Preface

In this book, we study the evolution of Riemannian metrics under the Ricci flow. This evolution equation was introduced in a seminal paper by R. Hamilton [44], following earlier work of Eells and Sampson [33] on the harmonic map heat flow. Using the Ricci flow, Hamilton proved that every compact three-manifold with positive Ricci curvature is diffeomorphic to a spherical space form. The Ricci flow has since been used to resolve longstanding open questions in Riemannian geometry and three-dimensional topology. In this text, we focus on the convergence theory for the Ricci flow in higher dimensions and its application to the Differentiable Sphere Theorem. The results we describe have all appeared in research articles. However, we have made an effort to simplify various arguments and streamline the exposition.

In Chapter 1, we give a survey of various sphere theorems in Riemannian geometry (see also [22]). We first describe the Topological Sphere Theorem of Berger and Klingenberg. We then discuss various generalizations of that theorem, such as the Diameter Sphere Theorem of Grove and Shiohama [42] and the Sphere Theorem of Micallef and Moore [60]. These results rely on the variational theory for geodesics and harmonic maps, respectively. We will discuss the main ideas involved in the proof; however, this material will not be used in later chapters. Finally, we state the Differentiable Sphere Theorem obtained by the author and R. Schoen [20].

In Chapter 2, we state the definition of the Ricci flow and describe the short-time existence and uniqueness theory. We then study how the Riemann curvature tensor changes when the metric evolves under the Ricci flow. This evolution equation will be the basis for all the *a priori* estimates established in later chapters.

In Chapter 3, we describe Shi's estimates for the covariant derivatives of the curvature tensor. As an application, we show that the Ricci flow cannot develop a singularity in finite time unless the curvature is unbounded. Moreover, we establish interior estimates for solutions of linear parabolic equations. These estimates play an important role in Sections 4.3 and 5.4.

In Chapter 4, we consider the Ricci flow on S^2. In Section 4.1, we show that any gradient Ricci soliton on S^2 has constant curvature. We then study solutions to the Ricci flow on S^2 with positive scalar curvature. A theorem of Hamilton [46] asserts that such a solution converges to a constant curvature metric after rescaling. A key ingredient in the proof is the monotonicity of Hamilton's entropy functional. This monotonicity formula will be discussed in Section 4.2. Alternative proofs of this theorem can be found in [4], [6], [48], or [82]. The arguments in [4] and [48] are based on a careful study of the isoperimetric profile, while the proofs in [6] and [82] employ PDE techniques.

In Chapter 5, we describe Hamilton's maximum principle for the Ricci flow and discuss the notion of a pinching set. We then describe a general convergence criterion for the Ricci flow. This criterion was discovered by Hamilton [45] and plays an important role in the study of Ricci flow.

In Chapter 6, we explain how Hamilton's classification of three-manifolds with positive Ricci curvature follows from the general theory developed in Chapter 5. We then describe an important curvature estimate, due to Hamilton and Ivey. This inequality holds for any solution to the Ricci flow in dimension 3.

In Chapter 7, we describe various curvature conditions which are preserved by the Ricci flow. We first prove that nonnegative isotropic curvature is preserved by the Ricci flow in all dimensions. This curvature condition originated in Micallef and Moore's work on the Morse index of harmonic two-spheres and plays a central role in this book. We then consider the condition that $M \times \mathbb{R}$ has nonnegative isotropic curvature. This condition is stronger than nonnegative isotropic curvature, and is also preserved by the Ricci flow. Continuing in this fashion, we consider the condition that $M \times \mathbb{R}^2$ has nonnegative isotropic curvature, and the condition that $M \times S^2(1)$ has nonnegative isotropic curvature. (Here, $S^2(1)$ denotes a two-dimensional sphere of constant curvature 1.) We show that these conditions are preserved by the Ricci flow as well.

In Chapter 8, we present the proof of the Differentiable Sphere Theorem. More generally, we show that every compact Riemannian manifold M with the property that $M \times \mathbb{R}$ has positive isotropic curvature is diffeomorphic to a spherical space form. This theorem is the main result of Chapter 8. It

can be viewed as a generalization of Hamilton's work in dimension 3 and was originally proved in [**17**].

In Chapter 9, we prove various rigidity theorems. In particular, we classify all compact Riemannian manifolds M with the property that $M \times \mathbb{R}$ has nonnegative isotropic curvature. Moreover, we show that any Einstein manifold with nonnegative isotropic curvature is necessarily locally symmetric. This generalizes classical results due to Berger [**10**], [**11**] and Tachibana [**84**]. In order to handle the borderline case, we employ a variant of Bony's strict maximum principle for degenerate elliptic equations.

The material presented in Chapters 2–9 is largely, though not fully, self-contained. In Section 2.2, we employ the existence and uniqueness theory for parabolic systems. In Section 4.2, we use the convergence theory for Riemannian manifolds developed by Cheeger and Gromov. Finally, in Chapter 9, we use Berger's classification of holonomy groups, as well as some basic facts about Kähler and quaternionic-Kähler manifolds.

There are some important aspects of Ricci flow which are not mentioned in this book. For example, we do not discuss Hamilton's differential Harnack inequality (cf. [**47**], [**49**]) or Perelman's crucial monotonicity formulae (see [**68**], [**69**]). A detailed exposition of Perelman's entropy functional can be found in [**63**] or [**85**]. A generalization of Hamilton's Harnack inequality is described in [**18**] (see also [**24**]).

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