

---

# PREFACE

---

This book provides a self-contained treatment of classical Fourier analysis at the upper undergraduate or beginning graduate level. I assume that the reader is familiar with the rudiments of Lebesgue measure and integral on the real line. My viewpoint is mostly classical and concrete, preferring explicit calculations to existential arguments. In some cases, several different proofs are offered for a given proposition to compare different methods.

The book contains more than 175 exercises that are an integral part of the text. It can be expected that a careful reader will be able to complete all of these exercises. Starred sections contain material that may be considered supplementary to the main themes of Fourier analysis. In this connection, it is fitting to comment on the role of Fourier analysis, which plays the dual role of queen and servant of mathematics. Fourier-analytic ideas have an inner harmony and beauty quite apart from any applications to number theory, approximation theory, partial differential equations, or probability theory. In writing this book it has been difficult to resist the temptation to develop some of these applications as a testimonial of the power and flexibility of the subject. The following list of “extra topics” are included in the starred sections: Stirling’s formula, Laplace asymptotic method, the isoperimetric inequality, equidistribution modulo one, Jackson/Bernstein theorems, Wiener’s density theorem, one-sided heat equation with Robin boundary condition, the uncertainty principle, Landau’s asymptotic lattice point formula, Gaussian sums and the Schrödinger equation, the central limit theorem, the Berry-Esséen theorem and the law of the iterated logarithm. While none of these topics is “mainstream Fourier analysis,” each of them has a definite relation to some part of the subject.

A word about the organization of the first two chapters, which are essentially independent of one another. Readers with some sophistication but little previous knowledge of Fourier series can begin with Chapter 2 and anticipate a self-contained treatment of the  $n$ -dimensional Fourier transform and many of its applications. By contrast, readers who wish an introductory treatment of Fourier series should begin with Chapter 1, which provides a reasonably complete introduction to Fourier analysis on the circle. In both cases I emphasize the Riesz-Fischer and Plancherel theorems, which demonstrate the

natural harmony of Fourier analysis with the Hilbert spaces  $L^2(\mathbb{T})$  and  $L^2(\mathbb{R}^n)$ . However much of modern harmonic analysis is carried out in the  $L^p$  spaces for  $p \neq 2$ , which is the subject of Chapter 3. Here we find the interpolation theorems of Riesz-Thorin and Marcinkiewicz, which are applied to discuss the boundedness of the Hilbert transform and its application to the  $L^p$  convergence of Fourier series and integrals. In Chapter 4 I merge the subjects of Fourier series and Fourier transforms by means of the Poisson summation formula in one and several dimensions. This also has applications to number theory and multiple Fourier series, as noted above.

Chapter 5 explores the application of Fourier methods to probability theory. Limit theorems for sums of independent random variables are equivalent to the study of iterated convolutions of a probability measure on the line, leading to the central limit theorem for convergence and the Berry-Esséen theorems for error estimates. These are then applied to prove the law of the iterated logarithm.

The final Chapter 6 deals with wavelets, which form a class of orthogonal expansions that can be studied by means of Fourier analysis—specifically the Plancherel theorem from Chapter 2. In contrast to Fourier series and integral expansions, which require one parameter (the frequency), wavelet expansions involve two indices—the scale and the location parameter. This allows additional freedom and leads to improved convergence properties of wavelet expansions in contrast with Fourier expansions. I include a brief application to Brownian motion, where the wavelet approach furnishes an easy access to the precise modulus of continuity of the standard Brownian motion.

Many of the topics in this book have been “class-tested” to a group of graduate students and faculty members at Northwestern University during the academic years 1998–2000. I am grateful to this audience for the opportunity to develop and improve my original efforts.

I owe a debt of gratitude to Paul Sally, Jr., who encouraged this project from the beginning. Gary Ostedt gave me full editorial support at the initial stages followed by Bob Pirtle and his efficient staff. Further thanks are due to Robert Fefferman, whose lectures provided much of the inspiration for the basic parts of the book. Further assistance and feedback was provided by Marshall Ash, William Beckner, Miron Bekker, Leonardo Colzani, Galia Dafni, George Gasper, Umberto Neri, Cora Sadosky, Aurel Stan, and Michael Taylor. Needless to say, the writing of Chapter 1 was strongly influenced by the classical treatise of Zygmund and the elegant text of Katznelson. The latter chapters were influenced in many ways by the books of Stein and Stein/Weiss. The final chapter on wavelets owes much to the texts of Hernandez/Weiss and Wojtaszczyk.

*Mark A. Pinsky*