CHAOTIC ELECTIONS!

A Mathematician Looks at Voting

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Preface

It is reasonable to treat this book as a personal aftershock which manifests the collision of two events. The first was a public event; it was the 2000 U.S. Presidential election. Everyone knows who was legally declared the President. But who *really* won? While each of us probably has a strong opinion, let's face it; nobody will ever know.

The second "colliding" event was a personal one; in January 2000 I published some papers ([**56, 57**]) which finally resolved to my satisfaction (well, at least for now) a two-centuries-old mathematical problem concerning the source and explanation of the paradoxes and problems of voting procedures. The resolution of this problem, which was initiated in the French Academy of Sciences by the mathematicians Jean Charles de Borda, Marie-Jean-Antoine-Nicolas de Caritat Condorcet, Pierre-Simon Laplace, and others near the end of the 1700s, attracted brief popular-press attention. As such it was easy for reporters to learn about my research if a contested election would ever arise.

Such an election surely did occur! The effect of the two forces colliding prompted several reporters to question whether mathematics could explain the election debate about Florida. To be honest, I don't think anyone other than those delightful characters in Lewis Carroll's "Alice in Wonderland" could fully explain what happened in Florida. On the other hand, mathematics does provide answers and guidance about what should be done in the future. It is worth enlisting the power of mathematics to shed new light on these issues which are of particularly crucial public interest.

The purpose of the book is to explain what can go wrong in elections and why. The first part of each chapter is directed toward a general reader with patience and a willingness to wade through some minimal mathematical notation. My advice to the reader is that when a chapter seems to getting a little heavy, read a bit more. If it becomes too heavy, as measured by a growing urge to relegate the book to a dusty pile of "future good intentions" or the waste basket, then just jump ahead to the next chapter. The reason for this advice is based on the structure of the book; the last part of each chapter is intended to give a reader who is a bit more mathematically sophisticated some insight into why the results are true.

Let me emphasize that this is *not* a research monograph; while professionals in this area may discover several new results, the book is *not* directed toward them. Instead, my intention is to provide a more readable exposition of my recent research results so that the reader can better appreciate what can happen in elections. I expect you will be surprised. Hopefully, these notions are described in a way so that the reader can develop an understanding of what can happen without being hindered by highly technical details. (Actual proofs are left for the original papers, or, maybe, a future, technical monograph.) Indeed, I hope that some readers become sufficiently disturbed about the dangers of our current voting procedure that they become activists in demanding a change.

Another goal of mine is to attract mathematicians to the growing and fascinating area of the mathematics of the social sciences. As some mathematicians may shy away only because of a lack of training about these topics, it may be instructive to briefly outline how I got interested and started in this area. After all, the mathematics of voting is not a research topic normally studied and represented in a mathematics department. Furthermore, as is true of most mathematicians, my training and earlier research interests are very far removed from this area; they emphasized techniques from analysis and dynamical systems and, in particular, questions about the Newtonian n-body problem. These bodies move, they don't vote.

Part of what attracted me came from being a mathematician. A particular delight and privilege of being a mathematician is that our training permits us to investigate and examine so many diverse aspects of our surrounding existence. Indeed, there are many other topics out there just waiting — almost begging — for a mathematical analysis to remove the mystery and provide guidance. Let me encourage mathematicians to investigate some of them. You have my promise; with patience and imagination, these studies can be fascinating!

As for my story, I can just say, "Beware of hobbies!" There I was, happily examining the evolution of Newton's universe, the effects of colliding particles in gravitational systems, and other dynamical effects when, almost as a hobby, I discovered voting problems.

It is not important to explain what initially attracted me; it happened. Instead, let me warn you; like an addictive drug, once you start fooling around in this area, it becomes difficult to quit. As is probably true of others, I expected to exercise superb discipline. At first, it was only on weekends — and strictly for recreational purposes — that I would experiment with small doses of election paradoxes. I knew I could handle it. I thought I had the willpower to just recreationally experience these issues and then return to the n-body problem the next morning. Very quickly, however, I was caught up with a personal compulsion to consume more and more of the mathematics of elections — the need extended even to evenings during the week. And then, well, I discovered that my dependency for a research fix in this area would start already most mornings, and it would continue through the day. I was hooked.

Without question, this area provides incredible temptations for any mathematician. To partially explain, when I first discovered this area, it was accepted that election procedures could cause paradoxical problems. But, somewhat surprisingly, only a few paradoxical examples were actually known and available. Be honest; if presented with this kind of intriguing but limited information, any mathematician would want to know whether this is the full story. Are these limited number of examples concocted, or are they real? Are there more? How many more? What are they? Can we find all of them? How likely are they? How about in actual elections; do we need to worry about these paradoxes?

The difficulty is that, unlike some politicians, a tempted mathematician almost always inhales the intellectual challenge. Almost by training, a mathematician needs to know, for instance, whether it is possible to characterize everything that can occur. In other words, a first, irresistible challenge for a mathematician is to determine and list *all possible election "paradoxes."*

To appreciate the temptation of this challenge, remember that a "paradox" is an unexpected, unexplained behavior. Thus, the challenge facing a mathematician is to discover and then catalogue *all possible election oddities that we do not expect to occur.* Can you think of a more enticing temptation to dangle before a mathematician? No wonder I partially dropped off the Newtonian n-body wagon.

But how does one discover everything that we don't expect to exist? When faced with a new difficulty, we tend to revert to those tools with which we are most comfortable. For me, coming from the dynamics of mathematical astronomy, these are the tools of "symbolic" or "chaotic dynamics." To see the relevance, remember that a particularly important success of this area of dynamical systems is how it allowed us to discover and even catalogue dynamical behavior that nobody expected to occur. The technical difficulty in applying this material to voting, however, is that tallying ballots does not involve dynamics with horseshoes and homoclinic points.

However, in a manner described at the end of Chapter 3, rather than the actual chaotic dynamics, concepts which are central to this area of "unexpected mathematical behavior" can be modified to identify all possible "election paradoxes" which could occur with any number of candidates, with any number of voters, and with any of the standard, basic, voting procedures. As described in Chapters 2 and 3, the conclusions are distinctly discouraging for any democracy. Although my later research uncovered other mathematical approaches which more efficiently explain these difficulties in a much sharper manner, the notions coming from chaotic dynamics continue to be the most illustrative to explain what happens and why.

Every so often, one finds one of those philosophical articles marveling about the "Unreasonable effectiveness of mathematics." While it is not appropriate for me to suddenly become a philosopher, let me remind the reader that these articles often mention how the abstract power of mathematics allows conclusions to be transferred from one research area to another.

Voting is no exception; the results described in this book can be extended in unexpected ways to other research topics. It turns out, for instance, that the discouraging assertions about voting methods extend to describe unexpected troubling conclusions about most aggregation procedures. After all, voting is just an aggregation method designed to assemble considerable amounts of data — the voters' preferences — into the simpler, digestible form of the election ranking. The same goal characterizes the fields of statistics, probability, game theory, economics, and most of the social sciences. I provide a taste of these connections in Chapter 6.

Another interesting connection between dynamics and voting is that many of the concepts which occur in chaotic dynamics have a parallel property with voting theory. For instance, one of the better known aspects of "chaos" is that familiar phrase "sensitivity with respect to initial conditions." This comment means that even a miniscule change in the starting position for the dynamics can have a huge effect. Eventually, this small starting change can result in a drastically different future for the dynamics. In voting theory, the parallel is where a slight change in how the voters mark their ballots — even if only one voter does so — can cause a surprising change in the outcome. For voting theory, one parallel aspect of this behavior is associated with "strategic" or "manipulative behavior"; this topic is discussed in Chapter 4.

To continue with how "dynamics" has influenced the development of "voting theory," recall how when "chaotic dynamics" became a widely discussed topic, it forced mathematicians to make a choice. They could throw up their collective hands in despair and then quit because the motion seems to be random where nothing is predictable, or they could dismiss this negative attitude by trying to uncover the underlying structure of the dynamics. Probably because of their eternal optimism and pragmatism, mathematicians did the latter.

Similarly, it is counterproductive to stop with dismay once we have identified and classified the many different kinds of allowable election paradoxes. Instead, the next step is to understand why they occur, how often, and whether it is possible to find some relief. As explained in Chapter 5, this search for relief is accomplished by using "symmetry" as a tool to determine what it is that the voters really want.

This is not the end of connections with chaos. In Chapter 6, I describe a mathematical problem that has had interesting ramifications in the United States. It is the source of the first Presidential veto; it is the reason we have 435 seats in the House of Representatives. While several attempts have been made to resolve this difficulty, corrective effects turn out to be much like trying to push an inflated balloon into a small package; push here and a new problem jumps out elsewhere. As indicated, the source of the difficulties is "chaos"; this time, the description is based on the true dynamical effect.

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