

PART I. ALGEBRAS AND GROUPS

This part contains the following five papers:

- [1] *On the structure of Hopf algebras* (with J. C. Moore), preprint (1959)
- [2] *On the structure of Hopf algebras* (with J. C. Moore) (1965)
- [3] *Remarks on infinite-dimensional Lie groups* (1986)
- [4] *The representation rings of some classical groups* (1963)
- [5] *Growth of finitely generated solvable groups* (1968)

These papers are quite diverse in content and focus, so they will be discussed separately.

- [1] *On the structure of Hopf algebras* (with J. C. Moore), preprint (1959)
- [2] *On the structure of Hopf algebras* (with J. C. Moore) (1965)

These two papers on Hopf algebras share their titles, authorship, and main theme, but they are significantly different in substance, style, and of course their dates of public release. The Annals publication [2] is the finished product of a decade of work, while the preprint [1] is the first widely accessible version of the work in progress, considered by some to be more user-friendly. In the words of Jim Stasheff [Topological Methods in Modern Mathematics, Publish or Perish, 1993], “Each has its virtues, and I hope both will be preserved for posterity.”

The main results describe the structure of graded connected primitively generated Hopf algebras A over a field K of characteristic zero. The primitive elements $P(A)$ form a graded Lie algebra, and for any graded Lie algebra L , the universal enveloping algebra $U(L)$ has the structure of a graded connected primitively generated Hopf algebra. A fundamental structure theorem asserts that PU and UP are isomorphic to the respective identity functors. In particular, A is canonically isomorphic, as Hopf algebra, to $U(P(A))$. In the course of proving this, Milnor and Moore establish a sharpened version of the Poincaré–Birkhoff–Witt Theorem: The associated graded algebra of $U(L)$ is Hopf algebra isomorphic to the symmetric algebra of the underlying graded vector space of L .

When A has commutative multiplication (and K is only required to be perfect), Milnor and Moore retrieve and strengthen classical theorems of Borel, Leray, and Samelson. For example, when A is further of finite type as a graded vector space, then A is a tensor product of single generator Hopf algebras.

The following topological application furnished much of the motivation of this work: If G is a pathwise connected homotopy associative H -space with unit, then the Hurewicz morphism $\lambda : \pi_*(G, K) \rightarrow H_*(G, K)$ of graded Lie algebras induces an isomorphism $U(\pi_*(G, K)) \rightarrow H_*(G, K)$ of Hopf algebras.

- [3] *Remarks on infinite-dimensional Lie groups* (1986)

This paper reports on a course of lectures at the conference on “Relativity, Groups, and Topology”, held at Les Houches in 1983. Its publication in the volume II of the voluminous proceedings perhaps deprived it of the wider exposure that it deserves. Lie theory is inherently a blend of analysis, geometry, and algebra. This

paper, notwithstanding its publication here, is mostly analysis, and some differential topology, rather than algebra.

A Lie group G is a group that is also a smooth (C^∞) manifold such that the product and inverse are smooth maps. Milnor explains at the outset that, “to make sense of this in the infinite-dimensional case we need to have a theory of infinite-dimensional smooth manifolds.” Such a theory is one of the main things developed in some detail in these notes. But prior to formal definitions and proofs the exposition is sprinkled with specific examples, providing not only motivation, but also cautions regarding the many added complexities and subtleties that appear in infinite dimensions. These show in particular that manifolds modeled on Banach spaces do not accommodate all of the important examples; instead, locally convex topological spaces seem appropriate. The basic examples include: the unitary group $U(\mathcal{H})$ of linear isometries of a Hilbert space \mathcal{H} , supporting the notion of unitary representations; the group $C^\infty(M, G)$ of all smooth maps from a smooth compact manifold M (possibly with boundary, for example the unit interval) to a Lie group G ; more generally, associated to a principal G -bundle $\pi : P \rightarrow M$, one has an associated bundle $\alpha : A \rightarrow M$ of groups $\alpha^{-1}(x) \cong G$, and the group $C^\infty(\alpha) = \text{Aut}(P)$ of smooth sections of α is the corresponding *gauge transformation group*; and finally the group $\text{Diff}(M)$ of all smooth diffeomorphisms of a smooth compact manifold M . For each of these groups H , there is a corresponding Lie algebra $L(H)$, and an exponential map $\text{EXP} : L(H) \rightarrow H$. For example $\text{Diff}(M)$ has the Lie algebra $\text{vect}(M)$ of smooth vector fields on M . All but the last example, $H = \text{Diff}(M)$, are *analytic* in the sense that the image of EXP is a neighborhood of 1 in H , with which one can introduce analytic coordinates for the product and inverse, for example using the Campbell–Baker–Hausdorff formula.

Subject to a mild regularity condition, believed to be always satisfied, it is shown that if G, H are Lie groups with G connected and simply connected, then any continuous Lie algebra homomorphism $L(G) \rightarrow L(H)$ arises from a unique Lie group homomorphism $G \rightarrow H$. Consequently, G is determined by $L(G)$. On the other hand, it is known that Ado’s Theorem from finite dimension does not generalize: There are Lie algebras with no corresponding group.

The paper concludes with remarks on the homotopy groups of Lie groups. For example, for $n \geq 5$, $\pi_0(\text{Diff}^+(S^n))$ is the set of oriented-diffeomorphism classes of smooth $(n + 1)$ -manifolds homeomorphic to S^{n+1} .

[4] *The representation rings of some classical groups* (1963)

These lecture notes from a 1963 course at Princeton provide a concise and self-contained calculation of the representation ring RG of some classical compact groups G . RG is a free \mathbb{Z} -module with basis the classes of irreducible complex G -modules, and with ring structure given by direct sum and tensor product. The general pattern goes as follows: Let T be a maximal torus of G . Then $RT \cong \mathbb{Z}[\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}]$ ($n = \dim(T)$), the restriction homomorphism identifies RG with a subring of RT , and it is just the ring of invariants of the action of the Weyl group on RT . This is first carried out in detail for the unitary groups $U(n)$ and special orthogonal groups $SO(n)$. For the case of the spin groups $\text{Spin}(n)$, the notes first define and develop the structure of the corresponding Clifford algebras.

[5] *Growth of finitely generated solvable groups* (1968)

This brief note, which we will put in context, is an addendum to the paper:

[JW] J. A. Wolf, Growth of finitely generated solvable groups, and curvature of Riemannian manifolds, *J. Diff. Geom.* **2** (1968), 421–446.

In turn [JW] was based on:

[JM1] J. Milnor, A note on curvature and the fundamental group, *J. Diff. Geom.* **2** (1968), 1–7; also in John Milnor, *Collected Papers, I, Geometry*, Publish or Perish, (1994), 53–61.¹

Related later papers are:

[JM2] J. Milnor, Advanced Problem 5603, *MAA Monthly* **75** (1968), 685–686.

[HB] H. Bass, On the degree of growth of a finitely generated nilpotent group, *Proc. Lond. Math. Soc.* **25** (1972), 603–614.

[RG] R. I. Grigorchuk, On the Milnor problem of group growth, *Dokl. Akad. Nauk. SSSR* **271** (1983), 30–33.

[MG] M. Gromov, Groups of polynomial growth and expanding maps, *Publ. Math. IHES* **53** (1981), 53–73.

[JT] J. Tits, Free subgroups of linear groups, *J. Algebra* **20** (1972), 250–270.

Milnor [JM1] introduced the growth function $g(r)$ of a group G ; it counts the number of elements of word length $\leq r$ with respect to a finite set S of generators of G . The group G is said to have *polynomial growth of degree $\leq d$* if $g(r) \leq ar^d$ for some $a > 0$, and *exponential growth* if $g(r) \geq b^r$ for some $b > 1$. Such properties depend only on G , not S , and are shared by subgroups of finite index.

Let M be a complete Riemannian n -manifold, with $G = \pi_1(M)$ acting (freely) on the universal cover \widetilde{M} . Choose $x \in \widetilde{M}$, and let $B(r)$ denote the ball of radius r in \widetilde{M} centered at x . Then it is shown in [JM1] that, for the fundamental group G , the function $g(r)$ above has the same asymptotic growth properties as the functions $\#(G \cdot x \cap B(r))$ and $\text{Vol}(B(r))$. The latter is related to curvature properties of M , and thus Milnor proved results relating curvature to the growth of $\pi_1(M)$. Results of this genre were further pursued by Wolf [JW].

Suppose that G has a nilpotent subgroup H of finite index, with descending central series $H = H_1 \geq H_2 \geq H_3 \geq \dots$. Let $r_i = \text{rank}(H_i/H_{i+1})$ and put

$$d(G) = \sum_{i \geq 1} i r_i \leq d'(G) = \sum_{i \geq 1} 2^{i-1} r_i.$$

(These depend only on G , not H .) Wolf in [JW] shows then that G has polynomial growth of degree $\geq d(G)$ and $\leq d'(G)$. Bass later showed, in [HB], that G has polynomial growth of degree exactly $d(G)$. Wolf further shows that if G is virtually polycyclic, then either G is virtually nilpotent, and hence has polynomial growth as above, or else it has exponential growth. Milnor's note [5] above shows in fact that any finitely generated solvable group is either virtually polycyclic or else of exponential growth.

Putting these results together we then have: If G is virtually solvable, then either G is virtually nilpotent, and hence of polynomial growth of degree $d(G)$, or else G is of exponential growth. Two questions naturally arise from this:

¹The ideas of this paper were partly anticipated by A. S. Svarc, A volume invariant of coverings, *Dokl. Akad. Nauk. SSSR* **105** (1955), 32–34.

- I. Must the growth of any finitely generated group be either polynomial or exponential?
- II. Conjecture: Every finitely generated group of polynomial growth is virtually nilpotent.

Question I was actually posed by Milnor as an Advanced Problem in the MAA Monthly [JM2]. It took 15 years before Grigorchuk [RG] found a counterexample, a group of “intermediate growth.”

As for Conjecture II, the results described above show that it holds for virtually solvable groups. It then follows that it is true for linear groups, using the “Tits alternative” [JT]: A finitely generated linear group G is either virtually solvable or else G contains a non-abelian free group (and is hence of exponential growth). Using this, Conjecture II was finally proved in a landmark paper of Gromov [MG] that has since made the ideas of metric geometry a fundamental tool of combinatorial group theory. Gromov’s idea was, roughly speaking, to embed the group G of polynomial growth in a Lie group L , and hence in a linear group. Relative to a finite set of generators of G , word length defines a kind of norm on G , and hence also a metric. Gromov obtains the sought after ambient Lie group L as a “limit” of a suitable sequence of micro-scalings of the metric space G , in the same way that we can picture \mathbb{R}^n as the limit of the sequence $(1/N!) \cdot \mathbb{Z}^n$ ($N \rightarrow \infty$).

PART II: THE CONGRUENCE SUBGROUP PROBLEM

This part contains the following five papers:

- [1] *On unimodular groups over number fields* (with H. Bass) (1965)
- [2] *Solution of the congruence subgroup problem for SL_n ($n \geq 3$) and Sp_{2n} ($n \geq 2$)* (with H. Bass and J-P. Serre) (1967)
- [3] *On a functorial property of the power residue symbol* (errata to the above) (1968)
- [4] *On polylogarithms, Hurwitz zeta functions, and the Kubert identities* (1983)
- [5] *Developments on the congruence subgroup problem after the work of Bass, Milnor, and Serre* by Gopal Prasad and Andrei S. Rapinchuk (2010)

Paper [1] represents a historically early stage of investigation that grew into the much more elaborated work on the congruence subgroup problem presented in [2]. Paper [3] corrects an error in a number theoretic calculation in [2]. One reason for including [1] here is that it marks the invention (by Milnor) of the so-called Mennicke symbol notation, which became a fundamental tool not only in [2], but also for many later calculations of the functor K_1 . The article [5] by Prasad and Rapinchuk provides a comprehensive survey of the state-of-the-art on the congruence subgroup problem for general simple algebraic groups over global fields. This introduction will focus instead on some of the initial impulses behind this work, and, in particular, on the solution of the congruence subgroup problem for SL_n , and its connections with both algebraic number theory and K -theory. (See also [B].)

Early investigations on the functor K_1 of algebraic K -theory brought into the foreground two quite different questions of prior interest that turned out to be somewhat related.

One was the desire to calculate the “Whitehead group” $Wh(\pi)$ of a discrete group π . This group houses an invariant of homotopy equivalences between spaces with fundamental group π ; it detects whether they are simple homotopy equivalences. (See [Mi].) $Wh(\pi)$ is the quotient of the abelianization $GL(\mathbb{Z}\pi)^{ab}$ of the infinite general linear group of the integral group ring of π , modulo the image of diagonal matrices with entries in $\pm\pi$. It was shown by Whitehead that the commutator subgroup of $GL(A)$, for any ring A , is the subgroup $E(A)$ generated by elementary matrices, and in fact $K_1(A)$ is defined to be $GL(A)/E(A)$.

The other question had to do with the congruence structure of arithmetic groups, like $SL_n(\mathbb{Z})$, and whether this accounts for all of the finite index subgroups; this is the *Congruence Subgroup Problem* (CSP). (See [5] for general formulations of the CSP.) We proceed now to explain the connection of the CSP with the functor K_1 . First some notation: for a ring A and ideal J , the kernel $GL_n(A, J)$ of $GL_n(A) \rightarrow GL_n(A/J)$ is a normal subgroup of $GL_n(A)$, called the *principal congruence subgroup of level J* . There is also a “relative elementary subgroup”, $E_n(A, J)$, and

$$K_1(A, J) = GL(A, J)/E(A, J)$$

is defined to be the direct limit of the sequence of maps

$$(*) \quad s_n : \mathrm{GL}_n(A, J)/\mathrm{E}_n(A, J) \rightarrow \mathrm{GL}_{n+1}(A, J)/\mathrm{E}_{n+1}(A, J).$$

Whitehead's arguments generalize to give the commutator formula,

$$\mathrm{E}(A, J) = [\mathrm{GL}(A), \mathrm{GL}(A, J)],$$

and so $K_1(A, J)$ is an abelian group. When A is commutative, the determinant furnishes a decomposition

$$K_1(A, J) = (1 + J)^\times \oplus SK_1(A, J),$$

where $(1 + J)^\times$ denotes the group of invertible elements of A in $1 + J$.

The first cases of the CSP concern $\mathrm{SL}_n(A)$, where A is the ring of integers in a number field (e.g. $A = \mathbb{Z}$) or the affine ring of an algebraic curve over a finite field. In this case A/J is finite for all ideals $J \neq 0$, and so $\mathrm{SL}_n(A, J)$ has finite index in $\mathrm{SL}_n(A)$. The CSP for $\mathrm{SL}_n(A)$ asks whether, conversely, every finite index subgroup of $\mathrm{SL}_n(A)$ is a congruence subgroup, i.e. contains some $\mathrm{SL}_n(A, J)$, $J \neq 0$. It was long known that the answer is decidedly negative for $\mathrm{SL}_2(\mathbb{Z})$; in fact $\mathrm{SL}_2(\mathbb{Z}, 2\mathbb{Z})$, of index 6, is a free group on two generators. But the situation was unclear for $n \geq 3$. The first progress was an affirmative answer for $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, proved independently by Mennicke [Me], and by Bass, Lazard, and Serre [BLS]. But the case of general A as above resisted.

The first stage of attack used stabilization methods from K -theory, to show that the maps $(*)$ are surjective for $n \geq 2$, and (much harder) injective for $n > 2$. Moreover, for $n \geq 3$, and $J \neq 0$,

- $\mathrm{E}_n(A, J) = [\mathrm{SL}_n(A), \mathrm{SL}_n(A, J)]$,
- $\mathrm{E}_n(A, J)$ has finite index in $\mathrm{SL}_n(A)$, and
- Every finite index subgroup of $\mathrm{SL}_n(A)$ contains $\mathrm{E}_n(A, J)$ for some $J \neq 0$.

Thus, the groups $SK_1(A, J)$ exactly measure the failure of the CSP for $\mathrm{SL}_n(A)$ when $n \geq 3$. There remained the problem of calculating these groups $SK_1(A, J)$.

It is here that number theory enters the picture, using the surjectivity of the map

$$s : \mathrm{SL}_2(A, J)/\mathrm{E}_2(A, J) \rightarrow SK_1(A, J).$$

For $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{SL}_2(A, J)$, its image $[\alpha]$ in $SK_1(A, J)$ depends only on the first row (a, b) of α , and so we can denote $[\alpha]$ by the "symbol" (called a "Mennicke symbol")

$$[\alpha] = \begin{bmatrix} b \\ a \end{bmatrix} \in SK_1(A, J).$$

Here (a, b) ranges over $(1 + J) \times J$. The final result gave the following presentation of $SK_1(A, J)$ in terms of these symbols.

$$\begin{aligned} \begin{bmatrix} b \\ a + tb \end{bmatrix} &= \begin{bmatrix} b \\ a \end{bmatrix} && \text{for all } t \in A; \\ \begin{bmatrix} b + ta \\ a \end{bmatrix} &= \begin{bmatrix} b \\ a \end{bmatrix} && \text{for all } t \in J; \\ \begin{bmatrix} b \\ a a' \end{bmatrix} &= \begin{bmatrix} b \\ a \end{bmatrix} \begin{bmatrix} b \\ a' \end{bmatrix} && \text{for all } a, a' \in 1 + J. \end{aligned}$$

The first two relations are easily seen, already in $SL_2(A, J)$, but the third, discovered by Mennicke, requires calculations in $SL_3(A, J)$. This presentation means that a homomorphism from $SK_1(A, J)$ to an abelian group μ is just a function $p(a, b)$ defined on $(1 + J) \times J$ and satisfying the above relations. Examples are furnished by the power residue symbols of class field theory, as follows: Suppose that A contains the group μ_m of m -th roots of unity. Then there is an m -th power residue symbol

$$\left(\frac{a}{b}\right)_m \in \mu_m$$

defined for $a, b \in A$ with a prime to mb . It follows easily from its definition that the symbol is multiplicative in both a and b , and depends on b only modulo a . But to assure that it also depends on a only modulo b , we need to place some restrictions on b . Assuming that the field of fractions F of A has no real places, this can be achieved by making J sufficiently divisible by m , and in fact we then get an isomorphism from $SK_1(A, J)$ to μ_m for a suitable m (depending on J , and increasing as J becomes smaller). If, however, F admits a real place then this construction fails, and in fact it can be shown then that $SK_1(A, J) = 0$ for all J .

This leads to a precise solution to the CSP for $SL_n(A)$ when $n \geq 3$, worked out in complete detail in [2]. The paper [2] contains a great deal of other material, including a similar treatment of $Sp_{2n}(A)$ for $n \geq 2$, and also connections with work of Cal Moore [Mo] on central extensions of these groups over local and global fields. Further, A in [2] is allowed, more generally, to be the ring of S -integers in a global field. Serre [S] later showed that for certain S there is even a positive solution to the CSP for $SL_2(A)$.

Finally, these results on $SK_1(A, J)$ contributed to calculations of $\text{Wh}(\pi)$ for finite groups π , with the help of induction methods from representation theory. The paper [4] presents some elaborations of number theoretic identities used in calculating a presentation of $K_1(\mathbb{Q}\pi)$ (See [B] for more details.)

[B] H. Bass, *John Milnor the algebraist*, in *Topological Methods in Modern Mathematics*, Publish or Perish (1993), 43–84.

[BLS] H. Bass, M. Lazard, and J-P. Serre, *Sous-groupes d'indice fini dans $rmSL(n, Z)$* , Bull. Amer. Math. Soc. **70** (1964), 385–392.

[Me] J. Mennicke, *Finite factor groups of the unimodular group*, Ann. Math. **81** (1965), 31–37.

[Mi] J. W. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1962), 358–426. Included also in *John Milnor Collected Papers, II: The Fundamental Group*, Publish or Perish (1995), 151–219.

[Mo] C. C. Moore, *Extensions and low dimensional cohomology of locally compact groups, I*, Trans. Amer. Math. Soc. **113** (1964), 40–63.

[S] J-P. Serre, *Le problème des groupes de congruence pour SL_2* , Ann. Math. **92** (1970), 489–527.

PART III: ALGEBRAIC K-THEORY AND QUADRATIC FORMS

This part contains the following five papers:

- [1] *On isometries of inner product spaces* (1969)
- [2] *Algebraic K-theory and quadratic forms* (1970)
- [3] *Symmetric inner product spaces over a Dedekind domain*, preprint (1970)
- [4] *Symmetric inner product spaces in characteristic 2* (1971)
- [5] *Developments in algebraic K-theory and quadratic forms after the work of Milnor*, by Alexander Merkurjev (2010)

Just as Milnor’s work presented in Part II helped launch a whole program of research on the congruence subgroup problem, the work presented here, particularly in [2] (see also [Mi2]), set the central agenda of research on algebraic K-theory and quadratic forms for the next three decades, culminating in the work of Suslin and Voevodsky. An excellent account of this is presented here by Alexander Merkurjev [5].

Milnor in [2] constructed the following homomorphisms,

$$H^n(F) \xleftarrow{h_n^F} k_n^M(F) \xrightarrow{s_n^F} GW_n(F).$$

His strong indications that they might be isomorphisms came to be known as the “Milnor Conjectures.” Merkurjev [5] narrates the elaborate and intricate story of their proofs.

Here F is a field, and $k_n^M(F) = K_n^M(F)/2K_n^M(F)$, where $K_n^M(F)$ is the degree n component of Milnor’s graded ring $K_*^M(F)$, generated by the group $F^\times = K_1^M(F)$, and defined by the relations of $K_2(F)$, with the product given by Steinberg symbols.

On the left, $H^n(F)$, when $\text{char}(F) \neq 2$, denotes the Galois cohomology $H^n(G, \mu_2)$ of the Galois group G of the separable closure F_{sep} of F , and μ_m denotes the group of m -th roots of unity in F_{sep} . The homomorphism h_n^F starts with the Kummer theory isomorphism from $k_1^M(F) = F^\times/F^{\times 2}$ to $H^1(F)$. The alternative description of $H^n(F)$ when $\text{char}(F) = 2$ can be found in [5].

On the right, $W(F)$ denotes the Witt ring of classes of non-degenerate symmetric bilinear forms over F (modulo “metabolic forms”), and $GW_*(F)$ denotes the associated graded ring with respect to powers of the fundamental ideal

$$I = \text{Ker} (W(F) \xrightarrow{\dim \bmod 2} \mathbb{Z}/2\mathbb{Z}).$$

The homomorphism s_n^F sends the class of $\alpha \in F^\times \bmod F^{\times 2}$ to the class of $(\langle a \rangle - \langle 1 \rangle) \in I \bmod I^2$, where $\langle a \rangle$ denotes the class of the 1-dimensional bilinear form with matrix (a) .

As Merkurjev explains, a basic difficulty in proving the Milnor Conjectures was that the computational tools available with Quillen’s K -theory $K_*^Q(F)$ were not available in Milnor’s K -theory because $K_*^M(F) \neq K_*^Q(F)$. A major breakthrough was made by Voevodsky, who showed that $K_*^M(F)$ could be identified with an appropriate motivic cohomology. Moreover he defined and made use of certain

motivic cohomology operations, using ideas from Milnor's work on the Steenrod algebra and its dual [Mi1].

In characteristic 2, the theories of symmetric bilinear forms and quadratic forms are not the same. The above discussion deals with the former. For the theory of quadratic forms over a field of characteristic 2, Milnor gives a complete algebraic analysis in [4]. (See also [B].)

[B] H. Bass, *John Milnor the algebraist*, in *Topological Methods in Modern Mathematics*, pp. 43–84, Publish or Perish, 1993.

[Mi1] J. Milnor, *The Steenrod algebra and its dual*, *Ann. Math.* **67** (1958), 150–171; see also *John Milnor Collected Papers IV, Homotopy, Homology, and Manifolds*, pp. 61–82, Amer. Math. Soc., Providence, R.I., 2009.

[Mi2] J. Milnor, *Introduction to Algebraic K-theory*, *Annals Math. Studies*, vol. 72, Princeton Univ. Press, 1971.