter whether the spaces are taken over the real or the complex field. In these cases we always work with real scalars without further notice. Complex scalars are used, again with no further notice, when the context requires it, e.g., when analytic functions or spectral theory are used, or when we apply the complex interpolation method. Our main interest is in separable Banach spaces. When separability makes no difference, we treat general Banach spaces; but when nonseparability poses additional difficulties, our general policy is to treat the separable case only. We do, however, state specifically that a space is separable whenever this is assumed.

We use standard notation and terminology. In particular we use the following notation for the classical Banach spaces. If K is a compact Hausdorff space, we denote by $C(K)$ the space of scalar-valued continuous functions on K, equipped with the supremum norm. The scalar field is usually assumed to be the real field and we do not indicate this in the notation. If K is a convergent sequence, we denote $C(K)$ by c. The hyperplane in c of sequences converging to 0 is denoted by c_0. For an infinite set Γ we denote by $c_0(\Gamma)$ the space of all scalar -valued functions f on Γ such that $\{\gamma : |f(\gamma)| > \varepsilon\}$ is finite for every $\varepsilon > 0$, equipped with the supremum norm. If Γ is countable, then $c_0(\Gamma)$ is just the space c_0.

Let $(\Omega, \mathcal{B}, \mu)$ be a measure space, i.e., $\mathcal{B}$ is a σ-algebra of subsets of Ω, and μ is a σ-additive nonnegative measure on $\mathcal{B}$. We denote by $L_p(\Omega, \mathcal{B}, \mu)$ for $1 \le p < \infty$ (respectively $L_\infty(\Omega, \mathcal{B}, \mu)$ for $p = \infty$) the space of equivalence classes of scalar-valued measurable functions f such that $\|f\|_p = \big(\int_\Omega |f|^p d\mu\big)^{1/p} < \infty$ (respectively $\|f\|_\infty = \sup\big\{t : \mu\{|f| > t\} > 0\big\} < \infty$). Once again the scalar field is usually assumed to be the real field. We usually suppress Ω and $\mathcal{B}$ from the notation and write $L_p(\mu)$. If μ is the Lebesgue measure on $[0,1]$, we denote $L_p(\mu)$ by $L_p[0,1]$ or simply by L_p. If μ is the counting measure on an abstract set Γ (i.e., $\mu(\gamma) = 1$ for every γ, and $\mathcal{B}$ is the σ-algebra of all subsets of Γ), we denote $L_p(\mu)$ by $l_p(\Gamma)$. If Γ is countable, or finite with cardinality n, we write l_p or l_p^n respectively.

Many other Banach spaces are defined throughout the book.

The vast majority of the material presented in this book appears for the first time in book form. A small part of the material in Chapters 1–5 is standard and is covered in many books. A larger part of the material in these chapters, well-known to experts but perhaps not to functional analysts in general, is covered in some specialized books (e.g. Deville, Godefroy and Zizler [**144**], Diestel and Uhl [**146**], Goebel and Kirk [**217**], Phelps [**461**]). However, much of the material even in these chapters also appears here for the first time in book form. Most of the material in Chapter 12 appears in Milman and Schechtman [**420**], and most of the material in the first section of Chapter 16 is contained in Castillo and Gonzalez [**114**].

A considerable amount of information is now known on the structure of Lipschitz and uniformly continuous functions, and on the relation of this subject to the geometry of Banach spaces. Many natural and fundamental problems in the area are, however, still open and some natural directions have not been studied at all. We hope that this book will help in making the results of geometric nonlinear functional analysis better known to the general community of functional analysts and that it will stimulate further research in the area.

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