

CHAPTER 1

Parameters

1.1. The automorphic Langlands group

We begin with some general motivation. We shall review some of the fundamental ideas that underlie the theoretical foundations laid by Langlands. This will help us put our theorems into perspective. It will also lead naturally to a formulation of some of the essential objects with which we need to work.

We take F to be a local or global field of characteristic 0. In other words, F is a finite extension of either the real field \mathbb{R} or a p -adic field \mathbb{Q}_p , or it is a finite extension of \mathbb{Q} itself. Suppose that G is a connected reductive algebraic group over F , which to be concrete we take to be a classical matrix group. For example, we could let G be the general linear group

$$G(N) = GL(N)$$

of invertible matrices of rank N over F .

In his original paper [L2], Langlands introduced what later became known as the L -group of G . This object is a semidirect product

$${}^L G = \widehat{G} \rtimes \Gamma_F$$

of a complex dual group \widehat{G} of G with the Galois group

$$\Gamma_F = \text{Gal}(\overline{F}/F)$$

of an algebraic closure \overline{F} of F . The action of Γ_F on \widehat{G} (called an L -action) is determined by its action on a based root datum for G and a corresponding splitting for \widehat{G} , according to the general theory of algebraic groups. (See [K3, §1.1–§1.3].) It factors through the quotient $\Gamma_{E/F}$ of Γ_F attached to any finite Galois extension $E \supset F$ over which G splits. We sometimes formulate the L -group by the simpler prescription

$${}^L G = {}^L G_{E/F} = \widehat{G} \rtimes \Gamma_{E/F},$$

since this suffices for many purposes. If $G = G(N)$, for example, the action of Γ_F on \widehat{G} is trivial. Since \widehat{G} is just the complex general linear group $GL(N, \mathbb{C})$ in this case, one can often work with the restricted form

$${}^L G = \widehat{G} = GL(N, \mathbb{C})$$

of the L -group.

Langlands' conjectures [L2] predicate a fundamental role for the L -group in the representation theory of G . Among other things, Langlands conjectured the existence of a natural correspondence

$$\phi \longrightarrow \pi$$

between two quite different kinds of objects. The domain consists of (continuous) L -homomorphisms

$$\phi : \Gamma_F \longrightarrow {}^L G,$$

taken up to conjugation by \widehat{G} . (An L -homomorphism between two groups that fibre over Γ_F is a homomorphism that commutes with the two projections onto Γ_F .) The codomain consists of irreducible representations π of $G(F)$ if F is local, and automorphic representations π of $G(\mathbb{A})$ if F is global, taken in each case up to the usual relation of equivalence of irreducible representations.

Recall that if F is global, the adèle ring is defined as a restricted tensor product

$$\mathbb{A} = \mathbb{A}_F = \prod_v F_v$$

of completions F_v of F . In this case, the Langlands correspondence should satisfy the natural local-global compatibility condition. Namely, if ϕ_v denotes the restriction of ϕ to the subgroup Γ_{F_v} of Γ_F (which is defined up to conjugacy), and π is a restricted tensor product

$$\pi = \widetilde{\bigotimes}_v \pi_v, \quad \phi_v \rightarrow \pi_v,$$

of representations that correspond to these localizations, then π should correspond to ϕ . We refer the reader to the respective articles [F] and [L6] for a discussion of restricted direct products and automorphic representations.

The correspondence $\phi \rightarrow \pi$, which remains conjectural, is to be understood in the literal sense of the word. For general G , it will not be a mapping. However, in the case $G = GL(N)$, the correspondence should in fact reduce to a well defined, injective mapping. For local F , this is part of what has now been established, as we will recall in §1.3. For global F , the injectivity would be a consequence of the required local-global compatibility condition and the theorem of strong multiplicity one, or rather its generalization in [JS] that we will also recall in §1.3. However, the correspondence will very definitely not be surjective.

In our initial attempts at motivation, we should not lose sight of the fact that the conjectural Langlands correspondence is very deep. For example, even though the mapping $\phi \rightarrow \pi$ is known to exist for $G = GL(1)$, it takes the form of the fundamental reciprocity laws of local and global class field theory. Its generalization to $GL(N)$ would amount to a formulation of nonabelian class field theory.

Langlands actually proposed the correspondence $\phi \rightarrow \pi$ with the Weil group W_F in place of the Galois group Γ_F . We recall that W_F is a locally compact group, which was defined separately for local and global F by Weil. It is equipped with a continuous homomorphism

$$W_F \longrightarrow \Gamma_F,$$

with dense image. If F is global, there is a commutative diagram

$$\begin{array}{ccc} W_F & \longrightarrow & \Gamma_{F_v} \\ \downarrow & & \downarrow \\ W_F & \longrightarrow & \Gamma_F \end{array}$$

for any completion F_v of F , with vertical embeddings defined up to conjugation. (See [T2].) In the Weil form of the Langlands correspondence, ϕ represents an L -homomorphism from W_F to ${}^L G$. The restriction mapping of continuous functions on Γ_F to continuous functions on W_F is injective. For this reason, the conjectural Langlands correspondence for Weil groups is a generalization of its version for Galois groups.

For $G = GL(N)$, the Weil form of the conjectural correspondence $\phi \rightarrow \pi$ again reduces to an injective mapping. (In the global case, one has to take π to be an *isobaric* automorphic representation, a natural restriction introduced in [L7] that includes all the representations in the automorphic spectral decomposition of $GL(N)$.) If $G = GL(1)$, it also becomes surjective. However, for nonabelian groups G , and in particular for $GL(N)$, the correspondence will again not be surjective. One of the purposes of Langlands' article [L7] was to suggest the possibility of a larger group, which when used in place of the Weil group, would give rise to a bijective mapping for $GL(N)$. Langlands formulated the group as a complex, reductive, proalgebraic group, in the spirit of the complex form of Grothendieck's motivic Galois group.

Kottwitz later pointed out that Langlands' group ought to have an equivalent but simpler formulation as a locally compact group L_F [K3]. It would come with a surjective mapping

$$L_F \longrightarrow W_F$$

onto the Weil group, whose kernel K_F should be compact and connected, and (in the optimistic view of some [A17]) even simply connected. If F is local, L_F would take the simple form

$$(1.1.1) \quad L_F = \begin{cases} W_F, & \text{if } F \text{ is archimedean,} \\ W_F \times SU(2), & \text{if } F \text{ is nonarchimedean.} \end{cases}$$

In this case, L_F is actually a *split* extension of W_F by a compact, simply connected group (namely, the trivial group $\{1\}$ if F is archimedean and the three-dimensional compact Lie group $SU(2) = SU(2, \mathbb{R})$ if F is p -adic.) If F is global, L_F remains hypothetical. Its existence is in fact one of the deepest

problems in the subject. Whatever form it does ultimately take, it ought to fit into a larger commutative diagram

$$\begin{array}{ccccc} L_{F_v} & \longrightarrow & W_{F_v} & \longrightarrow & \Gamma_{F_v} \\ \downarrow & & \downarrow & & \downarrow \\ L_F & \longrightarrow & W_F & \longrightarrow & \Gamma_F \end{array}$$

for any completion F_v , the vertical embedding on the left again being defined up to conjugation.

The hypothetical formal structure of L_F is thus compatible with an extension of the Langlands correspondence from W_F to L_F . This is what Langlands proposed in [L7] (for the proalgebraic form of L_F). The extension amounts to a hypothetical correspondence $\phi \rightarrow \pi$, in which ϕ now represents an L -homomorphism

$$\phi : L_F \longrightarrow {}^L G,$$

taken again up to \widehat{G} -conjugacy. Here it is convenient to use the Weil form of the L -group

$${}^L G = \widehat{G} \rtimes W_F,$$

for the action of W_F on \widehat{G} inherited from Γ_F . An L -homomorphism between two groups over W_F is again one that commutes with the two projections. There are some minor conditions on ϕ that are implicit here. For example, since W_F and L_F are no longer compact, one has to require that for any $\lambda \in L_F$, the image of $\phi(\lambda)$ in \widehat{G} be semisimple. If G is not quasisplit, one generally also requires that ϕ be *relevant* to G , in the sense that if its image lies in a parabolic subgroup ${}^L P$ of ${}^L G$, there is a corresponding parabolic subgroup P of G that is defined over F .

Suppose again that $G = GL(N)$. Then the hypothetical extended correspondence $\phi \rightarrow \pi$ again reduces to an injective mapping. However, this time it should also be surjective (provided that for global F , we take the image to be the set of isobaric automorphic representations). If F is local archimedean, so that $L_F = W_F$, the correspondence was established (for any G in fact) by Langlands [L11]. If F is a local p -adic field, so that $L_F = W_F \times SU(2)$, the correspondence was established for $GL(N)$ by Harris and Taylor [HT] and Henniart [He1]. For global F , the correspondence for $GL(N)$ is much deeper, and remains highly conjectural. We have introduced it here as a model to motivate the form of the theorems we seek for classical groups.

If G is more general than $GL(N)$, the extended correspondence $\phi \rightarrow \pi$ will not reduce to a mapping. It was to account for this circumstance that Langlands introduced what are now called L -packets. We recall that L -packets are supposed to be the equivalence classes for a natural relation that is weaker than the usual notion of equivalence of irreducible representations. (The supplementary relation is called *L -equivalence*, since it is arithmetic in nature, and is designed to preserve the L -functions and ε -factors of representations.) The extended correspondence $\phi \rightarrow \pi$ is supposed to project to

a well defined mapping $\phi \rightarrow \Pi_\phi$ from the set of parameters ϕ to the set of L -packets. For $G = GL(N)$, L -equivalence reduces to ordinary equivalence. The L -packets Π_ϕ then contain one element each, which is the reason that the correspondence $\phi \rightarrow \pi$ reduces to a mapping in this case.

The general construction of L -packets is part of Langlands' conjectural theory of endoscopy. It will be a central topic of investigation for this volume. We recall at this stage simply that the L -packet attached to a given ϕ will be intimately related to the centralizer

$$(1.1.2) \quad S_\phi = \text{Cent}(\text{Im}(\phi), \widehat{G})$$

in \widehat{G} of the image $\phi(L_F)$ of ϕ , generally through its finite quotient

$$(1.1.3) \quad \mathcal{S}_\phi = S_\phi / S_\phi^0 Z(\widehat{G})^\Gamma.$$

Following standard notation, we have written S_ϕ^0 for the connected component of 1 in the complex reductive group S_ϕ , $Z(\widehat{G})$ for the center of \widehat{G} , and $Z(\widehat{G})^\Gamma$ for the subgroup of invariants in $Z(\widehat{G})$ under the natural action of the Galois group $\Gamma = \Gamma_F$. For $G = GL(N)$, the groups S_ϕ will all be connected. Each quotient \mathcal{S}_ϕ is therefore trivial. The implication for other groups G is that we will have to find a way to introduce the centralizers S_ϕ , even though we have no hope of constructing the automorphic Langlands group L_F and the general parameters ϕ .

There is a further matter that must also be taken into consideration. Suppose for example that F is global and that $G = GL(N)$. The problem in this case is that the conjectural parametrization of automorphic representations π by N -dimensional representations

$$\phi : L_F \longrightarrow \widehat{G} = GL(N, \mathbb{C})$$

is not compatible with the spectral decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$. If ϕ is irreducible, π is supposed to be a cuspidal automorphic representation. Any such representation is part of the discrete spectrum (taken modulo the center). However, there are also noncuspidal automorphic representations in the discrete spectrum. These come from residues of Eisenstein series, and include for example the trivial one-dimensional representation of $G(\mathbb{A})$. Such automorphic representations will correspond to certain *reducible* N -dimensional representations of L_F . How is one to account for them?

The answer, it turns out, lies in the product of L_F with the supplementary group $SU(2) = SU(2, \mathbb{R})$. The representations in the discrete automorphic spectrum of $GL(N)$ should be attached to *irreducible unitary* N -dimensional representations of this product. The local constituents of these automorphic representations should again be determined by the restriction of parameters, this time from the product $L_F \times SU(2)$ to its subgroups $L_{F_v} \times SU(2)$. Notice that if v is a p -adic valuation, the localization

$$L_{F_v} \times SU(2) = W_{F_v} \times SU(2) \times SU(2)$$

contains two $SU(2)$ -factors. Each will have its own distinct role. In §1.3, we shall recall the general construction, and why it is the product $L_F \times SU(2)$ that governs the automorphic spectrum of $GL(N)$.

Similar considerations should apply to a more general connected group G over any F . One would consider L -homomorphisms

$$\psi : L_F \times SU(2) \longrightarrow {}^L G,$$

with relatively compact image in \widehat{G} (the analogue of the unitary condition for $GL(N)$). If F is global, the parameters should govern the automorphic spectrum of G . If F is local, they ought to determine corresponding local constituents. In either case, the relevant representations should occur in packets Π_ψ , which are larger and more complicated than L -packets, but which are better adapted to the spectral properties of automorphic representations. These packets should in turn be related to the centralizers

$$S_\psi = \text{Cent}(\text{Im}(\psi), \widehat{G})$$

and their quotients

$$\mathcal{S}_\psi = S_\psi / S_\psi^0 Z(\widehat{G})^\Gamma.$$

For $G = GL(N)$, the groups S_ψ remain connected. However, for other classical groups G we might wish to study, we must again be prepared to introduce parameters ψ and centralizers S_ψ without reference to the global Langlands group L_F .

The objects of study in this volume will be orthogonal and symplectic groups G . Our general goal will be to classify the representations of such groups in terms of those of $GL(N)$. In the hypothetical setting of the discussion above, the problem includes being able to relate the parameters ψ for G with those for $GL(N)$. As further motivation for what is to come, we shall consider this question in the next section. We shall analyze the self-dual, finite dimensional representations of a general group Λ_F . Among other things, this exercise will allow us to introduce endoscopic data, the internal objects for G that drive the classification, in concrete terms.

1.2. Self-dual, finite-dimensional representations

We continue to take F to be any local or global field of characteristic 0. For this section, we let Λ_F denote a general, unspecified topological group. The reader can take Λ_F to be one of the groups Γ_F , W_F or L_F discussed in §1.1, or perhaps the product of one of these groups with $SU(2)$. We assume only that Λ_F is equipped with a continuous mapping $\Lambda_F \rightarrow \Gamma_F$, with connected kernel and dense image.

We shall be looking at continuous, N -dimensional representations

$$r : \Lambda_F \longrightarrow GL(N, \mathbb{C}).$$

Any such r factors through the preimage of a finite quotient of Γ_F . We can therefore replace Λ_F by its preimage. In fact, one could simply take a large

finite quotient of Γ_F in place of Λ_F , which for the purposes of the present exercise we could treat as an abstract finite group.

We say that r is *self-dual* if it is equivalent to its contragredient representation

$$r^\vee(\lambda) = {}^t r(\lambda)^{-1}, \quad \lambda \in \Lambda_F,$$

where $x \rightarrow {}^t x$ is the usual transpose mapping. In other words, the equivalence class of r is invariant under the standard outer automorphism

$$\theta(x) = x^\vee = {}^t x^{-1}, \quad x \in GL(N),$$

of $GL(N)$. This condition depends only on the inner class of θ . It remains the same if θ is replaced by any conjugate

$$\theta_g(x) = g^{-1}\theta(x)g, \quad g \in GL(N).$$

We shall analyze the self-dual representations r in terms of orthogonal and symplectic subgroups of $GL(N, \mathbb{C})$.

We decompose a given representation r into a direct sum

$$r = \ell_1 r_1 \oplus \cdots \oplus \ell_r r_r,$$

for inequivalent irreducible representations

$$r_k : \Lambda_F \longrightarrow GL(N_k, \mathbb{C}), \quad 1 \leq k \leq r,$$

and multiplicities ℓ_k with

$$N = \ell_1 N_1 + \cdots + \ell_r N_r.$$

The representation is self-dual if and only if there is an involution $k \leftrightarrow k^\vee$ on the indices such that for any k , r_k^\vee is equivalent to r_{k^\vee} and $\ell_k = \ell_{k^\vee}$. We shall say that r is *elliptic* if it satisfies the further constraint that for each k , $k^\vee = k$ and $\ell_k = 1$. We shall concentrate on this case.

Assume that r is elliptic. Then

$$r = r_1 \oplus \cdots \oplus r_r,$$

for distinct irreducible, self-dual representations r_i of Λ_F of degree N_i . If i is any index, we can write

$$r_i^\vee(\lambda) = A_i r_i(\lambda) A_i^{-1}, \quad \lambda \in \Lambda_F,$$

for a fixed element $A_i \in GL(N_i, \mathbb{C})$. Applying the automorphism θ to each side of this equation, we then see that

$$r_i(\lambda) = A_i^\vee r_i^\vee(A_i^\vee)^{-1} = (A_i^\vee A_i) r_i(\lambda) (A_i^\vee A_i)^{-1}.$$

Since r_i is irreducible, the product $A_i^\vee A_i$ is a scalar matrix. We can therefore write

$${}^t A_i = c_i A_i, \quad c_i \in \mathbb{C}^*.$$

If we take the transpose of each side of this equation, we see further that $c_i^2 = 1$. Thus, c_i equals $+1$ or -1 , and the nonsingular matrix A_i is either symmetric or skew-symmetric. The mapping

$$x_i \longrightarrow (A_i^{-1})^t x_i A_i, \quad x_i \in GL(N_i),$$

represents the adjoint relative to the bilinear form defined by A_i . Therefore $r_i(\lambda)$ belongs to the corresponding orthogonal group $O(A_i, \mathbb{C})$ or symplectic group $Sp(A_i, \mathbb{C})$, according to whether c_i equals $+1$ or -1 .

Let us write I_O and I_S for the set of indices i such that c_i equals $+1$ and -1 respectively. We then write

$$r_\varepsilon(\lambda) = \bigoplus_{i \in I_\varepsilon} r_i(\lambda), \quad \lambda \in \Lambda_F,$$

$$A_\varepsilon = \bigoplus_{i \in I_\varepsilon} A_i,$$

and

$$N_\varepsilon = \sum_{i \in I_\varepsilon} N_i,$$

for ε equal to O or S . Thus A_O is a symmetric matrix in $GL(N_O, \mathbb{C})$, A_S is a skew-symmetric matrix in $GL(N_S, \mathbb{C})$, and r_O and r_S are representations of Λ_F that take values in the respective groups $O(A_O, \mathbb{C})$ and $Sp(A_S, \mathbb{C})$. We have established a canonical decomposition

$$r = r_O \oplus r_S$$

of the self-dual representation r into orthogonal and symplectic components.

It will be only the equivalence class of r that is relevant, so we are free to replace $r(\lambda)$ by its conjugate

$$B^{-1}r(\lambda)B$$

by a matrix $B \in GL(N, \mathbb{C})$. This has the effect of replacing the matrix

$$A = A_O \oplus A_S$$

by tBAB . In particular, we could take A_O to be any symmetric matrix in $GL(N_O, \mathbb{C})$, and A_S to be any skew-symmetric matrix in $GL(N_S, \mathbb{C})$. We may therefore put the orthogonal and symplectic groups that contain the images of r_O and r_S into standard form.

It will be convenient to adopt a slightly different convention for these groups. As our standard orthogonal group in $GL(N)$, we take

$$O(N) = O(N, J),$$

where

$$J = J(N) = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}$$

is the ‘‘second diagonal’’ in $GL(N)$. This is a group of two connected components, whose identity component is the special orthogonal group

$$SO(N) = \{x \in O(N) : \det(x) = 1\}.$$

As the standard symplectic group in $GL(N)$, defined for $N = 2N'$ even, we take the connected group

$$Sp(N) = Sp(N, J')$$

for the skew-symmetric matrix

$$J' = J'(N) = \begin{pmatrix} 0 & -J(N') \\ J(N') & 0 \end{pmatrix}.$$

The advantage of this formalism is that the set of diagonal matrices in either $SO(N)$ or $Sp(N)$ forms a maximal torus. Similarly, the set of upper triangular matrices in either group forms a Borel subgroup. The point is that if

$${}_t x = J^t x J = J^t x J^{-1}, \quad x \in GL(N),$$

denotes the transpose of x about the second diagonal, the automorphism

$$\text{Int}(J) \circ \theta : x \longrightarrow J\theta(x)J^{-1} = {}_t x^{-1}$$

of $GL(N)$ stabilizes the standard Borel subgroup of upper triangular matrices. Notice that there is a related automorphism

$$\tilde{\theta}(N) = \text{Int}(\tilde{J}) \circ \theta : x \longrightarrow \tilde{J}\theta(x)\tilde{J}^{-1},$$

defined by the matrix

$$(1.2.1) \quad \tilde{J} = \tilde{J}(N) = \begin{pmatrix} 0 & & & 1 \\ & \ddots & -1 & \\ & & \ddots & \\ (-1)^{N+1} & & & 0 \end{pmatrix},$$

which stabilizes the standard splitting in $GL(N)$ as well. Both of these automorphisms lie in the inner class of θ , and either one could have been used originally in place of θ .

Returning to our discussion, we can arrange that A equals $J(N_O) \oplus J'(N_S)$. It is best to work with the matrix

$$J_{O,S} = J(N_O, N_S) = \begin{pmatrix} 0 & & -J(N'_S) \\ & J(N_O) & \\ J(N'_S) & & 0 \end{pmatrix}, \quad N_S = 2N'_S,$$

obtained from the obvious embedding of $J(N_O) \oplus J'(N_S)$ into $GL(N, \mathbb{C})$. The associated elliptic representation r from the given equivalence class then maps Λ_F to the corresponding subgroup of $GL(N, \mathbb{C})$, namely the subgroup

$$O(N_O, \mathbb{C}) \times Sp(N_S, \mathbb{C})$$

defined by the embedding

$$(x, y) \longrightarrow \begin{pmatrix} y_{11} & 0 & y_{12} \\ 0 & x & 0 \\ y_{21} & 0 & y_{22} \end{pmatrix},$$

where y_{ij} are the four $(N'_S \times N'_S)$ -block components of the matrix $y \in Sp(N_S, \mathbb{C})$.

The symplectic part r_S of r is the simpler of the two. Its image is contained in the *connected* complex group

$$\widehat{G}_S = Sp(N_S, \mathbb{C}).$$

This in turn is the dual group of the split classical group

$$G_S = SO(N_S + 1).$$

The orthogonal part r_O of r is complicated by the fact that its image is contained only in the *disconnected* group $O(N_O, \mathbb{C})$. Its composition with the projection of $O(N_O, \mathbb{C})$ onto the group

$$O(N_O, \mathbb{C})/SO(N_O, \mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}$$

of components yields a character η on Λ_F of order 1 or 2. Since we are assuming that the kernel of the mapping of Λ_F to Γ_F is connected, η can be identified with a character on the Galois group Γ_F of order 1 or 2. This in turn determines an extension E of F of degree 1 or 2.

Suppose first that N_O is odd. In this case, the matrix $(-I)$ in $O(N_O)$ represents the nonidentity component, and the orthogonal group is a direct product

$$O(N_O, \mathbb{C}) = SO(N_O, \mathbb{C}) \times \mathbb{Z}/2\mathbb{Z}.$$

We write

$$SO(N_O, \mathbb{C}) = \widehat{G}_O,$$

where G_O is the split group $Sp(N_O - 1)$ over F . We then use η to identify the direct product

$${}^L(G_O)_{E/F} = \widehat{G}_O \times \Gamma_{E/F}$$

with a subgroup of $O(N_O, \mathbb{C})$, namely $SO(N_O, \mathbb{C})$ or $O(N_O, \mathbb{C})$, according to whether η has order 1 or 2. We thus obtain an embedding of the (restricted) L -group of G_O into $GL(N_O, \mathbb{C})$.

Assume next that N is even. In this case, the nonidentity component in $O(N)$ acts by an outer automorphism on $SO(N_O)$. We write

$$SO(N_O, \mathbb{C}) = \widehat{G}_O,$$

where G_O is now the corresponding quasisplit orthogonal group $SO(N_O, \eta)$ over F defined by η . In other words, G_O is the split group $SO(N_O)$ if η is trivial, and the nonsplit group obtained by twisting $SO(N_O)$ over E by the given outer automorphism if η is nontrivial. Let $\tilde{w}(N_O)$ be the permutation matrix in $GL(N_O)$ that interchanges the middle two coordinates, and leaves the other coordinates invariant. We take this element as a representative of the nonidentity component of $O(N_O, \mathbb{C})$. We then use η to identify the semidirect product

$${}^L(G_O)_{E/F} = \widehat{G}_O \rtimes \Gamma_{E/F}$$

with a subgroup of $O(N_O, \mathbb{C})$, namely $SO(N_O, \mathbb{C})$ or $O(N_O, \mathbb{C})$ as before. We again obtain an embedding of the (restricted) L -group of G_O into $GL(N_O, \mathbb{C})$.

We have shown that the elliptic self-dual representation r factors through the embedded subgroup

$${}^L G_{E/F} = {}^L(G_O)_{E/F} \times {}^L(G_S)_{E/F}$$

of $GL(N, \mathbb{C})$ attached to a quasisplit group

$$G = G_O \times G_S$$

over F . The group G is called a θ -twisted *endoscopic group* for $GL(N)$. It is determined by r , and in fact by the decomposition $N = N_O + N_S$ and the character $\eta = \eta_G$ (of order 1 or 2) attached to r . The same is true of the L -embedding

$$\xi = \xi_{O,S,\eta} : {}^L G = \widehat{G} \rtimes \Gamma_F \hookrightarrow {}^L G(N) = GL(N, \mathbb{C}) \times \Gamma_F,$$

obtained by inflating the embedding above to the full L -groups.

It is convenient to form the semidirect product

$$\widetilde{G}^+(N) = GL(N) \rtimes \langle \theta \rangle = G(N) \rtimes \langle \widetilde{\theta}(N) \rangle,$$

where $\langle \theta \rangle$ and $\langle \widetilde{\theta}(N) \rangle$ are the groups of order 2 generated by the automorphisms θ and $\widetilde{\theta}(N)$. We write

$$\widetilde{G}^0(N) = GL(N) \rtimes 1 = G(N) \rtimes 1$$

for the identity component, which we can of course identify with the general linear group $GL(N)$, and

$$(1.2.2) \quad \widetilde{G}(N) = GL(N) \rtimes \theta = G(N) \rtimes \widetilde{\theta}(N)$$

for the other connected component. Given r , and hence also the decomposition $N = N_O + N_S$, we can form the semisimple element

$$s = s_{O,S} = J_{O,S}^{-1} \rtimes \theta,$$

in the “dual set” $\widehat{\widetilde{G}}(N)$ of complex points $GL(N, \mathbb{C}) \rtimes \theta$. The complex group $\widehat{G} = \widehat{G}_O \times \widehat{G}_S$, attached to r as above, is then the connected centralizer of s in the group

$$\widehat{G}(N) = \widehat{\widetilde{G}}^0(N) = GL(N, \mathbb{C}).$$

The triplet (G, s, ξ) is called an *endoscopic datum* for $\widetilde{G}(N)$, since it becomes a special case of the terminology of [KS, p. 16] if we replace $\widetilde{G}(N)$ with either of the pairs $(\widetilde{G}^0(N), \widetilde{\theta}(N))$ or $(G(N), \theta) = (GL(N), \theta)$.

The endoscopic datum (G, s, ξ) we have introduced has the property of being elliptic. This is a consequence of our condition that the original self-dual representation r is elliptic. A general (nonelliptic) endoscopic datum for $\widetilde{G}(N)$ is again a triplet (G, s, ξ) , where G is a quasisplit group over F , s is a semisimple element in $\widehat{\widetilde{G}}(N)$ of which \widehat{G} is the connected centralizer in $\widehat{\widetilde{G}}^0(N)$, and ξ is an L -embedding of ${}^L G$ into the L -group ${}^L \widetilde{G}^0(N) = {}^L G(N)$ of $GL(N)$. (In the present setting, we are free to take either the Galois or Weil form of the L -groups.) We require that ξ equal the identity on \widehat{G} , and

that the projection onto $\widehat{G}^0(N)$ of the image of ξ lie in the full centralizer of s . The endoscopic group G (or datum (G, s, ξ)) for $\widetilde{G}(N)$ then said to be *elliptic* if $Z(\widehat{G})^\Gamma$, the subgroup of elements in the center $Z(\widehat{G})$ of \widehat{G} invariant under the action of the Galois group $\Gamma = \Gamma_F$, is finite.

The notion of *isomorphism* between two general endoscopic data is defined in [KS, p. 18]. In the case at hand, it is given by an element g in the dual group $\widehat{G}(N) = GL(N, \mathbb{C})$ whose action by conjugation is compatible in a natural sense with the two endoscopic data. We write

$$\widetilde{\text{Aut}}_N(G) = \text{Aut}_{\widetilde{G}(N)}(G)$$

for the group of isomorphisms of the endoscopic datum G to itself. The main role for this group is in its image

$$\widetilde{\text{Out}}_N(G) = \widetilde{\text{Aut}}_N(G)/\widetilde{\text{Int}}_N(G)$$

in the group of outer automorphisms of the group G over F . (Following standard practice, we often let the endoscopic group G represent a full endoscopic datum (G, s, ξ) , or even an isomorphism class of such data.) If G represents one of the elliptic endoscopic data constructed above, $\widetilde{\text{Out}}_N(G)$ is trivial if the integer N_O is odd or zero. In the remaining case that N_O is even and positive, $\widetilde{\text{Out}}_N(G)$ is a group of order 2, the nontrivial element being the outer automorphism induced by the nontrivial connected component of $O(N_O, \mathbb{C})$.

We write

$$\widetilde{\mathcal{E}}(N) = \mathcal{E}(\widetilde{G}(N))$$

for the set of isomorphism classes of endoscopic data for $\widetilde{G}(N)$, and

$$\widetilde{\mathcal{E}}_{\text{ell}}(N) = \mathcal{E}_{\text{ell}}(\widetilde{G}(N))$$

for the subset of classes in $\widetilde{\mathcal{E}}(N)$ that are elliptic. The data (G, s, ξ) , attached to equivalence classes of elliptic, self-dual representations r as above, form a complete set of representatives of $\widetilde{\mathcal{E}}_{\text{ell}}(N)$. The set $\widetilde{\mathcal{E}}_{\text{ell}}(N)$ is thus parametrized by triplets (N_O, N_S, η) , where $N_O + N_S = N$ is a decomposition of N into nonnegative integers with N_S even, and $\eta = \eta_G$ is a character of Γ_F of order 1 or 2 with the property that $\eta = 1$ if $N_O = 0$, and $\eta \neq 1$ if $N_O = 2$. (The last constraint is required in order that the datum be elliptic.) The goal of this volume is to describe the representations of the classical groups G in terms of those of $GL(N)$. The general arguments will be inductive. For this reason, the case in which \widehat{G} is either purely orthogonal or purely symplectic will have a special role. Accordingly, we write

$$\widetilde{\mathcal{E}}_{\text{sim}}(N) = \mathcal{E}_{\text{sim}}(\widetilde{G}(N))$$

for the set of elements in $\widetilde{\mathcal{E}}_{\text{ell}}(N)$ that are *simple*, in the sense that one of the integers N_O or N_S vanishes. We then have a chain of sets

$$(1.2.3) \quad \widetilde{\mathcal{E}}_{\text{sim}}(N) \subset \widetilde{\mathcal{E}}_{\text{ell}}(N) \subset \widetilde{\mathcal{E}}(N),$$

which are all finite if F is local, and all infinite if F is global.

The elements $G \in \tilde{\mathcal{E}}(N)$ are usually called twisted endoscopic data, since they are attached to the automorphism θ . We shall have to work also with ordinary (untwisted) endoscopic data, at least for the quasisplit orthogonal and symplectic groups G that represent elements in $\tilde{\mathcal{E}}_{\text{sim}}(N)$. An endoscopic datum G' for G is similar to what we have described above for $\tilde{G}(N)$. It amounts to a triplet (G', s', ξ') , where G' is a (connected) quasisplit group over F , s' is a semisimple element in \hat{G} of which \hat{G}' is the connected centralizer in \hat{G} , and ξ' is an L -embedding of ${}^L G'$ into ${}^L G$. We again require that ξ' equal the identity on \hat{G}' , and that its image lie in the centralizer of s' in ${}^L G$. (See [LS1, (1.2)], a special case of the general definition in [KS], which we have specialized further to the case at hand.) There is again the notion of isomorphism of endoscopic data, which allows us to form the associated finite group

$$\text{Out}_G(G') = \text{Aut}_G(G')/\text{Int}_G(G')$$

of outer automorphisms of any given G' . We write $\mathcal{E}(G)$ for the set of isomorphism classes of endoscopic data G' for G , and $\mathcal{E}_{\text{ell}}(G)$ for the subset of data that are elliptic, in the sense that $Z(\hat{G}')^F$ is finite. We then have a second chain of sets

$$(1.2.4) \quad \mathcal{E}_{\text{sim}}(G) \subset \mathcal{E}_{\text{ell}}(G) \subset \mathcal{E}(G),$$

where $\mathcal{E}_{\text{sim}}(G) = \{G\}$ is the subset consisting of G alone. Similar definitions apply to groups G that represent more general data in $\tilde{\mathcal{E}}(N)$.

It is easy to describe the set $\mathcal{E}_{\text{ell}}(G)$, for any $G \in \tilde{\mathcal{E}}_{\text{sim}}(N)$. It suffices to consider diagonal matrices $s' \in \hat{G}$ with eigenvalues ± 1 . For example, in the first case that $G = SO(N+1)$ and $\hat{G} = Sp(N, \mathbb{C})$ (with $N = N_S$ even), it is enough to take diagonal matrices of the form

$$s' = \begin{pmatrix} -I_1'' & & 0 \\ & I_2' & \\ 0 & & -I_1'' \end{pmatrix},$$

where I_1'' is the identity matrix of rank $N_1'' = N_1'/2$, and I_2' is the identity matrix of rank N_2' . The set $\mathcal{E}_{\text{ell}}(G)$ is parametrized by pairs (N_1', N_2') of even integers with $0 \leq N_1' \leq N_2'$ and $N = N_1' + N_2'$. The corresponding endoscopic groups are the split groups

$$G' = SO(N_1' + 1) \times SO(N_2' + 1),$$

with dual groups

$$\hat{G}' = Sp(N_1', \mathbb{C}) \times Sp(N_2', \mathbb{C}) \subset Sp(N, \mathbb{C}) = \hat{G}.$$

The group $\text{Out}_G(G')$ is trivial in this case unless $N_1' = N_2'$, in which case it has order 2.

The other cases are similar. In the second case that $G = Sp(N-1)$ and $\hat{G} = SO(N, \mathbb{C})$ (with $N = N_O$ odd), $\mathcal{E}_{\text{ell}}(G)$ is parametrized by pairs (N_1', N_2') of nonnegative even integers with $N = N_1' + (N_2' + 1)$, and characters

η' on Γ_F with $(\eta')^2 = 1$. The corresponding endoscopic groups are the quasisplit groups

$$G' = SO(N'_1, \eta') \times Sp(N'_2),$$

with dual groups

$$\widehat{G}' = SO(N'_1, \mathbb{C}) \times SO(N'_2 + 1, \mathbb{C}) \subset SO(N, \mathbb{C}) = \widehat{G}.$$

In the third case that $G = SO(N, \eta)$ and $\widehat{G} = SO(N, \mathbb{C})$ (with $N = N_O$ even), $\mathcal{E}_{\text{ell}}(G)$ is parametrized by pairs of even integers (N'_1, N'_2) with $0 \leq N'_1 \leq N'_2$ and $N = N'_1 + N'_2$, and pairs (η'_1, η'_2) of characters on Γ_F with $(\eta'_1)^2 = (\eta'_2)^2 = 1$ and $\eta = \eta'_1 \eta'_2$. The corresponding endoscopic groups are the quasisplit groups

$$G' = SO(N'_1, \eta'_1) \times SO(N'_2, \eta'_2),$$

with dual groups

$$\widehat{G}' = SO(N'_1, \mathbb{C}) \times SO(N'_2, \mathbb{C}) \subset SO(N, \mathbb{C}) = \widehat{G}.$$

In the second and third cases, each character η' has to be nontrivial if the corresponding integer N' equals 2, while if $N' = 0$, η' must of course be trivial. In these cases, the group $\text{Out}_G(G')$ has order 2 unless N is even, and $N'_1 = N'_2$ and $\eta'_1 = \eta'_2$, in which case it is a product of two groups of order 2, or $N'_1 = 0$, in which case the group is trivial.

Observe that in the even orthogonal case, where $\widehat{G} = SO(N, \mathbb{C})$ with N even, the endoscopic data $G' \in \mathcal{E}_{\text{ell}}(G)$ will not be able to isolate constituents r_i of r of *odd* dimension. The discrepancy is made up by a third kind of endoscopic datum. These are the twisted endoscopic data for the even orthogonal groups $G = SO(N, \eta)$ in $\tilde{\mathcal{E}}_{\text{sim}}(N)$. For any such G , let

$$(1.2.5) \quad \tilde{G} = G \rtimes \tilde{\theta}, \quad \tilde{\theta} = \text{Int}(\tilde{w}(N)),$$

be the nonidentity component in the semi-direct product of G with the group of order two generated by the outer automorphism $\tilde{\theta}$ of $SO(N)$. In this setting a (twisted) endoscopic datum is a triplet $(\tilde{G}', \tilde{s}', \tilde{\xi}')$, where \tilde{G}' is a quasisplit group over F , \tilde{s}' is a semisimple element in the “dual set” $\tilde{\widehat{G}} = \widehat{G} \rtimes \tilde{\theta}$ of which \widehat{G}' is the connected centralizer in \widehat{G} , and $\tilde{\xi}'$ is an L -embedding of ${}^L\tilde{G}'$ into ${}^L\tilde{G}$, all being subject also to further conditions and definitions as above. The subset $\mathcal{E}_{\text{ell}}(\tilde{G}) \subset \mathcal{E}(\tilde{G})$ of isomorphism classes of elliptic endoscopic data for \tilde{G} is parametrized by pairs of *odd* integers $(\tilde{N}'_1, \tilde{N}'_2)$, with $0 \leq \tilde{N}'_1 \leq \tilde{N}'_2$ and $N = \tilde{N}'_1 + \tilde{N}'_2$, and pairs of characters $(\tilde{\eta}'_1, \tilde{\eta}'_2)$ on Γ_F , with $(\tilde{\eta}'_1)^2 = (\tilde{\eta}'_2)^2 = 1$ and $\eta = \tilde{\eta}'_1 \tilde{\eta}'_2$. The corresponding endoscopic groups are the quasisplit groups

$$\tilde{G}' = Sp(\tilde{N}'_1 - 1) \times Sp(\tilde{N}'_2 - 1),$$

with dual groups

$$\widehat{\tilde{G}}' = SO(\tilde{N}'_1, \mathbb{C}) \times SO(\tilde{N}'_2, \mathbb{C}) \subset SO(N, \mathbb{C}) = \widehat{G}.$$

The family $\mathcal{E}_{\text{ell}}(\tilde{G})$ will have to be part of our analysis. However, its role will be subsidiary to that of the two primary families $\tilde{\mathcal{E}}_{\text{ell}}(N)$ and $\mathcal{E}_{\text{ell}}(G)$.

We have completed our brief study of elliptic self-dual representations r . Remember that we are regarding these objects as parameters, in the spirit of §1.1. We have seen that a parameter for $GL(N)$ factors into a product of two parameters for quasisplit classical groups. The products are governed by twisted endoscopic data $G \in \tilde{\mathcal{E}}_{\text{ell}}(N)$. They can be refined further according to ordinary endoscopic data $G' \in \mathcal{E}_{\text{ell}}(G)$. Thus, while the parameters will not be available (for lack of a global Langlands group L_F), the endoscopic data that control many of their properties will be. Before we can study the ramifications of this, we must first formulate a makeshift substitute for the parameters attached to our classical groups. We shall do so in §1.4, after a review in §1.3 of the representations of $GL(N)$ that will serve as parameters for this group.

We have considered only the self-dual representation r that are elliptic, since it is these objects that pertain directly to our theorems. Before going on, we might ask what happens if r is not elliptic. A moment's reflection reveals that any such r factors through subgroups of $GL(N, \mathbb{C})$ attached to several data $G \in \tilde{\mathcal{E}}_{\text{ell}}(N)$, in contrast to what we have seen in the elliptic case. This is because r also factors through a subgroup attached to a datum M in the complement of $\tilde{\mathcal{E}}_{\text{ell}}(N)$ in $\mathcal{E}(N)$, and because any such M can be identified with a proper Levi subgroup of several G . The analysis of general self-dual representations r is therefore more complicated, though still not very difficult. It is best formulated in terms of the centralizers

$$\tilde{S}_r(N) = S_r(\tilde{G}(N)) = \text{Cent}(\text{Im}(r), \hat{\tilde{G}}(N))$$

and

$$S_r = S_r(G) = \text{Cent}(\text{Im}(r), \hat{G})$$

of the images of r . We shall return to this matter briefly in §1.4, and then more systematically as part of the general theory of Chapter 4.

1.3. Representations of $GL(N)$

A general goal, for the present volume and beyond, is to classify representations of a broad class of groups in terms of those of general linear groups. What makes this useful is the fact that much of the representation theory of $GL(N)$ is both well understood and relatively simple. We shall review what we need of the theory.

Suppose first that F is local. In this case, we can replace the abstract group Λ_F of the last section by the local Langlands group L_F defined by (1.1.1). The local Langlands classification parametrizes irreducible representations of $GL(N, F)$ in terms of N -dimensional representations

$$\phi : L_F \longrightarrow GL(N, \mathbb{C}).$$

Before we state it formally, we should recall a few basic notions.

Given a finite dimensional (semisimple, continuous) representation ϕ of L_F , we can form the local L -function $L(s, \phi)$, a meromorphic function of $s \in \mathbb{C}$. We can also form the local ε -factor $\varepsilon(s, \phi, \psi_F)$, a monomial of the form ab^{-s} which also depends on a nontrivial additive character ψ_F of F . If F is archimedean, we refer the reader to the definition in [T2, §3]. If F is p -adic, we extend ϕ analytically to a representation of the product of W_F with the complexification $SL(2, \mathbb{C})$ of the subgroup $SU(2)$ of L_F . We can then form the representation

$$\mu_\phi(w) = \phi \left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix} \right), \quad w \in W_F,$$

of W_F , where $|w|$ is the absolute value on W_F , and the nilpotent matrix

$$N_\phi = \log \phi \left(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right).$$

The pair $V_\phi = (\mu_\phi, N_\phi)$ gives a representation of the Weil-Deligne group [T2, (4.1.3)], for which we define an L -function

$$L(s, \phi) = Z(V_\phi, q_F^{-s})$$

and ε -factor

$$\varepsilon(s, \phi, \psi_F) = \varepsilon(V_\phi, q_F^{-s}),$$

following notation in [T2, §4]. (We have written q_F here for the order of the residue field of F .) Of particular interest are the tensor product L -function

$$L(s, \phi_1 \times \phi_2) = L(s, \phi_1 \otimes \phi_2)$$

and ε -factor

$$\varepsilon(s, \phi_1 \times \phi_2, \psi_F) = \varepsilon(s, \phi_1 \otimes \phi_2, \psi_F),$$

attached to any pair of representations ϕ_1 and ϕ_2 of L_F .

We expect also to be able to attach local L -functions $L(s, \pi, r)$ and ε -factors $\varepsilon(s, \pi, r, \psi_F)$ to any connected reductive group G over F , where π ranges over irreducible representations of $G(F)$, and r is a finite dimensional representation of ${}^L G$. For general G , this has been done in only the simplest of cases. However, if G is a product $G(N_1) \times G(N_2)$ of general linear groups, there is a broader theory [JPS]. (See also [MW2, Appendice].) It applies to any representation $\pi = \pi_1 \times \pi_2$, in the case that r is the standard representation

$$(1.3.1) \quad r(g_1, g_2) : X \longrightarrow g_1 \cdot X \cdot {}^t g_2, \quad g_i \in G(N_i),$$

of

$$\widehat{G} = GL(N_1, \mathbb{C}) \times GL(N_2, \mathbb{C})$$

on the space of complex $(N_1 \times N_2)$ -matrices X . The theory yields functions

$$L(s, \pi_1 \times \pi_2) = L(s, \pi, r)$$

and

$$\varepsilon(s, \pi_1 \times \pi_2, \psi_F) = \varepsilon(s, \pi, r, \psi_F),$$

known as local Rankin-Selberg convolutions.

The local classification for $GL(N)$ is essentially characterized by being compatible with these local Rankin-Selberg convolutions. It has other important properties as well. Some of these relate to supplementary conditions we can impose on the parameters ϕ as follows.

Suppose for a moment that G is any connected group over F . We write $\Phi(G)$ for the set of \widehat{G} -orbits of (semisimple, continuous, G -relevant) L -homomorphisms

$$\phi : L_F \longrightarrow {}^L G,$$

and $\Pi(G)$ for the set of equivalence classes of irreducible (admissible) representations of $G(F)$. (See [Bo].) These sets come with parallel chains of subsets

$$\Phi_{2,\text{bdd}}(G) \subset \Phi_{\text{bdd}}(G) \subset \Phi(G)$$

and

$$\Pi_{2,\text{temp}}(G) \subset \Pi_{\text{temp}}(G) \subset \Pi(G).$$

In the second chain, $\Pi_{\text{temp}}(G)$ denotes the set of tempered representations in $\Pi(G)$, and

$$\Pi_{2,\text{temp}}(G) = \Pi_2(G) \cap \Pi_{\text{temp}}(G),$$

where $\Pi_2(G)$ is the set of representations in $\Pi(G)$ that are essentially square integrable, in the sense that after tensoring with the appropriate positive character on $G(F)$, they are square integrable modulo the centre of $G(F)$. Recall that $\Pi_{\text{temp}}(G)$ can be described informally as the set of representations $\pi \in \Pi(G)$ that occur in the spectral decomposition of $L^2(G(F))$. Similarly, $\Pi_{2,\text{temp}}(G)$ is the set of π that occur in the discrete spectrum (taken modulo the center). In the first chain, $\Phi_{\text{bdd}}(G)$ denotes the set of $\phi \in \Phi(G)$ whose image in ${}^L G$ projects onto a relatively compact subset of \widehat{G} , and

$$\Phi_{2,\text{bdd}}(G) = \Phi_2(G) \cap \Phi_{\text{bdd}}(G),$$

where $\Phi_2(G)$ is the set of parameters ϕ in $\Phi(G)$ whose image does not lie in any proper parabolic subgroup ${}^L P$ of ${}^L G$.

In the case $G = G(N) = GL(N)$ of present concern, we write $\Phi(N) = \Phi(GL(N))$ and $\Pi(N) = \Pi(GL(N))$, and follow similar notation for the corresponding subsets above. Then $\Phi(N)$ can be identified with the set of equivalence classes of (semisimple, continuous) N -dimensional representations of L_F . The subset

$$\Phi_{\text{sim}}(N) = \Phi_2(N)$$

consists of those representations that are irreducible, while $\Phi_{\text{bdd}}(N)$ corresponds to representations that are unitary. On the other hand, the set $\Pi_{\text{unit}}(N)$ of unitary representations in $\Pi(N)$ properly contains $\Pi_{\text{temp}}(N)$, if $N \geq 2$. It has an elegant classification [V2], [Tad1], but our point here is

that it is not parallel to the set of N -dimensional representations ϕ that are unitary. We do observe that

$$\Pi_{2,\text{temp}}(N) = \Pi_2(N) \cap \Pi_{\text{unit}}(N) = \Pi_{2,\text{unit}}(N),$$

so the notions of tempered and unitary are the same for square integrable representations.

If F is p -adic, we can write $\Pi_{\text{scusp,temp}}(N)$ (resp. $\Pi_{\text{scusp}}(N)$) for the set of *supercuspidal* representations in $\Pi_{2,\text{temp}}(N)$ (resp. $\Pi_2(N)$). It is also convenient to write $M_{\text{sim,bdd}}(N)$ (resp. $M_{\text{sim}}(N)$) for the set of parameters $\mu = \phi$ in $\Phi_{\text{sim,bdd}}(N)$ (resp. $\Phi_{\text{sim}}(N)$) that are trivial on the second factor $SU(2)$ of L_F . If F is archimedean, it is natural to take $\Pi_{\text{scusp,temp}}(N)$ and $\Pi_{\text{scusp}}(N)$ to be empty unless $GL(N, F)$ is compact modulo the center (which is a silly way of saying that $N = 1$), in which case we can take them to be the corresponding sets $\Pi_{2,\text{temp}}(GL(1)) = \Pi_{\text{temp}}(GL(1))$ and $\Pi_2(GL(1)) = \Pi(GL(1))$. With the parameter sets defined accordingly, we then have two parallel chains

$$(1.3.2) \quad M_{\text{sim,bdd}}(N) \subset \Phi_{\text{sim,bdd}}(N) \subset \Phi_{\text{bdd}}(N) \subset \Phi(N),$$

and

$$(1.3.3) \quad \Pi_{\text{scusp,temp}}(N) \subset \Pi_{2,\text{temp}}(N) \subset \Pi_{\text{temp}}(N) \subset \Pi(N),$$

for our given local field F .

The local classification for $G = GL(N)$ can now be formulated as follows.

THEOREM 1.3.1 (Langlands [L11], Harris-Taylor [HT], Henniart [He1], Scholze [Sch]). *There is a unique bijective correspondence $\phi \rightarrow \pi$ from $\Phi(N)$ onto $\Pi(N)$ such that*

$$(i) \quad \phi \otimes \chi \longrightarrow \pi \otimes (\chi \circ \det),$$

for any character χ in the group $\Phi(1) = \Pi(1)$,

$$(ii) \quad \det \circ \phi \longrightarrow \eta_\pi,$$

for the central character η_π of π , and

$$(iii) \quad \phi^\vee \longrightarrow \pi^\vee,$$

for the contragredient involutions \vee on $\Phi(N)$ and $\Pi(N)$, and such that if

$$\phi_i \longrightarrow \pi_i, \quad \phi_i \in \Phi(N_i), \quad i = 1, 2,$$

then

$$(iv) \quad L(s, \pi_1 \times \pi_2) = L(s, \phi_1 \times \phi_2)$$

and

$$(v) \quad \varepsilon(s, \pi_1 \times \pi_2, \psi_F) = \varepsilon(s, \phi_1 \times \phi_2, \psi_F).$$

Furthermore, the bijection is compatible with the two chains (1.3.2) and (1.3.3), in the sense that it maps each subset in (1.3.2) onto its counterpart in (1.3.3). \square

We now take F to be global. This brings us to the representations that will be the foundation of all that follows. They are the objects in the set

$$\mathcal{A}_{\text{cusp}}(N) = \mathcal{A}_{\text{cusp}}(GL(N))$$

of (equivalence classes of) unitary, cuspidal automorphic representations of $GL(N)$.

Assume first that G is an arbitrary connected group over the global field F . To suppress the noncompact part of the center, one often works with the closed subgroup

$$G(\mathbb{A})^1 = \{x \in G(\mathbb{A}) : |\chi(x)| = 1, \chi \in X^*(G)_F\}$$

of $G(\mathbb{A})$, where $X^*(G)_F$ is the additive group of characters of G defined over F . We recall that $G(F)\backslash G(\mathbb{A})^1$ has finite volume, and that there is a sequence

$$L_{\text{cusp}}^2(G(F)\backslash G(\mathbb{A})^1) \subset L_{\text{disc}}^2(G(F)\backslash G(\mathbb{A})^1) \subset L^2(G(F)\backslash G(\mathbb{A})^1),$$

of embedded, right $G(\mathbb{A})^1$ -invariant Hilbert spaces. In particular, the space $L_{\text{cusp}}^2(G(F)\backslash G(\mathbb{A})^1)$ of cuspidal functions in $L^2(G(F)\backslash G(\mathbb{A})^1)$ is contained in the subspace $L_{\text{disc}}^2(G(F)\backslash G(\mathbb{A})^1)$ that decomposes under the action of $G(\mathbb{A})^1$ into a direct sum of irreducible representations. We can then introduce a corresponding chain of subsets of irreducible automorphic representations

$$\mathcal{A}_{\text{cusp}}(G) \subset \mathcal{A}_2(G) \subset \mathcal{A}(G).$$

By definition, $\mathcal{A}_{\text{cusp}}(G)$, $\mathcal{A}_2(G)$ and $\mathcal{A}(G)$ denote the subsets of irreducible *unitary* representations π of $G(\mathbb{A})$ whose restrictions to $G(\mathbb{A})^1$ are irreducible constituents of the respective spaces L_{cusp}^2 , L_{disc}^2 and L^2 . We shall also write $\mathcal{A}_{\text{cusp}}^+(G)$ and $\mathcal{A}_2^+(G)$ for the analogues of $\mathcal{A}_{\text{cusp}}(G)$ and $\mathcal{A}_2(G)$ defined without the condition that π be unitary. (The definition of $\mathcal{A}(G)$ here is somewhat informal, and will be used only for guidance. It can in fact be made precise at the singular points in the continuous spectrum where there might be some ambiguity.)

We specialize again to the case $G = GL(N)$, taken now over the global field F . We write $\mathcal{A}(N) = \mathcal{A}(GL(N))$, with similar notation for the corresponding subsets above. Observe that $GL(N, \mathbb{A})^1$ is the group of adelic matrices $x \in GL(N, \mathbb{A})$ whose determinant has absolute value 1. In this case, the set $\mathcal{A}(N)$ is easy to characterize. It is composed of the induced representations

$$\pi = \mathcal{I}_P(\pi_1 \otimes \cdots \otimes \pi_r), \quad \pi_i \in \mathcal{A}_2(N_i),$$

where P is the standard parabolic subgroup of block upper triangular matrices in $G = GL(N)$ corresponding to a partition (N_1, \dots, N_r) of N , with the standard Levi subgroup of block diagonal matrices

$$M_P = GL(N_1) \times \cdots \times GL(N_r).$$

This follows from the theory of Eisenstein series [L1], [L5], [A1] (valid for any G), and the fact [Be] (special to $G = GL(N)$) that an induced

representation $\mathcal{I}_P^G(\sigma)$ is irreducible for any representation σ of $M_P(\mathbb{A})$ that is irreducible and unitary. Let us also write $\mathcal{A}^+(N)$ for the set of induced representations as above, but with the components π_i now taken from the larger sets $\mathcal{A}_2^+(N_i)$. These are global analogues of what are usually called standard representations. They can be reducible at certain points, although they typically remain irreducible. We thus have a chain of sets

$$(1.3.4) \quad \mathcal{A}_{\text{cusp}}(N) \subset \mathcal{A}_2(N) \subset \mathcal{A}(N) \subset \mathcal{A}^+(N)$$

of representations, for our given global field F . This is a rough global analogue of the local sequence (1.3.3). We have dropped the subscript “temp” in the global notation, since the local components of a constituent of $L^2(GL(N, F) \backslash GL(N, \mathbb{A})^1)$ need not be tempered, and added the superscript + to remind ourselves that $\mathcal{A}^+(N)$ contains full induced representations, rather than irreducible quotients.

There are two fundamental theorems on the automorphic representations of $GL(N)$ that will be essential to us. The first is the classification of automorphic representations by Jacquet and Shalika, while the second is the characterization by Mœglin and Waldspurger of the discrete spectrum in terms of cuspidal spectra. We shall review each in turn.

Suppose again that G is an arbitrary connected group over the global field F . An automorphic representation π of G is among other things, a weakly continuous, irreducible representation of $G(\mathbb{A})$. As such, it can be written as a restricted tensor product

$$\pi = \bigotimes_v \pi_v$$

of irreducible representations of the local groups $G(F_v)$, almost all which are unramified [F]. Recall that if π_v is unramified, the group $G_v = G/F_v$ is unramified. This means that F_v is a p -adic field, that G_v is quasisplit, and that the action of W_{F_v} on \widehat{G} factors through the infinite cyclic quotient

$$W_{F_v}/I_{F_v} = \langle \text{Frob}_v \rangle$$

of W_{F_v} by the inertia subgroup I_{F_v} with canonical generator Frob_v . Recall also that the local Langlands correspondence has long existed in this very particular context. The \widehat{G} -orbit of homomorphisms

$$\phi_v : L_{F_v} \longrightarrow {}^L G_v = \widehat{G} \rtimes \langle \text{Frob}_v \rangle$$

in $\Phi(G_v)$ to which π_v corresponds factors through the quotient

$$L_{F_v}/(I_{F_v} \times SU(2)) = W_{F_v}/I_{F_v}$$

of L_{F_v} . The resulting mapping

$$\pi_v \longrightarrow c(\pi_v) = \phi_v(\text{Frob}_v)$$

is a bijection from the set of unramified representations of $G(F_v)$ (relative to any given hyperspecial maximal compact subgroup $K_v \subset G(F_v)$) and the set of semisimple \widehat{G} -orbits in ${}^L G_v$ that project to the Frobenius generator

of W_{F_v}/I_{F_v} . Let $c_v(\pi)$ be the image of $c(\pi_v)$ in ${}^L G$ under the embedding of ${}^L G_v$ into ${}^L G$ that is defined canonically up to conjugation. In this way, the automorphic representation π of G gives rise to a family of semisimple conjugacy classes

$$c^S(\pi) = \{c_v(\pi) : v \notin S\}$$

in ${}^L G$, where S is some finite set of valuations of F outside of which G is unramified.

Let $\mathcal{C}_{\text{aut}}^S(G)$ be the set of families

$$c^S = \{c_v : v \notin S\}$$

of semisimple conjugacy classes in ${}^L G$ obtained in this way. That is, $c^S = c^S(\pi)$, for some automorphic representation π of G . We define $\mathcal{C}_{\text{aut}}(G)$ to be the set of equivalence classes of such families, c^S and $(c')^{S'}$ being equivalent if c_v equals c'_v for almost all v . We then have a mapping

$$\pi \longrightarrow c(\pi)$$

from the set of automorphic representations of G onto $\mathcal{C}_{\text{aut}}(G)$.

The families $c^S(\pi)$ arise most often in the guise of (partial) global L -functions. Suppose that r is a finite dimensional representation of ${}^L G$ that is unramified outside of S . The corresponding incomplete L -function is given by an infinite product

$$L^S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r_v)$$

of unramified local L -functions

$$L(s, \pi_v, r_v) = \det(1 - r(c(\pi_v))q_v^{-s})^{-1}, \quad v \notin S,$$

which converges for the real part of s large. It is at the ramified places $v \in S$ that the construction of local L -functions and ε -factors poses a challenge. As the reader is no doubt well aware, the completed L -function

$$L(s, \pi, r) = \prod_v L(s, \pi_v, r_v)$$

is expected to have analytic continuation as a meromorphic function of $s \in \mathbb{C}$ that satisfies the functional equation

$$(1.3.5) \quad L(s, \pi, r) = \varepsilon(s, \pi, r) L(1 - s, \pi, r^\vee),$$

where r^\vee is the contragredient of r , and

$$\varepsilon(s, \pi, r) = \prod_{v \notin S} \varepsilon(s, \pi_v, r_v, \psi_{F_v}).$$

Here, ψ_{F_v} stands for the localization of a nontrivial additive character ψ_F on \mathbb{A}/F that is unramified outside of S .

We return again to the case $G = GL(N)$. The elements in the set

$$\mathcal{C}_{\text{aut}}(N) = \mathcal{C}_{\text{aut}}(GL(N))$$

can be identified with families of semisimple conjugacy classes in $GL(N, \mathbb{C})$, defined up to the equivalence relation above. It is convenient to restrict the domain of the corresponding mapping $\pi \rightarrow c(\pi)$.

Recall that general automorphic representations for $GL(N)$ can be characterized as the irreducible constituents of induced representations

$$\rho = \mathcal{I}_P(\pi_1 \otimes \cdots \otimes \pi_r), \quad \pi_i = \mathcal{A}_{\text{cusp}}^+(N_i).$$

(See Proposition 1.2 of [L6], which applies to any G .) With this condition, the nonunitary induced representation ρ can have many irreducible constituents. However, it does have a canonical constituent. This is the irreducible representation

$$\pi = \bigotimes_v \pi_v,$$

where $\pi_v \in \Pi(GL(N)_v)$ is obtained from the local Langlands parameter

$$\phi_v = \phi_{1,v} \oplus \cdots \oplus \phi_{r,v}, \quad \phi_{i,v} \longrightarrow \pi_{i,v},$$

in $\Phi(GL(N)_v)$ by Theorem 1.3.1 (applied to both $GL(N)_v$ and its subgroups $GL(N_i)_v$). *Isobaric* representations are the automorphic representations π of $GL(N)$ obtained in this way. The equivalence class of π does not change if we reorder the cuspidal representations $\{\pi_i\}$. We formalize this property by writing

$$(1.3.6) \quad \pi = \pi_1 \boxplus \cdots \boxplus \pi_r, \quad \pi_i \in \mathcal{A}_{\text{cusp}}^+(N_i),$$

in the notation [L7, §2] of Langlands, where the right hand side is regarded as a formal, unordered direct sum.

THEOREM 1.3.2 (Jacquet-Shalika [JS]). *The mapping*

$$\pi = \pi_1 \boxplus \cdots \boxplus \pi_r \longrightarrow c(\pi),$$

from the set of equivalence classes of isobaric automorphic representations π of $GL(N)$ to the set of elements $c = c(\pi)$ in $\mathcal{C}_{\text{aut}}(N)$, is a bijection. \square

Global Rankin-Selberg convolutions $L(s, \pi_1 \times \pi_2)$ and ε -factors $\varepsilon(s, \pi_1 \times \pi_2)$ are defined as in the local case. They correspond to automorphic representations $\pi = \pi_1 \times \pi_2$ of a group $G = GL(N_1) \times GL(N_2)$, and the standard representation (1.3.1) of \widehat{G} . The analytic behaviour of these functions is now quite well understood [JPS], [MW2, Appendice]. In particular, $L(s, \pi_1 \times \pi_2)$ has analytic continuation with functional equation (1.3.5). Moreover, if π_1 and π_2 are cuspidal, $L(s, \pi_1 \times \pi_2)$ is an entire function of s unless N_1 equals N_2 , and π_2^\vee is equivalent to the representation

$$\pi_1(g_1) |\det g_1|^{-s_1}, \quad g_1 \in GL(N_1, \mathbb{A}),$$

for some $s_1 \in \mathbb{C}$, in which case $L(s, \pi_1 \times \pi_2)$ has only a simple pole at $s = s_1 + 1$. It is this property that is used to prove Theorem 1.3.2.

Theorem 1.3.2 can be regarded as a characterization of (isobaric) automorphic representations of $GL(N)$ in terms of simpler objects, families of semisimple conjugacy classes. However, it does not in itself characterize

the spectral properties of these representations. For example, it is not a priori clear that the representations in $\mathcal{A}(N)$, or even its subset $\mathcal{A}_2(N)$, are isobaric. The following theorem provides the necessary corroboration.

THEOREM 1.3.3 (Mœglin-Waldspurger [MW2]). *The representations $\pi \in \mathcal{A}_2(N)$ are parametrized by the set of pairs (m, μ) , where $N = mn$ is divisible by m , and μ is a representation in $\mathcal{A}_{\text{cusp}}(m)$. If P is the standard parabolic subgroup corresponding to the partition (m, \dots, m) of N , and σ_μ is the representation*

$$x \longrightarrow \mu(x_1) |\det x_1|^{\frac{n-1}{2}} \otimes \mu(x_2) |\det x_2|^{\frac{n-3}{2}} \otimes \cdots \otimes \mu(x_n) |\det x_n|^{-\frac{(n-1)}{2}}$$

of the Levi subgroup

$$M_P(\mathbb{A}) \cong \{x = (x_1, \dots, x_n) : x_i \in GL(m, \mathbb{A})\},$$

then π is the unique irreducible quotient of the induced representation $\mathcal{I}_P(\sigma_\mu)$. Moreover, the restriction of π to $GL(N, \mathbb{A})^1$ occurs in the discrete spectrum with multiplicity one. \square

Consider the representation $\pi \in \mathcal{A}_2(N)$ described in the theorem. For any valuation v , its local component π_v is the Langlands quotient of the local component $\mathcal{I}_P(\sigma_{\mu,v})$ of $\mathcal{I}_P(\sigma_\mu)$. It then follows from the definitions that π_v is the irreducible representation of $GL(N, F_v)$ that corresponds to the Langlands parameter of the induced representation $\mathcal{I}_P(\sigma_{\mu,v})$. In other words, the automorphic representation π is isobaric. We can therefore write

$$(1.3.7) \quad \pi = \mu\left(\frac{n-1}{2}\right) \boxplus \mu\left(\frac{n-3}{2}\right) \boxplus \cdots \boxplus \mu\left(-\frac{(n-1)}{2}\right),$$

in the notation (1.3.6), for cuspidal automorphic representations

$$\mu(i) : x_i \longrightarrow \mu(x) |x|^i, \quad x \in GL(m, \mathbb{A}).$$

Theorem 1.3.3 also provides a description of the automorphic representations in the larger set $\mathcal{A}(N)$. For as we noted above, $\mathcal{A}(N)$ consists of the set of irreducible induced representations

$$\pi = \mathcal{I}_P(\pi_1 \otimes \cdots \otimes \pi_r), \quad \pi_i \in \mathcal{A}_2(N_i),$$

where $N = N_1 + \cdots + N_r$ is again a partition of N , and P is the corresponding standard parabolic subgroup of $GL(N)$. It is easy to see from its irreducibility that π is also isobaric. We can therefore write

$$(1.3.8) \quad \pi = \pi_1 \boxplus \cdots \boxplus \pi_r, \quad \pi_i \in \mathcal{A}_2(N_i).$$

Observe that despite the notation, (1.3.8) differs slightly from the general isobaric representation (1.3.6). Its constituents π_i are more complex, since they are not cuspidal, but the associated induced representation is simpler, since it is irreducible.

There is another way to view Theorem 1.3.3. Since F is global, the Langlands group L_F is not available to us. If it were, we would expect its set $\Phi(N)$ of (equivalence classes of) N -dimensional representations

$$\phi : L_F \longrightarrow GL(N, \mathbb{C})$$

to parametrize the isobaric automorphic representations of $GL(N)$. A very simple case, for example, is the pullback $|\lambda|$ to L_F of the absolute value on the quotient

$$W_F^{\text{ab}} = GL(1, F) \backslash GL(1, \mathbb{A})$$

of L_F . This of course corresponds to the automorphic character on $GL(1)$ given by the original absolute value. We would expect the subset

$$\Phi_{\text{sim,bdd}}(N) = \Phi_{2,\text{bdd}}(N)$$

of irreducible unitary representations in $\Phi(N)$ to parametrize the unitary cuspidal automorphic representations of $GL(N)$. How then are we to account for the full discrete spectrum $\mathcal{A}_2(N)$? A convenient way to do so is to take the product of L_F with the group $SU(2) = SU(2, \mathbb{R})$.

Let us write $\Psi(N)$ temporarily for the set of (equivalence classes of) unitary N -dimension representations

$$\psi : L_F \times SU(2) \longrightarrow GL(N, \mathbb{C}).$$

According to Theorem 1.3.3, it would then be the subset

$$\Psi_{\text{sim}}(N) = \Psi_2(N)$$

of irreducible representations in $\Psi(N)$ that parametrizes $\mathcal{A}_2(N)$. Indeed, any $\psi \in \Psi_{\text{sim}}(N)$ decomposes uniquely as a tensor product $\mu \otimes \nu$ of irreducible representations of L_F and $SU(2)$. It therefore gives rise to a pair (m, μ) , in which $\mu \in \mathcal{A}_{\text{cusp}}(m)$ represents a unitary cuspidal automorphic representation of $GL(N)$ as well as the corresponding irreducible, unitary, m -dimensional representation of L_F . Conversely, given any such pair, we again identify $\mu \in \mathcal{A}_{\text{cusp}}(m)$ with the corresponding representation of L_F , and we take ν to be the unique irreducible representation of $SU(2)$ of degree $n = Nm^{-1}$.

We are identifying any finite dimensional representation of $SU(2)$ with its analytic extension to $SL(2, \mathbb{C})$. With this convention, we attach an N -dimensional representation

$$\phi_\psi : u \longrightarrow \psi \left(u, \begin{pmatrix} |u|^{\frac{1}{2}} & 0 \\ 0 & |u|^{-\frac{1}{2}} \end{pmatrix} \right), \quad u \in L_F,$$

of L_F to any $\psi \in \Psi(N)$. If $\psi = \mu \otimes \nu$ belongs to the subset $\Psi_{\text{sim}}(N)$, ϕ_ψ decomposes as a direct sum

$$u \longrightarrow \mu(u)|u|^{\frac{n-1}{2}} \oplus \cdots \oplus \mu(u)|u|^{-\frac{(n-1)}{2}},$$

to which we associate the induced representation $\mathcal{I}_P(\sigma_\mu)$ of Theorem 1.3.3. According to the rules of the hypothetical global correspondence from $\Phi(N)$ to isobaric automorphic representations of $GL(N)$, it is the unique irreducible quotient $\pi = \pi_\psi$ of $\mathcal{I}_P(\sigma_\mu)$ that is supposed to correspond to the parameter ϕ_ψ . This representation is unitary, as of course is implicit in Theorem 1.3.3. The mapping $\psi \rightarrow \pi_\psi$ is thus an explicit realization of the

bijjective correspondence from $\Psi_{\text{sim}}(N)$ to $\mathcal{A}_2(N)$ implied by the theorem of Mœglin and Waldspurger (and the existence of the group L_F).

More generally, suppose that ψ belongs to the larger set $\Psi(N)$. Then the isobaric representation π_ψ attached to the parameter $\phi_\psi \in \Phi(N)$ belongs to $\mathcal{A}(N)$. It can be described in the familiar way as the irreducible unitary representation induced from a unitary representation of a Levi subgroup. The mapping $\psi \rightarrow \pi_\psi$ becomes a bijection from $\Psi(N)$ to $\mathcal{A}(N)$. In general, the restriction $\psi \rightarrow \phi_\psi$ of parameters is an *injection* from $\Psi(N)$ into $\Phi(N)$. The role of the set $\Psi(N)$ we have just defined in terms of the supplementary group $SU(2)$ is thus to single out the subset of $\Phi(N)$ that corresponds to the subset $\mathcal{A}(N)$ of “globally tempered” automorphic representations.

We are also free to form larger sets

$$\Psi^+(N) \supset \Psi(N)$$

and

$$\Psi_{\text{sim}}^+(N) \supset \Psi_{\text{sim}}(N)$$

of representations of $L_F \times SU(2)$ by removing the condition that they be unitary. Any element $\psi \in \Psi^+(N)$ is then a direct sum of irreducible representations $\psi_i \in \Psi_{\text{sim}}^+(N_i)$. The components ψ_i should correspond to automorphic representations $\pi_i = \pi_{\psi_i}$ in $\mathcal{A}_2^+(N_i)$. The corresponding induced representation

$$\rho_\psi = \mathcal{I}_P(\pi_1 \otimes \cdots \otimes \pi_r)$$

then belongs to the set of $\mathcal{A}^+(N)$ of (possibly reducible) representations of $GL(N, \mathbb{A})$ introduced above. Notice that the extended mapping $\psi \rightarrow \phi_\psi$ from $\Psi^+(N)$ to $\Phi(N)$ is no longer injective. In particular, ρ_ψ will not in general be equal to the automorphic representation corresponding to ϕ_ψ . However, the mapping $\psi \rightarrow \rho_\psi$ will be a bijection from $\Psi^+(N)$ to $\mathcal{A}^+(N)$. In the interests of symmetry, we can also write

$$\Psi_{\text{cusp}}(N) = \Phi_{\text{sim,bdd}}(N)$$

for the set of representations ψ that are trivial on the factor $SU(2)$. This gives us a chain of sets

$$(1.3.9) \quad \Psi_{\text{cusp}}(N) \subset \Psi_{\text{sim}}(N) \subset \Psi(N) \subset \Psi^+(N)$$

that is parallel to (1.3.4). We will then have a bijective correspondence $\psi \rightarrow \rho_\psi$ that takes each set in (1.3.9) to its counterpart in (1.3.4).

The global parameter sets in the chain (1.3.9) are hypotheticalal, depending as they do on the global Langlands group L_F . However, their local analogues are not. They can be defined for the general linear group $G_v(N) = GL(N)_v$ over any completion F_v of F . Replacing L_F by L_{F_v} in the definitions above, we obtain local parameter sets

$$(1.3.10) \quad \Psi_{\text{cusp},v}(N) \subset \Psi_{\text{sim},v}(N) \subset \Psi_v(N) \subset \Psi_v^+(N).$$

We also obtain a restriction mapping $\psi \rightarrow \psi_v$ from the hypothetical global set $\Psi^+(N)$ to the local set $\Psi_v^+(N)$. The generalized Ramanujan conjecture

for $GL(N)$ asserts that this mapping takes $\Psi(N)$ to the subset $\Psi_v(N)$ of $\Psi_v^+(N)$. However, the conjecture is not known. For this reason, we will be forced to work with the larger local sets $\Psi_v^+(N)$ in our study of global spectra.

Our purpose in introducing the hypothetical families of parameters (1.3.9) has been to persuade ourselves that they correspond to well defined families of automorphic representations (1.3.4). This will inform the discussion of the next section. There we shall revisit the constructions of the last section for orthogonal and symplectic groups, but with the objects (1.3.4) in place of the parameter sets (1.3.9). Notice that the three left hand sets in (1.3.4) contain only isobaric automorphic representations. They therefore correspond bijectively with the three subsets

$$(1.3.11) \quad \mathcal{C}_{\text{sim}}(N) \subset \mathcal{C}_2(N) \subset \mathcal{C}(N) \subset \mathcal{C}_{\text{aut}}(N)$$

of $\mathcal{C}_{\text{aut}}(N)$ under the mapping of Theorem 1.3.2. We have written

$$\mathcal{C}_{\text{sim}}(N) = \mathcal{C}_{\text{cusp}}(N)$$

for the smallest of these sets, in part because it would be bijective with the hypothetical family

$$\Phi_{\text{sim,bdd}}(N) = \Psi_{\text{cusp}}(N)$$

of irreducible unitary, N -dimensional representations of the group L_F . The largest set $\mathcal{C}_{\text{aut}}(N)$ would of course be bijective with the family $\Phi(N)$ of all N -dimensional representations of L_F .

The sets (1.3.11) can all be expressed in terms of the smallest set $\mathcal{C}_{\text{sim}}(N)$ (or rather its analogue $\mathcal{C}_{\text{sim}}(m)$ for $m \leq N$). This is a consequence of Theorem 1.3.3, or if one prefers, its embodiment in the left hand three parameter in (1.3.9). The sets $\mathcal{C}_{\text{sim}}(N)$ thus contain all the global information for general linear groups. The notation we have chosen is meant to reflect the role of their elements as the *simple* building blocks. Our ultimate goal is to show that this fundamental role extends to the representations of orthogonal and symplectic groups.

1.4. A substitute for global parameters

We can now resume the discussion from §1.2. We shall use the cuspidal automorphic representations of $GL(N)$ as a substitute for the irreducible N -dimensional representations of the hypothetical global Langlands group L_F . This will allow us to construct objects that ultimately parametrize families of automorphic representations of classical groups.

Assume that F is global. In the last section, we wrote $\Psi_{\text{sim}}(N) = \Psi_2(N)$ temporarily for the set of equivalence classes of irreducible unitary representations of the group $L_F \times SU(2)$. From now on, we let

$$\Psi_{\text{sim}}(N) = \Psi_{\text{sim}}(G(N)) = \Psi_{\text{sim}}(GL(N))$$

stand for the set of formal tensor products

$$\psi = \mu \boxtimes \nu, \quad \mu \in \mathcal{A}_{\text{cusp}}(m),$$