



# FIELDS INSTITUTE MONOGRAPHS

THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES

## Lectures on Operator Theory

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## Preface

The articles in this volume are based for the most part on lectures given at The Fields Institute for Research in Mathematical Sciences by participants in the program, “Operator Algebras and Applications”, held during the year 1994–1995 in Waterloo, Ontario. (It is the ninth volume related to the proceedings of this program.)

The scientific organizing committee for this program consisted of Alain Connes, Man-Duen Choi, Kenneth R. Davidson, George A. Elliott (chairman), Peter A. Fillmore, David E. Handelman, Nigel Higson, Vaughan F. R. Jones, Ian F. Putnam, and Dan-Virgil Voiculescu.

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The editors trust that the record of these lectures will be pertinent, at least in some degree, to all future investigations into the subject.

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