

## Preliminaries

Let  $\mathbb{B}_n$  be the unit ball in  $\mathbb{C}^n$ . Let  $dz$  be Lebesgue measure on  $\mathbb{C}^n$  and let  $d\lambda_n(z) = |\mathbb{B}_n|^{-1} (1 - |z|^2)^{-n-1} dz$  be the invariant measure on the ball. For an integer  $m \geq 0$ , and for  $0 \leq \sigma < \infty$ ,  $m + \sigma > n/2$  we define the analytic Besov-Sobolev spaces  $B_2^\sigma(\mathbb{B}_n)$  to consist of those holomorphic functions  $f$  on the ball such that

$$\left\{ \sum_{k=0}^{m-1} |f^{(k)}(0)|^2 + \int_{\mathbb{B}_n} \left| (1 - |z|^2)^{m+\sigma} f^{(m)}(z) \right|^2 d\lambda_n(z) \right\}^{\frac{1}{2}} < \infty. \quad (1.1)$$

Here  $f^{(m)} = \left( \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} f \right)_{|\alpha|=m}$  is the  $m^{\text{th}}$  order complex derivative of  $f$ . The spaces  $B_2^\sigma(\mathbb{B}_n)$  are independent of  $m$  and are Hilbert spaces with the inner product

$$\begin{aligned} & \langle f, g \rangle \\ &= \sum_{|\alpha| < m} \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} f(0) \overline{\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} g(0)} \\ & \quad + \int_{\mathbb{B}_n} (1 - |z|^2)^{2(m+\sigma)} \left\{ \sum_{|\alpha|=m} \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} f(z) \overline{\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} g(z)} \right\} d\lambda_n(z). \end{aligned}$$

To place this scale of spaces in context, we mention without proof that they include the Dirichlet space  $\mathcal{D}(\mathbb{B}_n) = B_2^0(\mathbb{B}_n)$ , weighted Dirichlet-type spaces with  $0 < \sigma < 1/2$ , the Drury-Arveson Hardy space  $H_n^2 = B_2^{1/2}(\mathbb{B}_n)$  which is also known as the symmetric Fock space over  $\mathbb{C}^n$  (see [9] and [16]), the classical Hardy space  $H^2(\mathbb{B}_n) = B_2^{n/2}(\mathbb{B}_n)$  of holomorphic functions with square integrable boundary values, and the weighted Bergman spaces with  $\sigma > n/2$ . Alternatively, these Hilbert spaces can be viewed as part of the Hardy-Sobolev scale of spaces  $J_\gamma^2(\mathbb{B}_n)$ ,  $\gamma \in \mathbb{R}$ , consisting of all holomorphic functions  $f$  in the unit ball whose radial derivative  $R^\gamma f$  of order  $\gamma$  belongs to the Hardy space  $H^2(\mathbb{B}_n)$  ( $R^\gamma f = \sum_{k=0}^{\infty} (k+1)^\gamma f_k$  if  $f = \sum_{k=0}^{\infty} f_k$  is the homogeneous expansion of  $f$ ). The Hardy-Sobolev scale coincides with the Besov-Sobolev scale and we have

$$B_2^\sigma(\mathbb{B}_n) = J_\gamma^2(\mathbb{B}_n), \quad \sigma + \gamma = \frac{n}{2}, \quad 0 \leq \sigma \leq \frac{n}{2}.$$

Thus for  $0 < \sigma < \frac{n}{2}$ , the number  $\sigma$  measures the order of “antiderivative” required to belong to the Dirichlet space  $B_2(\mathbb{B}_n)$ , and  $\gamma = \frac{n}{2} - \sigma$  measures the order of the derivative required to belong to the classical Hardy space  $H^2(\mathbb{B}_n)$ .

In the range  $\frac{1}{2} \leq \sigma \leq \frac{n}{2}$ , the Hilbert spaces  $B_2^\sigma(\mathbb{B}_n) = J_\gamma^2(\mathbb{B}_n)$ ,  $\sigma + \gamma = \frac{n}{2}$ , compete for the honour of generalizing the classical Hardy space  $H^2(\mathbb{D})$  on the disk. At one extreme, the classical generalization  $H^2(\mathbb{B}_n) = B_2^{\frac{n}{2}}(\mathbb{B}_n)$  can be characterized by the square integrability of its boundary values, while at the other extreme, the

Drury-Arveson Hardy space  $H_n^2 = B_2^{\frac{1}{2}}(\mathbb{B}_n)$  can be identified with the symmetric Fock space over  $\mathbb{C}^n$  (see [9] and [16]), and enjoys many universal operator-theoretic properties [9] including the von Neumann inequality for multivariable contractions in a Hilbert space.

An excellent survey of Hilbert space developments in these areas up to now is the recent monograph of K. Seip [43].

### 1.1 The Hardy space

We begin with interpolation sequences and corona problems for the Hardy space  $H^2(\mathbb{D})$  on the unit disk  $\mathbb{D}$ , whose pointwise multiplier algebra is given by  $H^\infty(\mathbb{D})$  (Lemma 2.9 (p. 21)). Note first that the Hardy space  $H^2(\mathbb{D})$  coincides with the Besov-Sobolev space  $B_2^{\frac{1}{2}}(\mathbb{D})$ . Indeed, for  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ,  $z \in \mathbb{D}$ , the orthogonality relations

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\theta} d\theta = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \quad (1.2)$$

yield

$$\begin{aligned} \sum_{n=0}^{\infty} |a_n|^2 &= \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=0}^{\infty} a_n (r e^{i\theta})^n \right|^2 d\theta \\ &= \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(r e^{i\theta})|^2 d\theta \\ &\equiv \|f\|_{H^2(\mathbb{D})}^2, \end{aligned} \quad (1.3)$$

which defines the norm on the Hardy space  $H^2(\mathbb{D})$ , while the calculation

$$\int_0^1 (1-r^2)r^{2(n-1)}r dr = \frac{1}{2n} - \frac{1}{2n+2} = \frac{2}{4n(n+1)}$$

yields

$$\begin{aligned} \|f\|_{B_2^{\frac{1}{2}}(\mathbb{D})}^2 &= \int_{\mathbb{D}} \left| (1-|z|^2)^{1+\frac{1}{2}} f'(z) \right|^2 d\lambda_1(z) + |f(0)|^2 \\ &= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \sum_{n=1}^{\infty} n a_n (r e^{i\theta})^{n-1} \right|^2 (1-r^2)r dr + |a_0|^2 \\ &= 2 \sum_{n=1}^{\infty} |n a_n|^2 \int_0^1 (1-r^2)r^{2(n-1)}r dr + |a_0|^2 \\ &= |a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 \frac{4n^2}{4n(n+1)} \approx \sum_{n=0}^{\infty} |a_n|^2 = \|f\|_{H^2(\mathbb{D})}^2. \end{aligned} \quad (1.4)$$

Note also that (1.2) shows that  $\{f_r\}_{0 < r < 1}$  is Cauchy in  $L^2(\mathbb{T})$  as  $r \rightarrow 1$  where  $\mathbb{T}$  is the boundary  $\partial\mathbb{D}$  of the disk and  $f_r(z) = f(rz)$  for  $r < 1$  and  $z \in \mathbb{D}$ :

$$\|f_r - f_s\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=0}^{\infty} (r^n - s^n) a_n e^{in\theta} \right|^2 d\theta = \sum_{n=0}^{\infty} |r^n - s^n|^2 |a_n|^2 \rightarrow 0$$

as  $r, s \rightarrow 1$  by the dominated convergence theorem since  $\sum_{n=0}^{\infty} |a_n|^2 < \infty$  by (1.3). By the completeness of  $L^2(\mathbb{T})$ , there is  $f^* \in L^2(\mathbb{T})$  such that  $f^* = \lim_{r \rightarrow 1} f_r$

in  $L^2(\mathbb{T})$ . We easily compute by taking limits that the Fourier coefficients of  $f^*$  satisfy

$$\begin{aligned}\widehat{f^*}(n) &= \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta}) \overline{e^{in\theta}} d\theta = \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} f_r(e^{i\theta}) \overline{e^{in\theta}} d\theta \\ &= \begin{cases} a_n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases},\end{aligned}$$

and that the inner product on  $H^2(\mathbb{D})$  satisfies

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta}) \overline{g^*(e^{i\theta})} d\theta = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

where  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  and  $g(z) = \sum_{n=0}^{\infty} b_n z^n$ . We also have by taking limits the Cauchy formula

$$f(z) = \lim_{r \rightarrow 1} f_r(z) = \lim_{r \rightarrow 1} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_r(w)}{w-z} dw = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f^*(w)}{w-z} dw, \quad (1.5)$$

which can also be expressed in terms of the inner product as

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f^*(e^{i\theta})}{e^{i\theta} - z} i e^{i\theta} d\theta = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{f^*(e^{i\theta})}{1 - e^{-i\theta} z} d\theta = \langle f, k_z \rangle, \quad (1.6)$$

for  $z \in \mathbb{D}$  where

$$k_z(w) = \frac{1}{1 - \overline{z}w} = \sum_{n=0}^{\infty} \overline{z}^n w^n \in H^2(\mathbb{D}) \quad (1.7)$$

with  $\|k_z\|_{H^2(\mathbb{D})} = \sqrt{\sum_{n=0}^{\infty} |z|^{2n}} = \frac{1}{\sqrt{1-|z|^2}}$ .

The Hardy space is an example of a Hilbert function space (see Chapter 5). By this we mean that for  $z \in \mathbb{D}$ , the point evaluation functional  $\Lambda_z : H^2(\mathbb{D}) \rightarrow \mathbb{C}$  defined by  $\Lambda_z f = f(z)$  is continuous. Indeed, continuity is an immediate consequence of (1.5) which gives the growth estimate

$$|f(z)| \leq \|f^*\|_{L^2(\mathbb{T})} \frac{1}{\sqrt{1-|z|^2}}, \quad z \in \mathbb{D}.$$

Riesz' Theorem A.18 (p. 152) in Appendix A shows that there is a *unique*  $k_z \in H^2(\mathbb{D})$  such that  $\Lambda_z f = \langle f, k_z \rangle$ , and the function  $k_z(w)$  on  $\mathbb{D} \times \mathbb{D}$  is referred to as the *reproducing kernel* for  $H^2(\mathbb{D})$ . From the uniqueness in Riesz' theorem we see that the reproducing kernel  $k_z(w)$  for the Hardy space  $H^2(\mathbb{D})$  is given by (1.7).

## 1.2 The Dirichlet space

We will also consider interpolation and corona problems for the Dirichlet space  $\mathcal{D}(\mathbb{D}) = B_2^0(\mathbb{D})$  and its multiplier algebra  $M_{\mathcal{D}(\mathbb{D})}$ . The Dirichlet norm squared of  $f$  is defined to be

$$\|f\|_{\mathcal{D}(\mathbb{D})}^2 = |f(0)|^2 + \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 dx dy,$$

where

$$\int_{\mathbb{D}} |f'(z)|^2 dx dy = \int_{\mathbb{D}} \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} dx dy = \int_{\mathbb{D}} J_f dx dy = \int_{f(\mathbb{D})} du dv$$

is the area (counting multiplicities) of the image  $f(\mathbb{D})$  of the disc under  $f$  by the Cauchy-Riemann equations  $u_x = v_y$ ,  $u_y = -v_x$  if  $f = u + iv$ . In particular if  $f$  is a finite Blaschke product in the disk (see Theorem C.5 in Appendix C below),

$$B(z) = z^k \prod_{n=1}^{N-k} \frac{\alpha_n - z}{1 - \overline{\alpha_n}z} \frac{|\alpha_n|}{\alpha_n}, \quad 0 \leq k \leq N,$$

then  $B(e^{i\theta})$  wraps around the circle  $\mathbb{T} = \partial\mathbb{D}$  exactly  $N$  times and so the area (counting multiplicities) of the image  $B(\mathbb{D})$  is  $N\pi$ . A thorny consequence of this is that the Dirichlet space contains no *infinite* Blaschke products (since their images cover the disk infinitely often), and hence the zeroes of a Dirichlet space function cannot be factored out as is the case for a Hardy space function (Theorem C.14 in Appendix C).

Calculations using (1.2) that are similar to those for  $H^2(\mathbb{D})$  above, show that for  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  and  $g(z) = \sum_{n=0}^{\infty} b_n z^n$ , the inner product corresponding to the above norm satisfies

$$\begin{aligned} \langle f, g \rangle_{\mathcal{D}(\mathbb{D})} &= f(0)\overline{g(0)} + \frac{1}{\pi} \int_{\mathbb{D}} f'(z)\overline{g'(z)} dx dy z \\ &= a_0\overline{b_0} + \sum_{n=0}^{\infty} n a_n \overline{b_n}, \quad f, g \in \mathcal{D}(\mathbb{D}), \end{aligned}$$

and that the reproducing kernel  $k_z(w)$  for the Dirichlet space is given by

$$k_z(w) = 1 + \log \frac{1}{1 - \overline{z}w} = 1 + \sum_{n=1}^{\infty} \frac{1}{n} \overline{z}^n w^n,$$

where the branch of log is taken to satisfy  $\log 1 = 0$ . Indeed, with  $g = k_z$  we have  $b_0 = 1$  and  $b_n = \frac{1}{n} \overline{z}^n$  for  $n \geq 1$  and so

$$\langle f, k_z \rangle_{\mathcal{D}(\mathbb{D})} = a_0 + \sum_{n=1}^{\infty} n a_n \frac{1}{n} \overline{z}^n = a_0 + \sum_{n=1}^{\infty} a_n z^n = f(z).$$

This gives the growth estimate

$$|f(z)| = \left| \langle f, k_z \rangle_{\mathcal{D}(\mathbb{D})} \right| \leq \|f\|_{\mathcal{D}(\mathbb{D})} \|k_z\|_{\mathcal{D}(\mathbb{D})} = \|f\|_{\mathcal{D}(\mathbb{D})} \left( 1 + \log \frac{1}{1 - |z|^2} \right),$$

since

$$\|k_z\|_{\mathcal{D}(\mathbb{D})}^2 = \langle k_z, k_z \rangle = k_z(z) = 1 + \log \frac{1}{1 - |z|^2}.$$

Note that this growth estimate for functions in the Dirichlet space  $\mathcal{D}(\mathbb{D})$  is much smaller than that for functions in the Hardy space  $H^2(\mathbb{D})$ . In fact  $\mathcal{D}(\mathbb{D}) \subset H^2(\mathbb{D})$  and we will see in Chapter 4 that the boundary limits  $f^*$  of Dirichlet space functions  $f$  lie in a subspace  $B_2^{\frac{1}{2}}(\mathbb{T})$  of  $BMO(\mathbb{T})$ , the space of functions having bounded mean oscillation in the circle. By the John-Nirenberg inequality such functions are in  $\cap_{p < \infty} L^p(\mathbb{T})$  (Theorem B.10 (p. 180) and Corollary B.11 in Appendix B), thus close to being bounded, while the boundary limits of Hardy space functions are precisely those functions in  $L^2(\mathbb{T})$  with vanishing negative Fourier coefficients. The real parts of these latter functions are thus arbitrary real  $L^2$  functions. See Ross [36] for a nice expository article on the Dirichlet space.

### 1.3 Tree spaces

There is a standard decomposition of the unit disk  $\mathbb{D}$  into “curved tiles”  $\{K_j^n\}_{n,j}$  that is well adapted to the analysis of holomorphic functions on  $\mathbb{D}$ . The precise form of the tiles  $K_j^n$  is not important, only that they have certain geometric properties. Here is a simple description of an adequate decomposition that makes clear the nature of the required geometry.

Let

$$K_1^0 = B\left(0, \frac{1}{2}\right)$$

be the ball of radius  $\frac{1}{2}$  centered at the origin. Divide the annulus

$$A^1 = B\left(0, \frac{3}{4}\right) \setminus B\left(0, \frac{1}{2}\right)$$

into two congruent pieces, the upper half  $K_1^1$  and the lower half  $K_2^1$ :

$$\begin{aligned} K_1^1 &= A^1 \cap \{0 \leq \theta < \pi\}, \\ K_2^1 &= A^1 \cap \{\pi \leq \theta < 2\pi\}. \end{aligned}$$

Then divide the annulus

$$A^2 = B\left(0, \frac{7}{8}\right) \setminus B\left(0, \frac{3}{4}\right)$$

into four congruent pieces:

$$\begin{aligned} K_1^2 &= A^2 \cap \left\{0 \leq \theta < \frac{\pi}{2}\right\}, \\ K_2^2 &= A^2 \cap \left\{\frac{\pi}{2} \leq \theta < \pi\right\} \\ K_3^2 &= A^2 \cap \left\{\pi \leq \theta < \frac{3\pi}{2}\right\} \\ K_4^2 &= A^2 \cap \left\{\frac{3\pi}{2} \leq \theta < 2\pi\right\}. \end{aligned}$$

Now continue in this way to divide the annulus

$$A^n = B\left(0, 1 - \frac{1}{2^{n+1}}\right) \setminus B\left(0, 1 - \frac{1}{2^n}\right) \quad (1.8)$$

into  $2^n$  congruent pieces:

$$K_j^n = A^n \cap \left\{(j-1)\frac{2\pi}{2^n} \leq \theta < j\frac{2\pi}{2^n}\right\}, \quad 1 \leq j \leq 2^n.$$

With  $A^0$  defined by (1.8), we have the pairwise disjoint decomposition

$$\mathbb{D} = B(0, 1) = \bigcup_{n \geq 0} A^n = \bigcup_{n \geq 0} \bigcup_{1 \leq j \leq 2^n} K_j^n.$$

We now define a *tree structure*  $\mathcal{T}$  on this decomposition  $\{K_j^n\}_{n \geq 0, 1 \leq j \leq 2^n}$  of the disk.

Recall that a *tree*  $T$  is a connected loopless rooted graph  $T = \{V, o, E\}$ . A rooted graph  $T$  consists of a set  $V$  of points called “vertices”, a distinguished vertex  $o \in V$  called the root, and a subset  $E$  of unordered pairs in  $V \times V$  called “edges”.

The rooted graph  $T$  is connected if for each vertex  $v \in V$  there is a finite chain of edges joining the root  $o$  to  $v$ :

$$(o, v_1), (v_1, v_2), \dots, (v_m, v) \in E.$$

The graph  $T$  is loopless if there is no nontrivial closed cycle or loop of edges

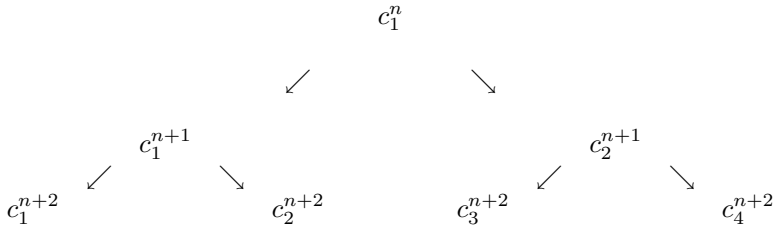
$$(v_1, v_2), (v_2, v_3), \dots, (v_m, v_1) \in E.$$

It is easy to see that for each vertex  $v \neq o$  in a connected loopless rooted graph  $T$  there is a *unique* chain of edges joining the root  $o$  to  $v$ . This defines a *direction* to each edge  $(v, w)$  by declaring  $v < w$  provided the unique chain of edges joining  $o$  to  $w$  ends in the edge  $(v, w)$ . In this case we say that  $v$  is the parent of  $w$  and that  $w$  is a child of  $v$ . Every vertex  $v \neq o$  has a unique parent denoted by  $Pv$ . We then extend the relation of inequality on vertices in the tree by declaring for any pair of vertices  $(v, w)$  that  $v < w$  if the chain of edges joining  $v = v_0$  to  $w = v_m$

$$(v, v_1) = (v_0, v_1), (v_1, v_2), \dots, (v_{m-1}, v_m) = (v_{m-1}, w) \in E,$$

(it is easy to see that such a chain exists and is unique), has  $v_{i-1} < v_i$  for  $1 \leq i \leq m$ . In other words,  $v = PP \cdots Pw$  where  $P$  is repeated  $m$  times.

We take  $\{K_j^n\}_{n \geq 0, 1 \leq j \leq 2^n}$  as the set of vertices in our graph  $\mathcal{T}$  with  $K_1^0$  as root - for psychological purposes it might help to replace the tile  $K_j^n$  by the point  $c_j^n$  at its center. Then define the pair  $(K_j^n, K_i^{n+1})$  to be an edge in  $E$  if the closure of the tiles  $\overline{K_j^n}$  and  $\overline{K_i^{n+1}}$  share an arc of positive length - in other words the tile  $K_i^{n+1}$  lies just outside of and touches the tile  $K_j^n$ . Again, for psychological purposes it might help to think of  $(K_j^n, K_i^{n+1})$  as the line segment joining  $c_j^n$  to  $c_i^{n+1}$ . With the choice of the origin as root, this directs the segment from  $c_j^n$  to  $c_i^{n+1}$ . We say that  $c_i^{n+1}$  is a *child* of  $c_j^n$  and that  $c_j^n$  is a *parent* of  $c_i^{n+1}$  when  $(K_j^n, K_i^{n+1}) \in E$ . A part of the tree  $\mathcal{T}$  looks like



We will often use the variable  $\alpha$  to denote a vertex  $c_j^n$  of  $\mathcal{T}$ . Then  $\alpha < \beta$  if  $\beta$  is a “descendent” of  $\alpha$  in the parent/child relationship, or in terms of tiles,  $\beta = K_i^{n+m}$  lies under the shadow of  $\alpha = K_j^n$  with the sun at the origin. In this connection we embed the tree  $\mathcal{T}$  into the disk by the natural map that sends  $\alpha = K_j^n \in \mathcal{T}$  to the point  $c(\alpha) = c_j^n \in \mathbb{D}$ . We also define the set  $K(\alpha)$  to be the corresponding tile  $K_j^n$ . Note that every vertex has exactly two children in  $\mathcal{T}$ , which is usually expressed by saying that the tree is homogeneous of degree two. The root  $o$  has no parent and all other vertices  $\alpha$  have one parent denoted  $P\alpha$ .

It turns out that the Dirichlet space  $\mathcal{D}(\mathbb{D})$  can be effectively modeled on the tree  $\mathcal{T}$  by the following Hilbert space of complex-valued functions  $f : \mathcal{T} \rightarrow \mathbb{C}$  on  $\mathcal{T}$ :

$$\mathcal{D}(\mathcal{T}) = \left\{ f = (f(\alpha))_{\alpha \in \mathcal{T}} : \sum_{\alpha \in \mathcal{T}} |\Delta f(\alpha)|^2 < \infty \right\},$$

with inner product

$$\langle f, g \rangle = \sum_{\alpha \in \mathcal{T}} \Delta f(\alpha) \overline{\Delta g(\alpha)},$$

and where the backward difference operator  $\Delta$  is defined on functions  $f$  by

$$\Delta f(\alpha) = \begin{cases} f(o) & \text{if } \alpha = o \\ f(\alpha) - f(P\alpha) & \text{if } \alpha \neq o \end{cases}.$$

For example, the restriction map  $\mathcal{R} : \mathcal{D}(\mathbb{D}) \rightarrow \mathcal{D}(\mathcal{T})$  defined by

$$\mathcal{R}f = (f(c(\alpha)))_{\alpha \in \mathcal{T}}$$

for  $f \in \mathcal{D}(\mathbb{D})$  turns out to be *continuous*. To see this let  $\alpha \in \mathcal{T}$ , and denote by  $B_\alpha$  the largest ball contained in  $K(\alpha)$  that is centered at  $c(\alpha)$ . In addition denote by  $H_\alpha$  the convex hull of  $B_\alpha$  and  $B_{P\alpha}$ . Then the mean value property for holomorphic functions and the fundamental theorem of calculus yield

$$\begin{aligned} & |f(\alpha) - f(P\alpha)| \\ &= |f(c(\alpha)) - f(c(P\alpha))| \\ &= \left| \frac{1}{|B_\alpha|} \int_{B_\alpha} f(z) dz - \frac{1}{|B_{P\alpha}|} \int_{B_{P\alpha}} f(\zeta) d\zeta \right| \\ &= \left| \frac{1}{|B_\alpha|} \frac{1}{|B_{P\alpha}|} \int_{B_\alpha} \int_{B_{P\alpha}} [f(z) - f(\zeta)] dz d\zeta \right| \\ &= \left| \frac{1}{|B_\alpha|} \frac{1}{|B_{P\alpha}|} \int_{B_\alpha} \int_{B_{P\alpha}} \int_0^1 (z - \zeta) \cdot \nabla f(tz + (1-t)\zeta) dt dz d\zeta \right| \\ &\leq \text{diam}(H_\alpha) \frac{1}{|B_\alpha|} \frac{1}{|B_{P\alpha}|} \int_{B_\alpha} \int_{B_{P\alpha}} \int_0^1 |f'(tz + (1-t)\zeta)| dt dz d\zeta \\ &\leq C \text{diam}(H_\alpha) \frac{1}{|H_\alpha|} \int_{H_\alpha} |f'(\omega)| d\omega, \end{aligned}$$

after making the change of variable  $\omega = tz + (1-t)\zeta$ . Now we compute that

$$\begin{aligned} \|\mathcal{R}f\|_{\mathcal{D}(\mathcal{T})}^2 &= |f(o)|^2 + \sum_{\alpha \in \mathcal{T}} |f(\alpha) - f(P\alpha)|^2 \\ &\leq |f(o)|^2 + C \sum_{\alpha \in \mathcal{T}} \frac{\text{diam}(H_\alpha)^2}{|H_\alpha|} \int_{H_\alpha} |f'(\omega)|^2 d\omega \\ &\leq |f(o)|^2 + C \int_{\mathbb{D}} |f'(\omega)|^2 d\omega \leq C \|f\|_{\mathcal{D}(\mathbb{D})}^2, \end{aligned}$$

since  $\text{diam}(H_\alpha)^2 \approx |H_\alpha|$  and the sets  $H_\alpha$  have finite overlap at most two in the disk.

A major advantage of the model space  $\mathcal{D}(\mathcal{T})$  is that the so-called Carleson measures for  $\mathcal{D}(\mathcal{T})$  are easily calculated; these are the positive measures  $\mu$  on  $\mathcal{T}$ , which here are the same as the nonnegative functions  $\mu$  on  $\mathcal{T}$ , for which we have an embedding of  $\mathcal{D}(\mathcal{T})$  into  $L^2(\mu)$ , i.e.

$$\|f\|_{L^2(\mu)}^2 \leq C \|f\|_{\mathcal{D}(\mathcal{T})}^2, \quad f \in \mathcal{D}(\mathcal{T}). \quad (1.9)$$

For a simple characterization of (1.9) see Chapter 6.1 below. Using (1.9), the continuity of the restriction operator  $\mathcal{R}$ , and an elementary duality argument gives

a characterization of the Carleson measures for the Dirichlet space  $\mathcal{D}(\mathbb{D})$  on the unit disk, i.e. of those positive measures  $\mu$  on  $\mathbb{D}$  such that  $\mathcal{D}(\mathbb{D})$  embeds into  $L^2(\mu)$ :

$$\|f\|_{L^2(\mu)}^2 \leq C \|f\|_{\mathcal{D}(\mathbb{D})}^2, \quad f \in \mathcal{D}(\mathbb{D}).$$

In turn this can then be used to characterize interpolating sequences for the Dirichlet space  $\mathcal{D}(\mathbb{D})$  ([24]). See Chapter 6 for this and more on Carleson measures and interpolation.

Trees have been used in analysis for some time, but possibly the first instance of their use in the spirit above occurs in the atomic decomposition of spaces of holomorphic functions in Coifman and Rochberg [17]. Rochberg has continued to investigate the connection between trees and functions spaces in a series of papers including those with Arcozzi and the author [4], [5], [6] and [7].

We end this preliminary section with some comments on the use of trees for other function spaces. The above tree model has an equally simple and effective analogue in the case of the spaces  $B_2^\sigma(\mathbb{D})$  when  $0 \leq \sigma < \frac{1}{2}$ . However, the model must be significantly changed in order to be of use for the Hardy space  $B_2^{\frac{1}{2}}(\mathbb{D}) = H^2(\mathbb{D})$ . See [4] for work in this direction. In higher dimensions, one can construct an analogue  $\mathcal{T}_n$  for the ball  $\mathbb{B}_n$  of the tree  $\mathcal{T}$  constructed above for the disk, but the construction is necessarily messy due to the fact that the sphere  $\mathbb{S}^k$  is not neatly tiled when  $k > 1$ . While the corresponding tree space  $\mathcal{D}(\mathcal{T}_n)$  remains effective for calculating the Carleson measures of the Dirichlet space  $B_2^0(\mathbb{B}_n) = \mathcal{D}(\mathbb{B}_n)$  on the ball, it is no longer an adequate model for characterizing interpolation for the Dirichlet space since the corresponding restriction map  $\mathcal{R}$  fails to be continuous from  $\mathcal{D}(\mathbb{B}_n)$  to  $\mathcal{D}(\mathcal{T}_n)$  when  $n > 1$ . This is rectified in [5] by introducing a holomorphic structure on the tree  $\mathcal{T}_n$  (that mirrors the holomorphic geometry of the ball) and redefining the model space  $\mathcal{D}(\mathcal{T}_n)$  to take this structure into account. The result is that the restriction operator is now continuous, and using this with some other special properties of the model, the Carleson measures and interpolating sequences for  $\mathcal{D}(\mathbb{B}_n)$  were characterized in [5]. See Chapter 5.4 for a characterization of interpolating sequences for this and certain other spaces in higher dimensions. Finally, the unstructured model  $\mathcal{D}(\mathcal{T}_n)$  extends to an effective model for calculating Carleson measures for the spaces  $B_2^\sigma(\mathbb{B}_n)$  with  $0 \leq \sigma < \frac{1}{2}$ . But again, this model breaks down at the Drury-Arveson Hardy space  $B_2^{\frac{1}{2}}(\mathbb{B}_n) = H_n^2$ . A different geometric structure is added to the tree  $\mathcal{T}_n$  in [6] to compute the Carleson measures for the Drury-Arveson Hardy space  $H_n^2$ .