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Higher Regulators, Algebraic *K*-Theory, and Zeta Functions of Elliptic Curves

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LECTURE 0

Introduction

0.1. Let K be a number field with ring of integers $\mathcal{O}_K$. Let

$$\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2},$$

so r_1 and r_2 are the numbers of real and complex places of K, and define

$$\ell = (\ell_1, \dots, \ell_{r_1+r_2}) \colon \mathcal{O}_K^* \to \mathbb{R}^{r_1+r_2} \tag{0.1.1}$$

to be the composite of the inclusion $\mathcal{O}_K^* \hookrightarrow (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R})^*$ with the map $(\mathcal{O}_K \otimes \mathbb{R})^* \to \mathbb{R}^{r_1+r_2}$ given by $\log|\ |$ (resp. $\log|\ |^2$) on the real (resp. complex) factors. The image $\ell(\mathcal{O}_K^*) = L$ is known to be a lattice of rank $r_1 + r_2 - 1$, and the *regulator* R_K is defined by

$$R_K = \frac{1}{\sqrt{r_1 + r_2}} \operatorname{Volume}(L). \tag{0.1.2}$$

If $\epsilon_1, \dots, \epsilon_{r_1+r_2-1}$ is a basis of $\mathcal{O}_K^*/\text{torsion}$, it is easy to show R_K is the absolute value of the determinant of any $(r_1 + r_2 - 1) \times (r_1 + r_2 - 1)$ minor of the matrix $\big(\ell_i(\epsilon_j)\big)_{1 \le i,j \le r_1+r_2}$. The *class number formula*

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1+r_2}\pi^{r_2} R_K h}{\sqrt{|D|}w} \tag{0.1.3}$$

relates the residue at $s = 1$ of the zeta function $\zeta_K(s)$ to the regulator R_K, the class number h, the number w of roots of 1, and the discriminant D of K. If one takes into account the functional equation satisfied by $\zeta_K(s)$, the class number formula can be rewritten ([**Lic73**])

$$\lim_{s \to 0} \zeta_K(s) s^{-(r_1+r_2-1)} = \frac{-hR_K}{w}. \tag{0.1.4}$$

Various conjectural generalizations of the class number formula have been proposed. One such, due to Lichtenbaum, would have

$$\lim_{s \to -m} \zeta_K(s)(s+m)^{-d_m} = \pm \frac{\# K_{2m}(\mathcal{O}_K)}{\# K_{2m+1}(\mathcal{O}_K)_{\text{tor}}} R_m(K) \tag{0.1.5}$$

for any non-negative integer m, where $K_*(\mathcal{O}_K)$ denotes the K-theory of Bass, Milnor, and Quillen, among others, $R_m(K)$ is a suitable regulator, and

$$d_m = \begin{cases} r_2 & m = 2n+1 > 0 \\ r_1 + r_2 & m = 2n > 0 \\ r_1 + r_2 - 1 & m = 0. \end{cases} \tag{0.1.6}$$

0.2. Unfortunately, Lichtenbaum's conjecture does not give the right value, even for $m = 1$ and $K = \mathbb{Q}$. However, various results suggest some formula of this sort should hold. In particular Borel has computed the rank of the $K_*(\mathcal{O}_K)$, proving

$$\operatorname{rk} K_n(\mathcal{O}_K) = \begin{cases} 0 & n = 2m \\ d_m & n = 2m+1. \end{cases} \tag{0.2.1}$$

So $\operatorname{rk} K_{2m+1}(\mathcal{O}_K) =$ order of zero of $\zeta_K(s)$ at $s = -m$. The indecomposable space I^{2m+1} of the continuous cohomology

$$H^{2m+1}_{\text{cont}}(\mathrm{SL}(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}), \mathbb{R})$$

has dimension d_m, and the Hurewitz map

$$K_*(\mathcal{O}_K) \to H_*\big(\mathrm{GL}(\mathcal{O}_K)\big)$$

gives a homomorphism

$$r\colon K_*(\mathcal{O}_K) \to \operatorname{Hom}(I^{2m+1}, \mathbb{R}). \tag{0.2.2}$$

Borel shows further that r embeds $K_*(\mathcal{O}_K)/\text{torsion}$ as a lattice of maximal rank, and that for a suitable choice of basis (obtained by relating $H^*_{\text{cont}}(\mathrm{SL}(\mathcal{O}_K \otimes \mathbb{R}), \mathbb{R})$ to the cohomology of the compact dual symmetric space of $\mathrm{SL}(\mathcal{O}_K \otimes \mathbb{R})$), $r\big(K_{2m+1}(\mathcal{O}_K)\big)$ has volume some rational multiple of

$$\pi^{-d_m} \lim_{s \to -m} \zeta_K(s)(s+m)^{-d_m} = \pi^{-d(m+1)} |D|^{1/2} \zeta_K(m+1). \tag{0.2.3}$$

The first four lectures are devoted to Borel's work, although we completely neglect the more difficult aspect, the computation of $\operatorname{rk} K_*(\mathcal{O}_K)$, in order to focus on the adelic Tamagawa-Weil techniques for determining the volume of $r\big(K_*(\mathcal{O}_K)\big)$. Lectures 1 and 2 recall necessary ideas from Weil's Princeton notes [**Wei61**], while Lectures 3 and 4 focus on Borel's preprint [**Bor77**].

0.3. The remaining lectures are more tentative, and the results are much less sweeping. They represent the author's attempt to extend Borel type regulator results from number fields to elliptic curves over number fields. After all, Quillen has defined higher K-groups $K_*(X)$

for any scheme X. One might hope for a computation of the rank d of $K_*(X)$ together with a regulator map

$$r\colon K_*(X) \to \mathbb{R}^d$$

such that $r\bigl(K_*(X)\bigr)$ was a lattice of maximal rank whose volume was related to the value of the Hasse-Weil zeta function of X. Unfortunately, such results do not come by magic, and it should be pointed out that appropriate generalizations of the two basic tools in Borel's arguments (viz., symmetric spaces and idèles) are not at this time available.

A conjectural generalization of the class number formula, due to Birch and Swinnerton-Dyer has impressive numerical evidence behind it. The idea is to describe the behavior of the Hasse-Weil zeta function of an elliptic curve E over a number field K near the point $s = 1$. The role of the regulator is played by the matrix associated to the height pairing on points of the curve defined over the given number field. In particular, one expects that the order of the zero of the Hasse-Weil zeta function at $s = 1$ should equal the rank of $E(K)$. Recently Coates-Wiles proved that when E has complex multiplication by the ring of integers in an imaginary quadratic number field k of class number 1, and when $K = \mathbb{Q}$ or k then the existence of points of infinite order in $E(K)$ implies the vanishing of the Hasse-Weil zeta function at $s = 1$. No one has been able to directly relate the height pairing matrix with the zeta function.

Oddly enough, if one looks at $s = 2$ rather than $s = 1$, things get easier. For one thing, instead of $K_0(E)$ or $E(K)$, the regulator will involve $K_2(E)$. It turns out to be possible to:

(a) write down explicitly a regulator map $K_2(E_{\mathbb{C}}) \to \mathbb{C}$ (not $\mathbb{R}$!);
(b) write down interesting elements in $K_2(E_K)$ analogous to cyclotomic units and related to points of finite order on E;
(c) evaluate the regulator map on the interesting elements and relate the resulting mess to $L(E, s)$, the Hasse-Weil zeta function, assuming E has complex multiplication.

The simplest case is when E is defined over $\mathbb{Q}$ and has complex multiplication by the ring of integers in an imaginary quadratic field (necessarily of class number 1 since $j(E) \in \mathbb{Q}$ generates the Hilbert class field). In the last lecture we write down an element $U \in K_2(E) \otimes_{\mathbb{Z}} \mathbb{Q}$, and show

$$\kappa R_q(U) = L(E, 2) \tag{0.3.1}$$

where R_q is the regulator (which depends "modularly" on the choice of $q = e^{2\pi i\tau}$) and κ is a certain constant involving a Gauss sum, the imaginary part of τ, and the conductor of E.

Some comments on (a). The construction is purely transcendental and we may think of E as defined over $\mathbb{C}$. We define a bilinear map

$$\begin{aligned} \mathbb{C}(E)^* \otimes_{\mathbb{Z}} \mathbb{C}(E)^* &\longrightarrow \textstyle\coprod_{p\in E} \mathbb{Z} \\ f \otimes g &\longmapsto (f)^- * (g) \end{aligned} \tag{0.3.2}$$

where (f) denotes the divisor of f, $\bigl(\sum n_i(a_i)\bigr)^- = \sum n_i(-a_i)$, and

$$\sum n_i(a_i) * \sum m_j(b_j) = \sum n_i m_j(a_i + b_j).$$

Thus any set-theoretic function $F\colon E_{\mathbb{C}} \to A$, A an Abelian group, induces a map $F^*\colon \mathbb{C}(E)^* \otimes \mathbb{C}(E)^* \to A$. It is natural to look for F such that F^* is a Steinberg symbol, i.e. $F^*\bigl(f \otimes (1-f)\bigr) = 0$ for any f. Such an F^* factors through $K_2\bigl(\mathbb{C}(E)\bigr)$ and we obtain by composition

$$K_2(E) \to K_2\bigl(\mathbb{C}(E)\bigr) \xrightarrow{F^*} A. \tag{0.3.3}$$

Actually, it is necessary to back off a step and first consider the number field case. This means replacing $K_2(E_{\mathbb{C}})$ by $K_3(\mathbb{C})$ and writing down an explicit recipe for the Borel regulator, which is done in Lectures 5–7. We view $K_3(\mathbb{C})$ as a direct summand of the relative K-group $K_2(\mathbb{P}^1_{\mathbb{C}}, \{0,\infty\})$ which can be thought of either as K_2 of a degenerate elliptic curve or as K_2 with compact supports of $\mathbb{G}_m$. The analogue of $\mathbb{C}(E)$ becomes the semi-local ring R of functions on $\mathbb{P}^1_{\mathbb{C}}$ regular at 0 and ∞, and $K_2\bigl(\mathbb{C}(E)\bigr)$ is replaced by $K_2(R,I)$, $I =$ ideal of $\{0,\infty\}$. The group structure on $\mathbb{P}^1_{\mathbb{C}} \setminus \{0,\infty\}$ enables us to define

$$(f)^- * (g) \in \coprod_{p\in\mathbb{P}^1_{\mathbb{C}}\setminus\{0,\infty\}} \mathbb{Z}$$

for $f \in 1 + I$, $g \in \mathbb{C}(\mathbb{P}^1)^*$.

The function $D\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{R}$,

$$D(x) = \arg(1-x)\log|x| - \operatorname{Im}\int_0^x \log(1-t)\frac{dt}{t} \tag{0.3.4}$$

was first written down by D. Wigner, who showed it was continuous and single-valued, vanished at 0, 1, ∞, and represented an $\mathbb{R}$-valued continuous 3-cohomology class for the group $\mathrm{SL}_2(\mathbb{C})$. We show

$$D\bigl((f)^- * (1-f)\bigr) = 0 \tag{0.3.5}$$

for $f \in 1 + I$. Results of Keune on generators and relations for relative K_2 imply that D induces a map $K_2(R,I) \to \mathbb{R}$, and we may consider the composition (also denoted D)

$$K_3(\mathbb{C}) \to K_2(R,I) \to \mathbb{R}.$$

In Lecture 7 we do some number theoretic computations in the spirit of Kummer for cyclotomic fields, using D.

Returning to the elliptic case, we write $E = E_{\mathbb{C}} = \mathbb{C}^*/q^{\mathbb{Z}}$, $q = e^{2\pi i\tau}$, $\operatorname{Im}\tau > 0$. Using the relation $D(x^{-1}) = -D(x)$, we show that the infinite sum

$$\sum_{n\in\mathbb{Z}} D(xq^n) \overset{\text{dfn}}{=} D_q(x) \tag{0.3.6}$$

induces a continuous function $D_q\colon E \to \mathbb{R}$. In Lecture 9, we show

$$D_q\big((f)^- * (1-f)\big) = 0$$

for any elliptic function f, so D_q induces a map

$$\begin{tikzcd} K_2(E) \arrow[r] \arrow[rr, bend right, "D_q"'] & K_2\big(\mathbb{C}(E)\big) \arrow[r] & \mathbb{R}. \end{tikzcd}$$

We may also consider the function $J\colon \mathbb{C} \to \mathbb{R}$,

$$J(x) = \log|x| \cdot \log|1-x|.$$

One checks $J\big((f)^- * (1-f)\big) = 0$ for $f \in 1 + I$, but the resulting map $J\colon K_2(R,I) \to \mathbb{R}$ appears less interesting than D because $K_3(\mathbb{C})$ maps to $\operatorname{Ker} J \subset K_2(R,I)$. Also the function

$$J_q = \sum_{n=0}^{\infty} J(xq^n) - \sum_{n=1}^{\infty} J(x^{-1}q^n) \tag{0.3.7}$$

is well-defined and continuous on $\mathbb{C}^*$, but is not invariant under $x \mapsto xq$. Given elliptic functions f, g, however, we may choose liftings $(\tilde{f})$, $(\tilde{g})$ to divisors on $\mathbb{C}^*$. It turns out that

$$J_q(f \otimes g) \overset{\text{dfn}}{=} J_q\big((\tilde{f})^- * (\tilde{g})\big)$$

is independent of liftings and $J_q\big(f \otimes (1-f)\big) = 0$. The regulator map is

$$R_q \overset{\text{dfn}}{=} J_q + \sqrt{-1}D_q. \tag{0.3.8}$$

Comments on (b); *interesting elements in* $K_2(E)$. When E is defined over K, there is an exact localization sequence

$$K_2(E) \to K_2\big(K(E)\big) \xrightarrow[\text{tame symbol}]{} \coprod_{x\in E} K(x)^*.$$

Suppose the points of order N, E_N, are defined over K, and let f and g be functions on E defined over K whose divisors are supported on E_N. Then one shows there is an expression

$$\{f,g\}^N \cdot \prod \{f_i, c_i\}, \quad f_i \in K(E)^*, c_i \in K^*$$

lying in the kernel of the tame symbol. The regulator map R_q is trivial on symbols with one entry constant, so it is possible to calculate its value on such an expression without specifying the f_i or the c_i. Let f be the function with poles of order 1 at each non-zero point of E_N and a zero of order $N^2 - 1$ at 0. Take $g = g_a$ to have a zero of order N at $a \in E_N$ and a pole of order N at 0. The resulting element in Image $\big(K_2(E) \to K_2\big(K(E)\big)\big)$ is denoted S_a. When K is a number field, the kernel of $K_2(E) \to K_2\big(K(E)\big)$ can be shown to be torsion, so $S_a \in K_2(E)/\text{torsion}$.

Comments on (c): *relation with* $L(E, 2)$. The idea is to imitate the following classical computation of Kummer. Let

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

be a Dirichlet L-series with conductor ℓ. Write

$$\widehat{\chi}(k) = \frac{1}{\ell} \sum_{r=0}^{\ell-1} \chi(r) e^{2\pi i r k/\ell}.$$

It is easy to see

$$L(s, \chi) = \sum_{k=0}^{\ell-1} \widehat{\chi}(k) \sum_{n=1}^{\infty} \frac{e^{-2\pi i n k/\ell}}{n^s}$$

so if χ is non-trivial we can write

$$L(1, \chi) = -\sum_{k=0}^{\ell-1} \widehat{\chi}(k) \log \left| \frac{1 - e^{-2\pi i k/\ell}}{1 - e^{2\pi i/\ell}} \right|$$

The point is that $(1 - e^{-2\pi i k/\ell})/(1 - e^{-2\pi i/\ell})$ is a unit in $\mathcal{O}_{\mathbb{Q}(e^{2\pi i/\ell})}$, and such units generate a subgroup of finite index in the full group of units. Since $\zeta_{\mathbb{Q}(e^{2\pi i/\ell})}(s)$ is a product of the $L(s, \chi)$, the above formula expresses the relation between the regulator and the residue of the zeta function at $s = 1$. One does not see the class number; it appears as the index of the above cyclotomic units in the full group of units.

The picture becomes even simpler if we consider a real quadratic extension K of $\mathbb{Q}$, and let χ be its character. Then $\lim_{s \to 1} \zeta_K(s) = L(1, \chi)$. The character χ is even and

$$\widehat{\chi}(k) = \chi(k)^{-1} \widehat{\chi}(1)$$

so we find

$$\lim_{s \to 1} \zeta_K(s)(s-1) = -\widehat{\chi}(1) \log |U| \tag{0.3.9}$$

where

$$U = \prod_{k \in (\mathbb{Z}/\ell\mathbb{Z})^*} \left(\frac{1 - e^{-2\pi i k/\ell}}{1 - e^{-2\pi i/\ell}} \right)^{\chi(k)^{-1}} \in \mathcal{O}_K^*$$

is the norm of a unit

$$\frac{1 - e^{-2\pi i k/\ell}}{1 - e^{-2\pi i/\ell}} \in \mathcal{O}^*_{\mathbb{Q}(e^{2\pi i/\ell})}.$$

This formula should be compared with the formula (0.3.1) above.