Preface

In this book we introduce a class of marked Riemann surfaces $(X; A_1, B_1, \dots)$ of infinite genus that are constructed by pasting plane domains and handles together. Here, $A_1, B_1, \dots$ is a canonical homology basis on X. The asymptotic holomorphic structure is specified by a set of geometric/analytic hypotheses. The analytic hypotheses primarily restrict the distribution and size of the cycles $A_1, A_2, \dots$. The entire classical theory of compact Riemann surfaces up to and including the Torelli Theorem extends to this class. In our generalization, compact surfaces correspond to the special case in which all but finitely many of the A_j, $j \geq 1$, have length zero.

The choice of geometric/analytic hypotheses was guided by two requirements. First, the classical theory of compact Riemann surfaces could be developed in this new context. Secondly, a number of interesting examples satisfy the hypotheses, in particular, the heat curve $\mathcal{H}(q)$ associated to $q \in L^2(\mathbb{R}^2/\Gamma)$. Here, Γ is the lattice

$$\Gamma = (0, 2\pi)\mathbb{Z} \oplus (\omega_1, \omega_2)\mathbb{Z}$$

where $\omega_1 > 0$, $\omega_2 \in \mathbb{R}$ and $\mathcal{H}(q)$ is the set of all points $(\xi_1, \xi_2) \in \mathbb{C}^* \times \mathbb{C}^*$ for which there is a nontrivial distributional solution $\psi(x_1, x_2)$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ of the "heat equation"

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_2^2}\right)\psi + q(x_1, x_2)\psi = 0$$

satisfying

$$\psi(x_1 + \omega_1, x_2 + \omega_2) = \xi_1 \psi(x_1, x_2), \quad \psi(x_1, x_2 + 2\pi) = \xi_2 \psi(x_1, x_2).$$

The general theory is used to express the solution of the Kadomcev–Petviashvilli equation with real analytic, periodic initial data q explicitly in terms of the theta function on the infinite-dimensional Jacobian variety corresponding to $\mathcal{H}(q)$. It is evident that the solution is almost periodic in time. In [**Me1, Me2**], this result is improved to finitely differentiable initial data.

This book is divided into four parts. We begin with a discussion, in a very general setting, of L^2-cohomology, exhaustions with finite charge and theta series. In the second part, the geometrical/analytical hypotheses are introduced. Then, an analogue of the classical theory is developed, starting with the construction of a normalized basis of square integrable holomorphic one forms and concluding with the proof of a Torelli theorem. The third part is devoted to a number of examples. Finally, the Kadomcev–Petviashvilli equation is treated in the fourth part.

We speculate that our theory can be extended to surfaces with double points, that a corresponding "Teichmueller theory" can be developed and that there is an infinite-dimensional "Teichmueller space" in which "finite genus" curves are dense.