Introduction

This text presents an introduction to algebra suitable for upper-level undergraduate or beginning graduate courses. While there is a very extensive offering of textbooks at this level, in my experience teaching this material I have invariably felt the need for a self-contained text that would start 'from zero' (in the sense of not assuming that the reader has had substantial previous exposure to the subject) but that would impart from the very beginning a rather modern, categorically minded viewpoint and aim at reaching a good level of depth. Many textbooks in algebra brilliantly satisfy some, but not all, of these requirements. This book is my attempt at providing a working alternative.

There is a widespread perception that categories should be avoided at first blush, that the abstract language of categories should not be introduced until a student has toiled for a few semesters through example-driven illustrations of the nature of a subject like algebra. According to this viewpoint, categories are only tangentially relevant to the main topics covered in a beginning course, so they can simply be mentioned occasionally for the general edification of the reader, who will in time learn about them (by osmosis?). Paraphrasing a reviewer of a draft of the present text, 'Discussions of categories at this level are the reason why God created appendices.'

It will be clear from a cursory glance at the table of contents that I think otherwise. In this text, categories are introduced on page 18, after a scant reminder of the basic language of naive set theory, for the main purpose of providing a context for universal properties. These are in turn evoked constantly as basic definitions are introduced. The word 'universal' appears at least 100 times in the first three chapters.

I believe that awareness of the categorical language, and especially some appreciation of universal properties, is particularly helpful in approaching a subject such as algebra 'from the beginning'. The reader I have in mind is someone who has reached a certain level of mathematical maturity—for example, who needs no special assistance in grasping an induction argument—but may have only been exposed to algebra in a very cursory manner. My experience is that many upper-level undergraduates or beginning graduate students at Florida State University and at comparable institutions fit this description. For these students, seeing the many introductory concepts in algebra as instances of a few powerful ideas (encapsulated in suitable universal properties) helps to build a comforting unifying context for these notions. The amount of categorical language needed for this catalyzing function is very limited; for example, functors are not really necessary in this acclimatizing stage.

Thus, in my mind the benefit of this approach is precisely that it helps a true beginner, if it is applied with due care. This is my experience in the classroom, and it is the main characteristic feature of this text. The very little categorical language introduced at the outset informs the first part of the book, introducing in general terms groups, rings, and modules. This is followed by a (rather traditional) treatment of standard topics such as Sylow theorems, unique factorization, elementary linear algebra, and field theory. The last third of the book wades into somewhat deeper waters, dealing with tensor products and Hom (including a first introduction to Tor and Ext) and including a final chapter devoted to homological algebra. Some familiarity with categorical language appears indispensable to me in order to appreciate this latter material, and this is hopefully uncontroversial. Having developed a feel for this language in the earlier parts of the book, students find the transition into these more advanced topics particularly smooth.

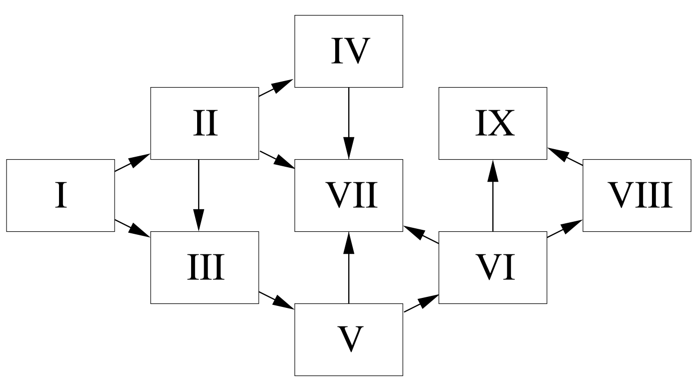
A first version of this book was essentially a careful transcript of my lectures in a run of the (three-semester) algebra sequence at FSU. The chapter on homological algebra was added at the instigation of Ed Dunne, as were a very substantial number of the exercises. The main body of the text has remained very close to the original 'transcript' version: I have resisted the temptation of expanding the material when revising it for publication. I believe that an effective introductory textbook (this is Chapter 0, after all...) should be realistic: it must be possible to cover in class what is covered in the book. Otherwise, the book veers into the 'reference' category; I never meant to write a reference book in algebra, and it would be futile (of me) to try to ameliorate excellent available references such as Lang's 'Algebra'.

The problem sets will give an opportunity to a teacher, or any motivated reader, to get quite a bit beyond what is covered in the main text. To guide in the choice of exercises, I have marked with a $\triangleright$ those problems that are directly referenced from the text, and with a $\neg$ those problems that are referenced from other problems. A minimalist teacher may simply assign all and only the $\triangleright$ problems; these do nothing more than anchor the understanding by practice and may be all that a student can realistically be expected to work out while juggling TA duties and two or three other courses of similar intensity as this one. The main body of the text, together with these exercises, forms a self-contained presentation of essential material. The other exercises, and especially the threads traced by those marked with $\neg$, will offer the opportunity to cover other topics, which some may well consider just as essential: the modular group, quaternions, nilpotent groups, Artinian rings, the Jacobson radical, localization, Lagrange's theorem on four squares, projective space and

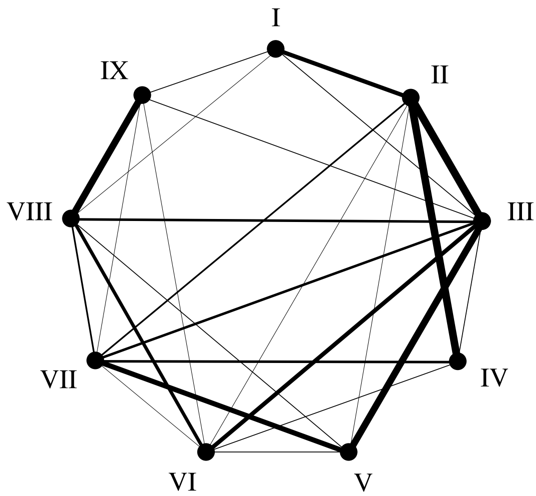
Grassmannians, Nakayama's lemma, associated primes, the spectral theorem for normal operators, etc., are some examples of topics that make their appearance in the exercises. Often a topic is presented over the course of several exercises, placed in appropriate sections of the book. For example, 'Wedderburn's little theorem' is mentioned in Remark III.1.16 (that is: Remark 1.16 in Chapter III); particular cases are presented in Exercises III.2.11 and IV.2.17, and the reader eventually obtains a proof in Exercise VII.5.14, following preliminaries given in Exercises VII.5.12 and VII.5.13. The ¬ label and perusal of the index should facilitate the navigation of such topics. To help further in this process, I have decorated every exercise with a list (added in square brackets) of the places in the book that refer to it. For example, an instructor evaluating whether to assign Exercise V.2.25 will be immediately aware that this exercise is quoted in Exercise VII.5.18, proving a particular case of Dirchlet's theorem on primes in arithmetic progressions, and that this will in turn be quoted in §VII.7.6, discussing the realization of abelian groups as Galois groups over $\mathbb{Q}$.

I have put a high priority on the requirement that this should be a self-contained text which essentially crosses all t's and dots all i's, and does not require that the reader have access to other texts while working through it. I have therefore made a conscious effort to *not* quote other references: I have avoided as much as possible the exquisitely tempting escape route 'For a proof, see ....' This is the main reason why this book is as thick as it is, even if so many topics are *not* covered in it. Among these, commutative algebra and representation theory are perhaps the most glaring omissions. The first is represented to the extent of the standard basic definitions, which allow me to sprinkle a little algebraic geometry here and there (for example, see §VII.2), and of a few slightly more advanced topics in the exercises, but I stopped short of covering, e.g., primary decompositions. The second is missing altogether. It is my hope to complement this book with a 'Chapter 1' in an undetermined future, where I will make amends for these and other shortcomings.

By its nature, this book should be quite suitable for self-study: readers working on their own will find here a self-contained starting point which should work well as a prelude to future, more intensive, explorations. Such readers may be helped by the following '9-fold way' diagram of logical interdependence of the chapters:



This may however better reflect my original intention than the final product. For a more objective gauge, this alternative diagram captures the web of references from a chapter to earlier chapters, with the thickness of the lines representing (roughly) the number of references:



With the self-studying reader especially in mind, I have put extra effort into providing an extensive index. It is not realistic to make a fanfare for each and every new term introduced in a text of this size by an official 'definition'; the index should help a lone traveler find the way back to the source of unfamiliar terminology.

Internal references are handled in a hopefully transparent way. For example, Remark III.1.16 refers to Remark 1.16 in Chapter III; if the reference is made from within Chapter III, the same item is called Remark 1.16. The list in brackets following an exercise indicates other exercises or sections in the book referring to that exercise. For example, Exercise 3.1 in Chapter I is followed by [5.1, §VIII.1.1, §IX.1.2, IX.1.10]: this alerts the reader that there are references to this problem in Exercise 5.1 in Chapter I, section 1.1 in Chapter VIII, section 1.2 in Chapter IX, and Exercise 1.10 in Chapter IX (and nowhere else).

Acknowledgments. My debt to Lang's book, to David Dummit and Richard Foote's 'Abstract Algebra,' or to Artin's 'Algebra' will be evident to anyone who is familiar with these sources. The chapter on homological algebra owes much to David Eisenbud's appendix on the topic in his 'Commutative Algebra', to Gelfand-Manin's 'Methods of homological algebra', and to Weibel's 'An introduction to homological algebra'. But in most cases it would simply be impossible for me to retrace the original source of an expository idea, of a proof, of an exercise, or of a specific pedagogical emphasis: these are all likely offsprings of ideas from any one of these and other influential references and often of associations triggered by following the manifold strands of the World Wide Web. This is another reason why, in a spirit of equanimity, I resolved to essentially avoid references altogether. In any case, I believe all the material I have presented here is standard, and I only retain absolute ownership of every error left in the end product.

I am very grateful to my students for the constant feedback that led me to write this book in this particular way and who contributed essentially to its success in my classes. Some of the students provided me with extensive lists of typos and outright mistakes, and I would especially like to thank Kevin Meek, Jay Stryker, and Yong Jae Cha for their particularly helpful comments. I had the opportunity to try out the material on homological algebra in a course given at Caltech in the fall of 2008, while on a sabbatical from FSU, and I would like to thank Caltech and the audience of the course for their hospitality and the friendly atmosphere. Thanks are also due to MSRI for hospitality during the winter of 2009, when the last fine-tuning of the text was performed.

A few people spotted big and small mistakes in preliminary versions of this book, and I will mention Georges Elencwajg, Xia Liao, and Mirroslav Yotov for particularly precious contributions. I also commend Arlene O'Sean and the staff at the AMS for the excellent copyediting and production work.

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