Introduction

THIS book is concerned with *simply-connected,* closed oriented 4–manifolds and is focused on *smooth* 4–manifolds. However, for a better understanding of smooth 4–manifolds one needs to have some perspective on their mathematical context.

In their immediate neighborhood, smooth 4–manifolds are included in the much wider class of topological 4–manifolds: we will need to see how the world of topological manifolds is different. At this moment, the topological realm is in fact much better understood than the smooth realm, while the latter has just started to unveil its wildness. Contrasting the two territories is necessary for gaining the proper perspective. A first remark is that, by softening our outlook from differential to topological, we make many smooth manifolds look topologically the same.[5] A second remark is that most topological manifolds do not admit smooth structures whatsoever.

In the opposite direction, not by weakening the structure but by strengthening its rigidity, lies the realm of 4–manifolds that admit complex structures, namely the empire of complex surfaces. These are also better understood than smooth 4–manifolds and are an excellent source of examples. The extra rigidity of the complex realm ensures that many complex surfaces that look the same as smooth 4–manifolds are different as complex manifolds. And, of course, most smooth 4–manifolds do not admit any complex structures whatsoever.

5. You should probably think of this as analogous to considering topological spaces only up to homotopy equivalence: many quite different spaces look homotopically the same.

In order to gain the proper perspective on 4–manifolds, it is also inevitable to peek at what happens in other dimensions. In lower dimensions, manifolds of dimension 1 are a bore, manifolds of dimension 2 have been well-understood for quite a while, while manifolds of dimension 3 (modulo the Poincaré conjecture) are essentially governed by their fundamental groups (but of course are far from being completely deciphered). In any case, the distinction between smooth and topological manifolds (or complex manifolds, for dimension 2) does not exist in lower dimensions, and simply-connected manifolds are uninteresting.

On the other hand, in dimensions 5 and higher, a theory of a different flavor has been developed, taking advantage of the extra room available. For simply-connected high-dimensional manifolds, the main technical tool is the h–cobordism theorem, discovered in the 1960s. Its power in helping clear the waters in high dimensions cannot be understated, and its author, S. Smale, received a Fields Medal for discovering it.

Such a powerful tool available from just one dimension higher than the realm of 4–manifolds can only tempt one to extend it to our land as well. An examination of its high-dimensional proof reveals that it hinges on embedding 2–dimensional disks in the manifold, which is easy in dimension 5 or more, but not in dimension 4. Eventually, M. Freedman was able to prove in 1981 the h–cobordism theorem for dimension 4, but at the price of dropping differentiability and softening to the more flexible domain of topological manifolds. This enabled him to quickly obtain a complete classification of simply-connected topological 4–manifolds, and earned him a Fields Medal.

In contrast, just one year later S.K. Donaldson showed that the realm of smooth 4–manifolds is not yet understood. Making use of differential-geometric methods, he showed that most topological 4–manifolds do not admit any smooth structures. Later, he exhibited smoothly-distinct 4–manifolds that look the same topologically, and even infinite families of such. These results led to a Fields Medal as well.

After about ten more years, in 1994, N. Seiberg and E. Witten came up with a different approach to Donaldson's insights, which was much easier to use and thus proved to be quite more powerful.[6] While Donaldson's methods worked best on complex surfaces, the Seiberg–Witten techniques are more flexible, and led to new striking results. Among them is a method (due to R. Fintushel and R. Stern) for modifying many 4–manifolds in a manner that alters their smooth structure but does not change their topological type.

6. Witten already held a Fields Medal for previous work.

The paradoxical result of all these advances is that they just made more and more obvious the level of our current ignorance, opening windows toward vast fields of unsuspected phenomena for which we presently do not have powerful enough methods of exploration. As a simple example, currently there are no tools for studying smooth manifolds homeomorphic to the 4–dimensional sphere: there might be infinitely many distinct such creatures, or just good old S^4.

It's a wide and wild world out there.

There goes the neighborhood