Chapter 1

A survey of sphere theorems in geometry

1.1. Riemannian geometry background

Let M be a smooth manifold of dimension n, and let g be a Riemannian metric on M. The Levi-Civita connection is defined by

$$\begin{aligned} 2\,g(D_X Y, Z) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \end{aligned}$$

for all vector fields X, Y, Z. The connection D is torsion-free and metric-compatible; that is,

$$D_X Y - D_Y X = [X,Y]$$

and

$$X(g(Y,Z)) = g(D_X Y, Z) + g(Y, D_X Z)$$

for all vector fields X, Y, Z. The Riemann curvature tensor of (M,g) is defined by

$$g\big(D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, W\big) = -R(X,Y,Z,W).$$

Hence, if we write $D^2_{X,Y} Z = D_X D_Y Z - D_{D_X Y} Z$, then we obtain

$$\begin{aligned} D^2_{X,Y} Z - D^2_{Y,X} Z &= D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \\ &= -\sum_{k=1}^n R(X,Y,Z,e_k)\, e_k. \end{aligned}$$

The Levi-Civita connection on (M,g) induces a connection on tensor bundles. For example, if S is a $(0,4)$-tensor, then the covariant derivative

$D_X S$ is given by

$$\begin{aligned}(D_X S)(U, V, W, Z) &= X(S(U, V, W, Z)) \\ &\quad - S(D_X U, V, W, Z) - S(U, D_X V, W, Z) \\ &\quad - S(U, V, D_X W, Z) - S(U, V, W, D_X Z)\end{aligned}$$

for all vector fields U, V, W, Z. Moreover, we denote by $D^2_{X,Y} S$ the second covariant derivative of S:

$$D^2_{X,Y} S = D_X D_Y S - D_{D_X Y} S.$$

Note that $D^2_{X,Y} S$ is tensorial in X and Y. The difference $D^2_{X,Y} S - D^2_{Y,X} S$ can be expressed in terms of the Riemann curvature tensor of (M, g). For example, if S is a $(0, 4)$-tensor, then we have

$$\begin{aligned}&(D^2_{X,Y} S)(U, V, W, Z) - (D^2_{Y,X} S)(U, V, W, Z) \\ &= -S(D^2_{X,Y} U - D^2_{Y,X} U, V, W, Z) - S(U, D^2_{X,Y} V - D^2_{Y,X} V, W, Z) \\ &\quad - S(U, V, D^2_{X,Y} W - D^2_{Y,X} W, Z) - S(U, V, W, D^2_{X,Y} Z - D^2_{Y,X} Z),\end{aligned}$$

hence

$$\begin{aligned}&(D^2_{X,Y} S)(U, V, W, Z) - (D^2_{Y,X} S)(U, V, W, Z) \\ &= \sum_{k=1}^n R(X, Y, U, e_k)\, S(e_k, V, W, Z) + \sum_{k=1}^n R(X, Y, V, e_k)\, S(U, e_k, W, Z) \\ &\quad + \sum_{k=1}^n R(X, Y, W, e_k)\, S(U, V, e_k, Z) + \sum_{k=1}^n R(X, Y, Z, e_k)\, S(U, V, W, e_k).\end{aligned}$$

Finally, the Laplacian of a tensor field S is defined by

$$\Delta S = \sum_{k=1}^n D^2_{e_k, e_k} S,$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M.

The Riemann curvature tensor satisfies certain algebraic identities. We state these identities without proof:

Proposition 1.1. *The curvature tensor satisfies*

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = R(Z, W, X, Y) \tag{1}$$

and

$$R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0 \tag{2}$$

for all vector fields X, Y, Z, W.

The identity (2) is known as the first Bianchi identity.

In light of (1), we may view R as a symmetric bilinear form on the space of two-forms. For each point $p \in M$, the curvature operator $R : \wedge^2 T_p M \times \wedge^2 T_p M \to \mathbb{R}$ is defined by

$$R(X \wedge Y, Z \wedge W) = R(X, Y, Z, W)$$

for all vectors $X, Y, Z, W \in T_p M$.

Definition 1.2. We say that (M, g) has nonnegative curvature operator if $R(\varphi, \varphi) \geq 0$ for all points $p \in M$ and all two-forms $\varphi \in \wedge^2 T_p M$.

Definition 1.3. We say that (M, g) has two-nonnegative curvature operator if $R(\varphi, \varphi) + R(\psi, \psi) \geq 0$ for all points $p \in M$ and all two-forms $\varphi, \psi \in \wedge^2 T_p M$ satisfying $|\varphi|^2 = |\psi|^2$ and $\langle \varphi, \psi \rangle = 0$.

We next recall the notion of sectional curvature. To that end, we consider a point $p \in M$ and a two-dimensional plane $\pi \subset T_p M$. The sectional curvature of π is defined by

$$K(\pi) = \frac{R(X, Y, X, Y)}{|X|^2 \, |Y|^2 - \langle X, Y \rangle^2} = \frac{R(X \wedge Y, X \wedge Y)}{|X \wedge Y|^2},$$

where $\{X, Y\}$ is a basis of π. It is straightforward to verify that this definition is independent of the particular choice of the basis $\{X, Y\}$.

Finally, we review the definition of the Ricci and scalar curvature. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame on M. The Ricci tensor of (M, g) is defined by

$$\text{Ric}(X, Y) = \sum_{k=1}^n R(X, e_k, Y, e_k).$$

The scalar curvature of (M, g) is given by the trace of the Ricci tensor, i.e.

$$\text{scal} = \sum_{k=1}^n \text{Ric}(e_k, e_k).$$

Finally, the trace-free Ricci tensor of (M, g) is defined by

$$\overset{\circ}{\text{Ric}}(X, Y) = \text{Ric}(X, Y) - \frac{1}{n} \, \text{scal} \, g(X, Y).$$

We now describe the second Bianchi identity. This identity relates the covariant derivatives of the Riemann curvature tensor:

Proposition 1.4. *We have*

$$(D_X R)(Y, Z, V, W) + (D_Y R)(Z, X, V, W) + (D_Z R)(X, Y, V, W) = 0$$

for all vector fields X, Y, Z, V, W.

The second Bianchi identity implies the following identity for the covariant derivatives of the Ricci tensor:

Proposition 1.5. *We have*

$$\sum_{k=1}^{n} (D_{e_k} \mathrm{Ric})(X, e_k) = \frac{1}{2} X(\mathrm{scal}) \tag{3}$$

and

$$\sum_{k=1}^{n} (D_{e_k} \overset{\circ}{\mathrm{Ric}})(X, e_k) = \frac{n-2}{2n} X(\mathrm{scal}) \tag{4}$$

for every vector field X.

Proof. Using the second Bianchi identity, we obtain

$$\begin{aligned} X(\mathrm{scal}) &= \sum_{k,l=1}^{n} (D_X R)(e_k, e_l, e_k, e_l) \\ &= \sum_{k,l=1}^{n} (D_{e_k} R)(X, e_l, e_k, e_l) + \sum_{k,l=1}^{n} (D_{e_l} R)(e_k, X, e_k, e_l) \\ &= \sum_{k=1}^{n} (D_{e_k} \mathrm{Ric})(X, e_k) + \sum_{l=1}^{n} (D_{e_l} \mathrm{Ric})(X, e_l). \end{aligned}$$

From this, the identity (3) follows. The identity (4) is an immediate consequence of (3). □

As a consequence, we obtain the following result, which is known as Schur's lemma:

Corollary 1.6. *Let (M, g) be a Riemannian manifold of dimension $n \geq 3$. Suppose that the trace-free Ricci tensor of (M, g) vanishes. Then* $\mathrm{Ric} = \rho\, g$ *for some constant ρ.*

In the remainder of this section, we discuss the notion of curvature pinching. We distinguish between global pinching and pointwise pinching:

Definition 1.7. Let (M, g) be a Riemannian manifold, and let $\delta \in (0, 1)$. We say that (M, g) is strictly δ-pinched in the global sense if the sectional curvatures of (M, g) lie in the interval $(\delta, 1]$. Moreover, we say that (M, g) is weakly δ-pinched in the global sense if the sectional curvatures of (M, g) lie in the interval $[\delta, 1]$.

Definition 1.8. Let (M, g) be a Riemannian manifold, and let $\delta \in (0, 1)$. We say that (M, g) is strictly δ-pinched in the pointwise sense if $0 < \delta\, K(\pi_1) < K(\pi_2)$ for all points $p \in M$ and all two-dimensional planes $\pi_1, \pi_2 \subset T_pM$. Moreover, we say that (M, g) is weakly δ-pinched in the pointwise sense if $0 \leq \delta\, K(\pi_1) \leq K(\pi_2)$ for all points $p \in M$ and all two-planes $\pi_1, \pi_2 \subset T_pM$.

The following important inequality was established by M. Berger:

Proposition 1.9 (M. Berger [**9**]). *Let (M, g) be a Riemannian manifold, and let p be an arbitrary point in M. Moreover, suppose that $\underline{\kappa} \leq K(\pi) \leq \overline{\kappa}$ for all two-dimensional planes $\pi \subset T_pM$. Then*

$$R(e_1, e_2, e_3, e_4) \leq \frac{2}{3}\,(\overline{\kappa} - \underline{\kappa})$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_pM$.

Proof. We shall express $R(e_1, e_2, e_3, e_4)$ in terms of sectional curvatures. To that end, we observe that

$$\begin{aligned}
&R(e_1 + e_3, e_2 + e_4, e_1 + e_3, e_2 + e_4) - R(e_1 + e_3, e_2 - e_4, e_1 + e_3, e_2 - e_4) \\
&- R(e_1 - e_3, e_2 + e_4, e_1 - e_3, e_2 + e_4) + R(e_1 - e_3, e_2 - e_4, e_1 - e_3, e_2 - e_4) \\
&= 8\, R(e_1, e_2, e_3, e_4) + 8\, R(e_1, e_4, e_3, e_2)
\end{aligned}$$

and

$$\begin{aligned}
&R(e_1 + e_4, e_2 + e_3, e_1 + e_4, e_2 + e_3) - R(e_1 + e_4, e_2 - e_3, e_1 + e_4, e_2 - e_3) \\
&- R(e_1 - e_4, e_2 + e_3, e_1 - e_4, e_2 + e_3) + R(e_1 - e_4, e_2 - e_3, e_1 - e_4, e_2 - e_3) \\
&= 8\, R(e_1, e_2, e_4, e_3) + 8\, R(e_1, e_3, e_4, e_2).
\end{aligned}$$

We now subtract the second identity from the first one. Using the first Bianchi identity, we obtain

$$\begin{aligned}
&R(e_1 + e_3, e_2 + e_4, e_1 + e_3, e_2 + e_4) - R(e_1 + e_3, e_2 - e_4, e_1 + e_3, e_2 - e_4) \\
&- R(e_1 - e_3, e_2 + e_4, e_1 - e_3, e_2 + e_4) + R(e_1 - e_3, e_2 - e_4, e_1 - e_3, e_2 - e_4) \\
&- R(e_1 + e_4, e_2 + e_3, e_1 + e_4, e_2 + e_3) + R(e_1 + e_4, e_2 - e_3, e_1 + e_4, e_2 - e_3) \\
&+ R(e_1 - e_4, e_2 + e_3, e_1 - e_4, e_2 + e_3) - R(e_1 - e_4, e_2 - e_3, e_1 - e_4, e_2 - e_3) \\
&= 16\, R(e_1, e_2, e_3, e_4) + 8\, R(e_1, e_4, e_3, e_2) - 8\, R(e_1, e_3, e_4, e_2) \\
&= 24\, R(e_1, e_2, e_3, e_4).
\end{aligned}$$

By assumption, the sectional curvatures of (M, g) all lie in the interval $[\underline{\kappa}, \overline{\kappa}]$. Consequently, we have

$$\begin{aligned}
R(e_1 + e_3, e_2 + e_4, e_1 + e_3, e_2 + e_4) &\leq 4\,\overline{\kappa}, \\
R(e_1 + e_3, e_2 - e_4, e_1 + e_3, e_2 - e_4) &\geq 4\,\underline{\kappa}, \\
R(e_1 - e_3, e_2 + e_4, e_1 - e_3, e_2 + e_4) &\geq 4\,\underline{\kappa}, \\
R(e_1 - e_3, e_2 - e_4, e_1 - e_3, e_2 - e_4) &\leq 4\,\overline{\kappa}, \\
R(e_1 + e_4, e_2 + e_3, e_1 + e_4, e_2 + e_3) &\geq 4\,\underline{\kappa}, \\
R(e_1 + e_4, e_2 - e_3, e_1 + e_4, e_2 - e_3) &\leq 4\,\overline{\kappa}, \\
R(e_1 - e_4, e_2 + e_3, e_1 - e_4, e_2 + e_3) &\leq 4\,\overline{\kappa}, \\
R(e_1 - e_4, e_2 - e_3, e_1 - e_4, e_2 - e_3) &\geq 4\,\underline{\kappa}.
\end{aligned}$$

Putting these facts together, we obtain $24\,R(e_1, e_2, e_3, e_4) \leq 16\,(\overline{\kappa} - \underline{\kappa})$. This completes the proof. □

1.2. The Topological Sphere Theorem

The Sphere Theorem in global differential geometry has a long history, going back to a question of H. Hopf. In 1951, H.E. Rauch [**71**] showed that a compact, simply connected manifold which is δ-pinched in the global sense is homeomorphic to a sphere ($\delta \approx 0.75$). Furthermore, Rauch posed the question of what the optimal pinching constant should be. This question was answered around 1960 by the Topological Sphere Theorem of M. Berger and W. Klingenberg:

Theorem 1.10 (M. Berger [**8**]; W. Klingenberg [**56**])**.** *Let (M, g) be a compact, simply connected Riemannian manifold which is strictly $1/4$-pinched in the global sense. Then M is homeomorphic to S^n.*

The pinching constant in Theorem 1.10 is optimal. To see this, consider the manifolds $\mathbb{CP}^m$, $\mathbb{HP}^m$, and $\mathbb{OP}^2$, equipped with their standard metrics. These spaces have sectional curvatures varying between 1/4 and 1, and are referred to as the compact symmetric spaces of rank one (see [**53**]).

M. Berger has classified all compact, simply connected manifolds which are weakly 1/4-pinched in the global sense.

Theorem 1.11 (M. Berger [**8**])**.** *Let (M, g) be a compact, simply connected Riemannian manifold which is weakly $1/4$-pinched in the global sense. Then M is either homeomorphic to S^n or isometric to a symmetric space.*

The proof of Theorem 1.10 relies on comparison geometry techniques (see e.g. [**26**], Chapter 6). An alternative argument, due to M. Gromov, can be found in [**34**] (see also [**3**]).

1.3. The Diameter Sphere Theorem

In this section, we discuss the Diameter Sphere Theorem of Grove and Shiohama. The argument presented here relies on the variational theory for geodesics, and is due to M. Berger (see [**26**], Theorem 6.13).

Lemma 1.12. *Let (M,g) be a complete Riemannian manifold, and let q be a point in M. Suppose that $\gamma : (-\varepsilon, 0] \to M$ is a smooth path satisfying $d(\gamma(s), q) \geq d(\gamma(0), q) + \mu s$ for all $s \in (-\varepsilon, 0]$. Then there exists a vector $v \in T_{\gamma(0)}M$ such that $\exp_{\gamma(0)}(v) = q$, $|v| = d(\gamma(0), q)$, and $\langle \gamma'(0), v \rangle \geq -\mu\, |v|$.*

Proof. Since (M,g) is complete, we can find a vector $v \in T_{\gamma(0)}M$ such that $\exp_{\gamma(0)}(v) = q$ and $|v| = d(\gamma(0), q)$. If $v = 0$, the assertion is trivial. Hence, it suffices to consider the case $v \neq 0$. We can find a smooth map $\alpha : [0,1] \times (-\varepsilon, 0] \to M$ such that $\alpha(0,s) = \gamma(s)$ for all $s \in (-\varepsilon, 0]$, $\alpha(1,s) = q$ for all $s \in (-\varepsilon, 0]$, and $\alpha(t, 0) = \exp_{\gamma(0)}(tv)$ for all $t \in [0,1]$. Then

$$L(\alpha(\cdot, s)) \geq d(\gamma(s), q) \geq d(\gamma(0), q) + \mu s$$

for all $s \in (-\varepsilon, 0]$, and the inequality is sharp for $s = 0$. Using the formula for the first variation of arc length (cf. [**26**]), we obtain

$$-\frac{1}{|v|}\,\langle \gamma'(0), v \rangle = \frac{d}{ds} L(\alpha(\cdot, s))\Big|_{s=0} \leq \mu.$$

From this, the assertion follows. □

Lemma 1.13. *Let (M,g) be a complete Riemannian manifold, and let q be a point in M. Suppose that $\gamma : [0, \varepsilon) \to M$ is a smooth path satisfying $d(\gamma(s), q) \leq d(\gamma(0), q) + \mu s$ for all $s \in [0, \varepsilon)$. Then there exists a vector $v \in T_{\gamma(0)}M$ such that $\exp_{\gamma(0)}(v) = q$, $|v| = d(\gamma(0), q)$, and $\langle \gamma'(0), v \rangle \geq -\mu\, |v|$.*

Proof. Choose k sufficiently large, and define

$$s_k = \inf\Big\{ s \in [0, \varepsilon) : d(\gamma(s), q) \leq d(\gamma(0), q) + \Big(\mu + \frac{1}{k}\Big)\, s - \frac{1}{k^2} \Big\}.$$

Clearly, $s_k \in (0, \frac{1}{k}]$. Moreover, we have

$$d(\gamma(s), q) \geq d(\gamma(s_k), q) + \Big(\mu + \frac{1}{k}\Big)\,(s - s_k)$$

for all $s \in [0, s_k]$. By Lemma 1.12, we can find a vector $v_k \in T_{\gamma(s_k)}M$ such that $\exp_{\gamma(s_k)}(v_k) = q$, $|v_k| = d(\gamma(s_k), q)$, and $\langle \gamma'(s_k), v_k \rangle \geq -(\mu + \frac{1}{k})\, |v_k|$. If we pass to the limit as $k \to \infty$, the assertion follows. □

Proposition 1.14. *Let (M,g) be a compact Riemannian manifold of dimension n with sectional curvature $K \geq 1$. Suppose that p and q are two points in M such that $d(p,q) = \operatorname{diam}(M,g) > \frac{\pi}{2}$. Moreover, suppose that $\gamma : [0,1] \to M$ is a geodesic satisfying $\gamma(0) = \gamma(1) = p$. Then γ has Morse index at least $n-1$.*

Proof. By assumption, we have $d(\gamma(s), q) \leq d(\gamma(0), q)$ for all $s \in [0,1]$. By Lemma 1.13, there exists a vector $v \in T_pM$ such that $\exp_p(v) = q$, $|v| = d(p,q)$, and $\langle \gamma'(0), v \rangle \geq 0$.

We claim that $L(\gamma) > \pi$. To prove this, we argue by contradiction. If $L(\gamma) \leq \pi$, then the hinge version of Toponogov's theorem (see e.g. [**26**], Theorem 2.2B) implies that

$$\begin{aligned}\cos(d(\gamma(1), q)) &\geq \cos(L(\gamma))\,\cos(d(\gamma(0), q)) \\ &\quad + \sin(L(\gamma))\,\sin(d(\gamma(0), q))\,\cos(\sphericalangle(\gamma'(0), v)).\end{aligned}$$

By assumption, we have $L(\gamma) \in (0, \pi]$ and $d(\gamma(0), q) \in (\frac{\pi}{2}, \pi]$. Moreover, the inequality $\langle \gamma'(0), v \rangle \geq 0$ implies $\cos(\sphericalangle(\gamma'(0), v)) \geq 0$. Putting these facts together, we obtain

$$\cos(d(\gamma(1), q)) \geq \cos(L(\gamma))\,\cos(d(\gamma(0), q)) > \cos(d(\gamma(0), q)).$$

This contradicts the fact that $\gamma(0) = \gamma(1)$. Consequently, we have $L(\gamma) > \pi$.

Let $\mathscr{H}$ be the space of all vector fields of the form $V(s) = \sin(\pi s)\, X(s)$, where X is a parallel vector field along γ satisfying $\langle \gamma'(s), X(s) \rangle = 0$ for all $s \in [0,1]$. Then

$$\begin{aligned}I(V,V) &= \int_0^1 \big(\big|D_{\frac{d}{ds}} V(s)\big|^2 - R(\gamma'(s), V(s), \gamma'(s), V(s))\big)\, ds \\ &\leq (\pi^2 - L(\gamma)^2) \int_0^1 |V(s)|^2\, ds\end{aligned}$$

for all $V \in \mathscr{H}$. Since $L(\gamma) > \pi$, the restriction of I to $\mathscr{H}$ is negative definite. This implies $\mathrm{ind}(\gamma) \geq \dim \mathscr{H} = n - 1$. □

Combining Proposition 1.14 with the variational theory for geodesics, we can draw the following conclusion:

Theorem 1.15 (K. Grove, K. Shiohama [**42**]). *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$ with sectional curvature $K \geq 1$ and diameter $\mathrm{diam}(M, g) > \frac{\pi}{2}$. Then M is homeomorphic to S^n.*

Proof. We claim that M is $(n-1)$-connected. Suppose this false. Then there exists an integer $k \in \{1, \ldots, n-1\}$ such that $\pi_k(M) \neq 0$. Let us fix two points $p, q \in M$ such that $d(p,q) = \mathrm{diam}(M, g) > \frac{\pi}{2}$. Since $\pi_k(M) \neq 0$, there exists a geodesic $\gamma : [0,1] \to M$ such that $\gamma(0) = \gamma(1) = p$ and $\mathrm{ind}(\gamma) < k$. On the other hand, we have $\mathrm{ind}(\gamma) \geq n - 1$ by Proposition 1.14. This is a contradiction.

Therefore, M is $(n-1)$-connected. This implies that M is a homotopy sphere. Hence, it follows from results of Freedman and Smale that M is homeomorphic to S^n (cf. [**36**], Theorem 1.6, and [**81**], Theorem A). □

1.4. The Sphere Theorem of Micallef and Moore

In this section, we describe a generalization of Theorem 1.10 due to Micallef and Moore. Let (M, g) be a Riemannian manifold of dimension $n \geq 4$, and let f be a smooth map from S^2 into M. In the following, we identify S^2 with $\mathbb{R}^2 \cup \{\infty\}$ via stereographic projection. Moreover, we denote by (x, y) the standard Cartesian coordinates on $\mathbb{R}^2$. The energy of f is defined by

$$\mathscr{E}(f) = \frac{1}{2} \int_{S^2} \Big(\Big|\frac{\partial f}{\partial x}\Big|^2 + \Big|\frac{\partial f}{\partial y}\Big|^2 \Big)\, dx\, dy.$$

A map $f : S^2 \to M$ is called harmonic if $D_{\frac{\partial}{\partial x}} \frac{\partial f}{\partial x} + D_{\frac{\partial}{\partial y}} \frac{\partial f}{\partial y} = 0$.

Throughout this section, we assume that $f : S^2 \to M$ is a nonconstant harmonic map. This implies that f is a critical point of the functional $\mathscr{E}$. Moreover, the second variation of $\mathscr{E}$ is given by

$$\begin{aligned} I(W, W) &= \int_{S^2} \Big[\big|D_{\frac{\partial}{\partial x}} W\big|^2 + \big|D_{\frac{\partial}{\partial y}} W\big|^2 \Big]\, dx\, dy \\ &\quad - \int_{S^2} \Big[R\Big(\frac{\partial f}{\partial x}, W, \frac{\partial f}{\partial x}, W\Big) + R\Big(\frac{\partial f}{\partial y}, W, \frac{\partial f}{\partial y}, W\Big) \Big]\, dx\, dy \end{aligned}$$

for all vector fields $W \in \Gamma(f^*(TM))$. Here, R denotes the Riemann curvature tensor of (M, g).

For each point $p \in M$, we denote by $T_p^{\mathbb{C}} M = T_p M \otimes_{\mathbb{R}} \mathbb{C}$ the complexified tangent space to M at p. We may extend the inner product $g : T_p M \times T_p M \to \mathbb{R}$ to a complex bilinear form $g : T_p^{\mathbb{C}} M \times T_p^{\mathbb{C}} M \to \mathbb{C}$. Similarly, the Riemann curvature tensor extends to a complex multilinear form $R : T_p^{\mathbb{C}} M \times T_p^{\mathbb{C}} M \times T_p^{\mathbb{C}} M \times T_p^{\mathbb{C}} M \to \mathbb{C}$.

We next define

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{1}{2} \Big(\frac{\partial f}{\partial x} - i\, \frac{\partial f}{\partial y} \Big) \in \Gamma(f^*(T^{\mathbb{C}} M)), \\ \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \Big(\frac{\partial f}{\partial x} + i\, \frac{\partial f}{\partial y} \Big) \in \Gamma(f^*(T^{\mathbb{C}} M)). \end{aligned}$$

Moreover, for each section $W \in \Gamma(f^*(T^{\mathbb{C}} M))$ we define

$$\begin{aligned} D_{\frac{\partial}{\partial z}} W &= \frac{1}{2} \big(D_{\frac{\partial}{\partial x}} W - i\, D_{\frac{\partial}{\partial y}} W \big), \\ D_{\frac{\partial}{\partial \bar{z}}} W &= \frac{1}{2} \big(D_{\frac{\partial}{\partial x}} W + i\, D_{\frac{\partial}{\partial y}} W \big). \end{aligned}$$

We may extend the index form $I : \Gamma(f^*(TM)) \times \Gamma(f^*(TM)) \to \mathbb{R}$ to a complex bilinear form $I : \Gamma(f^*(T^{\mathbb{C}} M)) \times \Gamma(f^*(T^{\mathbb{C}} M)) \to \mathbb{C}$. The complexified index form can be rewritten as follows:

Proposition 1.16. *For each section $W \in \Gamma(f^*(T^{\mathbb{C}}M))$, we have*

$$I(W,\overline{W}) = 4 \int_{S^2} g\big(D_{\frac{\partial}{\partial \bar{z}}} W, D_{\frac{\partial}{\partial z}} \overline{W}\big)\, dx\, dy - 4 \int_{S^2} R\Big(\frac{\partial f}{\partial z}, W, \frac{\partial f}{\partial \bar{z}}, \overline{W}\Big)\, dx\, dy.$$

Proof. By definition of I, we have

$$\begin{aligned} I(W,\overline{W}) &= \int_{S^2} \Big[g\big(D_{\frac{\partial}{\partial x}} W, D_{\frac{\partial}{\partial x}} \overline{W}\big) + g\big(D_{\frac{\partial}{\partial y}} W, D_{\frac{\partial}{\partial y}} \overline{W}\big)\Big]\, dx\, dy \\ &\quad - \int_{S^2} \Big[R\Big(\frac{\partial f}{\partial x}, W, \frac{\partial f}{\partial x}, \overline{W}\Big) + R\Big(\frac{\partial f}{\partial y}, W, \frac{\partial f}{\partial y}, \overline{W}\Big)\Big]\, dx\, dy. \end{aligned}$$

This implies

$$\begin{aligned} I(W,\overline{W}) &= 2 \int_{S^2} \Big[g\big(D_{\frac{\partial}{\partial z}} W, D_{\frac{\partial}{\partial \bar{z}}} \overline{W}\big) + g\big(D_{\frac{\partial}{\partial \bar{z}}} W, D_{\frac{\partial}{\partial z}} \overline{W}\big)\Big]\, dx\, dy \\ &\quad - 2 \int_{S^2} \Big[R\Big(\frac{\partial f}{\partial \bar{z}}, W, \frac{\partial f}{\partial z}, \overline{W}\Big) + R\Big(\frac{\partial f}{\partial z}, W, \frac{\partial f}{\partial \bar{z}}, \overline{W}\Big)\Big]\, dx\, dy. \end{aligned}$$

Using the first Bianchi identity, we obtain

$$\begin{aligned} &\int_{S^2} \Big[g\big(D_{\frac{\partial}{\partial z}} W, D_{\frac{\partial}{\partial \bar{z}}} \overline{W}\big) - g\big(D_{\frac{\partial}{\partial \bar{z}}} W, D_{\frac{\partial}{\partial z}} \overline{W}\big)\Big]\, dx\, dy \\ &= \int_{S^2} g\big(D_{\frac{\partial}{\partial z}} D_{\frac{\partial}{\partial \bar{z}}} W - D_{\frac{\partial}{\partial \bar{z}}} D_{\frac{\partial}{\partial z}} W, \overline{W}\big)\, dx\, dy \\ &= -\int_{S^2} R\Big(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar{z}}, W, \overline{W}\Big)\, dx\, dy \\ &= \int_{S^2} \Big[R\Big(\frac{\partial f}{\partial \bar{z}}, W, \frac{\partial f}{\partial z}, \overline{W}\Big) - R\Big(\frac{\partial f}{\partial z}, W, \frac{\partial f}{\partial \bar{z}}, \overline{W}\Big)\Big]\, dx\, dy. \end{aligned}$$

Putting these facts together, the assertion follows. □

Since $f : S^2 \to M$ is a harmonic map, we have $D_{\frac{\partial}{\partial \bar{z}}} \frac{\partial f}{\partial z} = 0$. Hence, $\frac{\partial f}{\partial z}$ is a holomorphic section of the bundle $f^*(T^{\mathbb{C}}M)$.

Proposition 1.17. *Suppose that $W \in \Gamma(f^*(T^{\mathbb{C}}M))$ is holomorphic. Then $g(\frac{\partial f}{\partial z}, W) = 0$ at each point on S^2. In particular, we have $g(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial z}) = 0$ at each point on S^2.*

Proof. By assumption, we have $D_{\frac{\partial}{\partial \bar{z}}} \frac{\partial f}{\partial z} = D_{\frac{\partial}{\partial \bar{z}}} W = 0$. Consequently, the inner product $g(\frac{\partial f}{\partial z}, W)$ is a holomorphic function on S^2. This implies that the function $g(\frac{\partial f}{\partial z}, W)$ is constant. Since f is a smooth map from S^2 into M, the section $\frac{\partial f}{\partial z}$ vanishes at the north pole on S^2. Therefore, the function $g(\frac{\partial f}{\partial z}, W)$ vanishes at the north pole on S^2. Thus, we conclude that $g(\frac{\partial f}{\partial z}, W) = 0$ at each point on S^2. □

Proposition 1.18. *There exists a holomorphic subbundle $E \subset f^*(T^{\mathbb{C}}M)$ such that $\operatorname{rank}_{\mathbb{C}} E \geq n - 2$, $c_1(E) = 0$, and $\frac{\partial f}{\partial z} \notin \Gamma(E)$.*

Proof. By a theorem of Grothendieck [**40**], there exist holomorphic line bundles $L_1, \ldots, L_n \subset f^*(T^{\mathbb{C}}M)$ such that $f^*(T^{\mathbb{C}}M) = L_1 \oplus L_2 \oplus \cdots \oplus L_n$. We assume that the line bundles $L_1, \ldots, L_n$ are chosen so that

$$c_1(L_1) \geq c_1(L_2) \geq \ldots \geq c_1(L_n).$$

The line bundles $L_1, \ldots, L_n$ are not unique. However, the sequence of Chern classes $c_1(L_1), \ldots, c_1(L_n)$ is uniquely determined (i.e. it is independent of the particular choice of $L_1, \ldots, L_n$).

Recall that $f^*(T^{\mathbb{C}}M)$ is the complexification of a real vector bundle. Hence, the bundle $f^*(T^{\mathbb{C}}M)$ is isomorphic to its dual bundle, which, in turn, is isomorphic to $L_n^* \oplus L_{n-1}^* \oplus \cdots \oplus L_1^*$. Since the sequence of Chern classes is unique, we conclude that

$$c_1(L_k) = c_1(L_{n-k+1}^*) = -c_1(L_{n-k+1})$$

for $k = 1, \ldots, n$.

We now write $\frac{\partial f}{\partial z} = W_1 + \cdots + W_n$, where $W_j \in \Gamma(L_j)$ for $j = 1, \ldots, n$. Since f is nonconstant, we can find an integer $k \in \{1, \ldots, n\}$ such that W_k does not vanish identically. We then define $E = \bigoplus_{j \in J} L_j$, where $J = \{1, \ldots, n\} \setminus \{k, n-k+1\}$. It is straightforward to verify that E has all the required properties. □

Theorem 1.19 (M. Micallef, J.D. Moore [**60**])**.** *Let (M, g) be a Riemannian manifold of dimension $n \geq 4$. Let us assume that $R(\zeta, \eta, \bar{\zeta}, \bar{\eta}) > 0$ for all points $p \in M$ and all linearly independent vectors $\zeta, \eta \in T_p^{\mathbb{C}}M$ satisfying $g(\zeta, \zeta) = g(\zeta, \eta) = g(\eta, \eta) = 0$. Finally, let $f : S^2 \to M$ be a nonconstant harmonic map. Then f has Morse index at least $[\frac{n-2}{2}]$.*

Proof. Let $E \subset f^*(T^{\mathbb{C}}M)$ be the holomorphic subbundle constructed in Proposition 1.18, and let $\mathscr{H}$ be the space of holomorphic sections of E. Given two sections $W_1, W_2 \in \mathscr{H}$, the inner product $g(W_1, W_2)$ is a holomorphic function on S^2. Consequently, the function $g(W_1, W_2)$ is constant. This gives a complex bilinear form

$$\mathscr{H} \times \mathscr{H} \to \mathbb{C}, \quad (W_1, W_2) \mapsto g(W_1, W_2).$$

By the Riemann-Roch theorem, we have $\dim_{\mathbb{C}} \mathscr{H} \geq n - 2$. Hence, there exists a subspace $\mathscr{H}_0 \subset \mathscr{H}$ such that $\dim_{\mathbb{C}} \mathscr{H}_0 \geq [\frac{n-2}{2}]$ and $g(W, W) = 0$ for all $W \in \mathscr{H}_0$.

We now consider an arbitrary section $W \in \mathscr{H}_0$. Since W is holomorphic, it follows from Proposition 1.17 that $g(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial z}) = g(\frac{\partial f}{\partial z}, W) = 0$ at each point on S^2. Moreover, we have $g(W, W) = 0$ at each point on S^2. Using the curvature assumption, we obtain

$$R\Big(\frac{\partial f}{\partial z}, W, \frac{\partial f}{\partial \bar{z}}, \overline{W}\Big) \geq 0,$$

with equality if and only if $\frac{\partial f}{\partial z}$ and W are linearly dependent. Since W is holomorphic, we conclude that

$$I(W,\overline{W}) = -4 \int_{S^2} R\Big(\frac{\partial f}{\partial z}, W, \frac{\partial f}{\partial \bar{z}}, \overline{W}\Big)\, dx\, dy \leq 0$$

by Proposition 1.16. We next analyze the case of equality. If $I(W,\overline{W}) = 0$, then $W = \psi\, \frac{\partial f}{\partial z}$ for some meromorphic function $\psi : S^2 \to \mathbb{C}$. Since $W \in \Gamma(E)$ and $\frac{\partial f}{\partial z} \notin \Gamma(E)$, the function ψ vanishes identically. Thus, we conclude that $I(W,\overline{W}) < 0$ for every nonvanishing section $W \in \mathscr{H}_0$.

We now complete the proof of Theorem 1.19. Suppose that f has Morse index $m < [\frac{n-2}{2}]$. Then $\dim_{\mathbb{C}} \mathscr{H}_0 \geq [\frac{n-2}{2}] > m$. Consequently, there exists a nonvanishing section $W \in \mathscr{H}_0$ which is orthogonal to the first m eigenfunctions of the second variation operator. Since $W \in \mathscr{H}_0$, we have $I(W,\overline{W}) < 0$. On the other hand, we have $I(W,\overline{W}) \geq 0$ since W is orthogonal to the first m eigenfunctions of the second variation operator. This is a contradiction. □

Combining Theorem 1.19 with the variational theory for harmonic maps (see e.g. [**75**]) yields the following result:

Theorem 1.20 (M. Micallef, J.D. Moore [**60**]). *Let (M,g) be a compact, simply connected Riemannian manifold of dimension $n \geq 4$. Suppose that $R(\zeta,\eta,\bar{\zeta},\bar{\eta}) > 0$ for all points $p \in M$ and all linearly independent vectors $\zeta,\eta \in T_p^{\mathbb{C}} M$ satisfying $g(\zeta,\zeta) = g(\zeta,\eta) = g(\eta,\eta) = 0$. Then M is homeomorphic to S^n.*

Proof. We claim that M is $(n-1)$-connected. Suppose this is false. Then there exists an integer $k \in \{2,\ldots,n-1\}$ such that $\pi_k(M) \neq 0$ and $\pi_j(M) = 0$ for $j = 1,\ldots,k-1$. Using the Hurewicz theorem (cf. [**16**], Chapter VII, Corollary 10.8), we obtain $H_k(M,\mathbb{Z}) \neq 0$ and $H_j(M,\mathbb{Z}) = 0$ for $j = 1,\ldots,k-1$. Hence, the universal coefficient theorem for cohomology implies that $H^j(M,\mathbb{Z}) = 0$ for $j = 1,\ldots,k-1$ (see [**16**], Chapter V, Corollary 7.3). Using Poincaré duality (see e.g. [**16**], Chapter VI, Corollary 8.4), we conclude that $H_{n-j}(M,\mathbb{Z}) = 0$ for $j = 1,\ldots,k-1$. Since $H_k(M,\mathbb{Z}) \neq 0$, it follows that $k \leq [\frac{n}{2}]$.

We now apply an existence theorem for harmonic two-spheres. Since $k \geq 2$ and $\pi_k(M) \neq 0$, there exists a nonconstant harmonic map $f : S^2 \to M$ with Morse index less than $k-1$ (see [**75**], Chapter VII, Theorem 2). On the other hand, it follows from Theorem 1.19 that f has Morse index at least $[\frac{n-2}{2}]$. Putting these facts together, we conclude that $k > [\frac{n}{2}]$. This is a contradiction.

Therefore, M is $(n-1)$-connected. Consequently, M is a homotopy sphere. It now follows from work of Freedman [**36**] and Smale [**81**] that M is homeomorphic to S^n. □

1.5. Exotic Spheres and the Differentiable Sphere Theorem

It is known that there exist smooth manifolds which are homeomorphic, but not diffeomorphic, to S^n. The first examples of such exotic spheres were constructed by J. Milnor in 1956:

Theorem 1.21 (J. Milnor [**62**]). *There exists a smooth manifold M which is homeomorphic, but not diffeomorphic, to S^7.*

In light of Theorem 1.21, it is natural to ask whether a compact, simply connected manifold which is strictly 1/4-pinched in the global sense is diffeomorphic to S^n. This question is known as the Differentiable Sphere Theorem and has been studied extensively. The first results in this direction were established in 1966 by D. Gromoll [**38**] and E. Calabi. Gromoll showed that a compact, simply connected Riemannian manifold which is $\delta(n)$-pinched in the global sense is diffeomorphic to S^n. The pinching constant $\delta(n)$ depends only on the dimension and converges to 1 as $n \to \infty$. In 1977, M. Sugimoto, K. Shiohama, and H. Karcher [**83**] proved an analogous result with a pinching constant independent of n ($\delta = 0.87$). The pinching constant was later improved by E. Ruh [**72**] ($\delta = 0.80$), and by K. Grove, H. Karcher, and E. Ruh [**41**] ($\delta = 0.76$). E. Ruh [**73**] proved the Differentiable Sphere Theorem under a pointwise pinching condition, with a pinching constant converging to 1 as $n \to \infty$.

Using the Ricci flow, R. Hamilton proved the following fundamental result:

Theorem 1.22 (R. Hamilton [**44**]). *Let (M, g) be a compact three-manifold with positive Ricci curvature. Then M is diffeomorphic to a spherical space form.*

The proof of Theorem 1.22 will be presented in Section 6.1. The key idea is to evolve the metric g by the Ricci flow and to show that the evolving metrics approach a metric of constant sectional curvature after rescaling. The proof relies on suitable pointwise curvature estimates, which are obtained using the maximum principle.

Various authors have obtained convergence results for the Ricci flow in higher dimensions. These results are all based on the general framework developed by R. Hamilton in [**44**] and [**45**]. G. Huisken [**54**] showed that the Ricci flow evolves metrics with sufficiently pinched curvature to constant curvature metrics. Similar results were obtained by C. Margerin [**57**] and

S. Nishikawa [**65**]. In dimension 4, R. Hamilton [**45**] proved a convergence theorem for initial metrics with positive curvature operator. This result was extended to arbitrary dimensions by C. Böhm and B. Wilking [**14**]. Other important results in this direction were established by B. Andrews and H. Nguyen [**5**], H. Chen [**28**], and C. Margerin [**58**].

A. Chang, M. Gursky, and P. Yang [**25**] proved a conformally invariant sphere theorem in dimension 4. This result only requires an integral pinching condition; furthermore, the pinching constant is sharp. The proof relies on a combination of conformal techniques and the Ricci flow. The key idea is to deform the given metric to a conformally equivalent metric which satisfies the assumptions of Margerin's theorem [**58**]. The Ricci flow then provides a deformation to a metric of constant sectional curvature.

In 2007, the author and R. Schoen proved the Differentiable Sphere Theorem with the optimal pinching constant ($\delta = 1/4$). This result is a special case of a more general theorem:

Theorem 1.23 (S. Brendle [**17**]). *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$. Suppose that $R(\zeta, \eta, \bar{\zeta}, \bar{\eta}) > 0$ for all points $p \in M$ and all linearly independent vectors $\zeta, \eta \in T_p^{\mathbb{C}} M$ satisfying $g(\zeta, \zeta)\, g(\eta, \eta) - g(\zeta, \eta)^2 = 0$. Then M is diffeomorphic to a spherical space form.*

Using Proposition 1.9, one can show that any manifold (M, g) which is strictly 1/4-pinched in the pointwise sense satisfies the curvature assumption in Theorem 1.23. Hence, we obtain the following result, which was first proved in [**20**]:

Corollary 1.24 (S. Brendle, R. Schoen [**20**]). *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$ which is strictly 1/4-pinched in the pointwise sense. Then M is diffeomorphic to a spherical space form.*

Finally, we have the following rigidity result:

Theorem 1.25 (S. Brendle, R. Schoen [**21**]). *Let M be a compact Riemannian manifold of dimension $n \geq 4$ which is weakly 1/4-pinched in the pointwise sense. Then M is either diffeomorphic to a spherical space form or isometric to a locally symmetric space.*

Using results from [**21**], P. Petersen and T. Tao [**70**] obtained a classification of manifolds with almost 1/4-pinched curvature.

The proof of Theorem 1.23 uses the Ricci flow and will be presented in Section 8.4. The proof of Theorem 1.25 will be described in Section 9.8.

Chapter 2

Hamilton's Ricci flow

2.1. Definition and special solutions

In this section, we state the definition of the Ricci flow, and discuss some basic examples.

Definition 2.1. Let M be a manifold, and let $g(t)$, $t \in [0, T)$, be a one-parameter family of Riemannian metrics on M. We say that $g(t)$ is a solution to the Ricci flow if

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}_{g(t)}.$$

In the remainder of this section, we describe various special solutions to the Ricci flow.

2.1.1. Einstein manifolds. Let (M, g_0) be a Riemannian manifold. We say that g_0 is an Einstein metric if $\operatorname{Ric}_{g_0} = \rho\, g_0$ for some constant ρ. In that case, the metrics

$$g(t) = (1 - 2\rho t)\, g_0$$

form a solution to the Ricci flow.

2.1.2. Ricci solitons. Let (M, g_0) be a Riemannian manifold. We say that (M, g_0) is a Ricci soliton if there exists a constant ρ and a vector field ξ such that

$$\operatorname{Ric}_{g_0} + \frac{1}{2} \mathscr{L}_\xi g_0 = \rho\, g_0,$$

where $\mathscr{L}_\xi g_0$ denotes the Lie derivative of g_0 along the vector field ξ. Depending on the sign of ρ, a Ricci soliton is called shrinking ($\rho > 0$), steady ($\rho = 0$), or expanding ($\rho < 0$). If the vector field ξ is the gradient of a function, we say that (M, g_0) is a gradient Ricci soliton.

Suppose that (M, g_0) is a Ricci soliton. For each point $p \in M$, we denote by $\varphi_t(p)$ the unique solution of the ordinary differential equation (ODE)

$$\frac{\partial}{\partial t}\varphi_t(p) = \frac{1}{1-2\rho t}\,\xi\big|_{\varphi_t(p)}$$

with initial condition $\varphi_0(p) = p$. This defines a one-parameter family of diffeomorphisms $\varphi_t : M \to M$. Then the metrics

$$g(t) = (1-2\rho t)\,\varphi_t^*(g_0)$$

form a solution to the Ricci flow.

2.1.3. The cigar soliton. The simplest example of a Ricci soliton is the cigar soliton on $\mathbb{R}^2$. For each $t \in (-\infty, \infty)$, we define a metric $g(t)$ on $\mathbb{R}^2$ by

$$g_{ij}(t) = \frac{4}{e^t + |x|^2}\,\delta_{ij}$$

for $x \in \mathbb{R}^2$. The scalar curvature of $g(t)$ is given by

$$\mathrm{scal}_{g(t)} = \frac{e^t}{e^t + |x|^2}.$$

This implies

$$\frac{\partial}{\partial t}g(t) = -\mathrm{scal}_{g(t)}\,g(t) = -2\,\mathrm{Ric}_{g(t)}.$$

Consequently, the metrics $g(t)$, $t \in (-\infty, \infty)$, form a solution to the Ricci flow. Moreover, we have $g(t) = \varphi_t^*(g(0))$, where $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\varphi_t(x) = e^{-\frac{t}{2}}\,x$. Thus, $g(0)$ is a steady Ricci soliton.

2.1.4. The Rosenau solution. There is an interesting closed-form solution to the Ricci flow on S^2. For each $t \in (-\infty, 0)$, we define a metric $g(t)$ on $\mathbb{R}^2$ by

$$g_{ij}(t) = \frac{8\,\sinh(-t)}{1 + 2\cosh(-t)\,|x|^2 + |x|^4}\,\delta_{ij}$$

for $x \in \mathbb{R}^2$. Note that $g(t)$ extends to a smooth metric on S^2. The scalar curvature of $g(t)$ is given by

$$\mathrm{scal}_{g(t)} = \frac{\cosh(-t)}{\sinh(-t)} - \frac{2\,\sinh(-t)\,|x|^2}{1 + 2\cosh(-t)\,|x|^2 + |x|^4}.$$

This implies

$$\frac{\partial}{\partial t}g(t) = -\mathrm{scal}_{g(t)}\,g(t) = -2\,\mathrm{Ric}_{g(t)}.$$

Consequently, the metrics $g(t)$, $t \in (-\infty, 0)$, form a solution to the Ricci flow.

2.2. Short-time existence and uniqueness

In this section, we describe a short-time existence and uniqueness theorem for the Ricci flow. This theorem was first proved by R. Hamilton in 1982. The proof of this result is subtle, as the Ricci flow fails to be strictly parabolic. In order to overcome this obstacle, Hamilton employed the Nash-Moser inverse function theorem. DeTurck [**32**] subsequently gave an alternative proof of Theorem 2.8, which avoids the use of the Nash-Moser theorem. In the remainder of this section, we outline the main ideas in DeTurck's argument (see also [**49**], Section 6). We begin with a definition:

Definition 2.2. Let f be a smooth map from a Riemannian manifold (M, g) into a Riemannian manifold (N, h). The harmonic map Laplacian of f is defined by

$$\Delta_{g,h} f = \sum_{k=1}^{n} (D_{e_k} df)(e_k),$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on (M, g). Here, the differential df is viewed as a section of the vector bundle $TM^* \otimes f^*(TN)$, and D denotes the induced connection on that bundle. Note that $\Delta_{g,h} f$ is a section of the vector bundle $f^*(TN)$.

Clearly, the harmonic map Laplacian is invariant under the action of the diffeomorphism group of M.

Lemma 2.3. *Let f be a smooth map from a Riemannian manifold (M, g) into a Riemannian manifold (N, h), and let φ be a diffeomorphism from M to itself. Then*

$$(\Delta_{\varphi^*(g),h} (f \circ \varphi))\big|_p = (\Delta_{g,h} f)\big|_{\varphi(p)} \in T_{f(\varphi(p))}N$$

for all points $p \in M$.

In order to show that the Ricci flow has a unique solution on a short time interval, we replace the Ricci flow by an equivalent evolution equation which is strictly parabolic. This evolution equation is known as the Ricci-DeTurck flow.

Definition 2.4. Let M be a compact manifold, and let h be a fixed background metric on M. Moreover, suppose that $\tilde{g}(t)$, $t \in [0, T)$, is a one-parameter family of Riemannian metrics on M. We say that $\tilde{g}(t)$ is a solution of the Ricci-DeTurck flow if

$$\frac{\partial}{\partial t} \tilde{g}(t) = -2 \operatorname{Ric}_{\tilde{g}(t)} - \mathscr{L}_{\xi_t} \tilde{g}(t),$$

where $\xi_t = \Delta_{\tilde{g}(t),h} \operatorname{id}$.

While the Ricci flow is only weakly parabolic, the Ricci-DeTurck flow turns out to be strictly parabolic. As a consequence, we obtain an existence and uniqueness result for the Ricci-DeTurck flow:

Proposition 2.5. *Let M be a compact manifold, and let h be a fixed background metric on M. Given any initial metric g_0, there exist a real number $T > 0$ and a smooth one-parameter family of metrics $\tilde{g}(t)$, $t \in [0,T)$, such that $\tilde{g}(t)$ is a solution of the Ricci-DeTurck flow and $\tilde{g}(0) = g_0$. Moreover, the solution $\tilde{g}(t)$ is unique.*

Proof. In local coordinates, the Ricci tensor of $\tilde{g}$ is given by

$$\begin{aligned}\mathrm{Ric}_{\tilde{g}} = -\frac{1}{2} \sum_{i,j,k,l=1}^{n} \tilde{g}^{ik} \, (\partial_i \partial_k \tilde{g}_{jl} - \partial_i \partial_l \tilde{g}_{jk} - \partial_j \partial_k \tilde{g}_{il} + \partial_j \partial_l \tilde{g}_{ik}) \, dx^j \otimes dx^l \\ + \text{lower order terms.}\end{aligned}$$

Moreover, the vector field $\xi = \Delta_{\tilde{g},h} \, \mathrm{id}$ can be written in the form

$$\xi = \sum_{i,k,l=1}^{n} \tilde{g}^{ik} \left((\Gamma^h)_{ik}^l - (\Gamma^{\tilde{g}})_{ik}^l \right) \partial_l,$$

where $\Gamma^{\tilde{g}}$ and Γ^h denote the Christoffel symbols associated with the metrics $\tilde{g}$ and h, respectively. This implies

$$\begin{aligned}\xi = -\frac{1}{2} \sum_{i,j,k,l=1}^{n} \tilde{g}^{ik} \, \tilde{g}^{jl} \left(\partial_i \tilde{g}_{jk} + \partial_k \tilde{g}_{ij} - \partial_j \tilde{g}_{ik} \right) \partial_l \\ + \text{lower order terms.}\end{aligned}$$

From this, we deduce that

$$\begin{aligned}\mathscr{L}_\xi \tilde{g} = - \sum_{i,j,k,l=1}^{n} \tilde{g}^{ik} \, (\partial_i \partial_l \tilde{g}_{jk} + \partial_j \partial_k \tilde{g}_{il} - \partial_j \partial_l \tilde{g}_{ik}) \, dx^j \otimes dx^l \\ + \text{lower order terms.}\end{aligned}$$

Putting these facts together, we obtain

$$\begin{aligned}-2 \, \mathrm{Ric}_{\tilde{g}} - \mathscr{L}_\xi \tilde{g} = \sum_{i,j,k,l=1}^{n} \tilde{g}^{ik} \, \partial_i \partial_k \tilde{g}_{jl} \, dx^j \otimes dx^l \\ + \text{lower order terms.}\end{aligned}$$

This shows that the Ricci-DeTurck flow is strictly parabolic. Hence, the assertion follows from standard existence and uniqueness theorems for parabolic systems. □

There is a one-to-one correspondence between solutions to the Ricci flow and solutions to the Ricci-DeTurck flow. In the first step, we show that any solution of the Ricci-DeTurck flow gives rise to a solution of the Ricci flow.

Proposition 2.6. *Fix a compact manifold M and a background metric h on M. Assume that $\tilde{g}(t)$, $t \in [0,T)$, is a one-parameter family of metrics on M satisfying*

$$\frac{\partial}{\partial t}\tilde{g}(t) = -2\,\mathrm{Ric}_{\tilde{g}(t)} - \mathscr{L}_{\xi_t}\tilde{g}(t),$$

where $\xi_t = \Delta_{\tilde{g}(t),h}\,\mathrm{id}$. Moreover, let φ_t, $t \in [0,T)$, be a one-parameter family of diffeomorphisms satisfying

$$\frac{\partial}{\partial t}\varphi_t(p) = \xi_t\big|_{\varphi_t(p)}$$

for all points $p \in M$ and all $t \in [0,T)$. Then the metrics $g(t) = \varphi_t^*(\tilde{g}(t))$, $t \in [0,T)$, form a solution to the Ricci flow.

Proof. Using the identity $g(t) = \varphi_t^*(\tilde{g}(t))$, we obtain

$$\frac{\partial}{\partial t}g(t) = \varphi_t^*\Big(\frac{\partial}{\partial t}\tilde{g}(t) + \mathscr{L}_{\xi_t}\tilde{g}(t)\Big),$$

hence

$$\frac{\partial}{\partial t}g(t) + 2\,\mathrm{Ric}_{g(t)} = \varphi_t^*\Big(\frac{\partial}{\partial t}\tilde{g}(t) + 2\,\mathrm{Ric}_{\tilde{g}(t)} + \mathscr{L}_{\xi_t}\tilde{g}(t)\Big) = 0.$$

Therefore, the metrics $g(t)$ form a solution to the Ricci flow. □

In the second step, we assume that a solution to the Ricci flow is given and construct a solution to the Ricci-DeTurck flow.

Proposition 2.7. *Fix a compact manifold M and a background metric h on M. Assume that $g(t)$, $t \in [0,T)$, is a solution to the Ricci flow on M. Moreover, we assume that φ_t, $t \in [0,T)$, is a one-parameter family of diffeomorphisms on M evolving under the harmonic map heat flow*

$$\frac{\partial}{\partial t}\varphi_t = \Delta_{g(t),h}\,\varphi_t.$$

For each $t \in [0,T)$, we define a metric $\tilde{g}(t)$ by $\varphi_t^*(\tilde{g}(t)) = g(t)$. Then

$$\frac{\partial}{\partial t}\tilde{g}(t) = -2\,\mathrm{Ric}_{\tilde{g}(t)} - \mathscr{L}_{\xi_t}\tilde{g}(t),$$

where $\xi_t = \Delta_{\tilde{g}(t),h}\,\mathrm{id}$. Furthermore, we have

$$\frac{\partial}{\partial t}\varphi_t(p) = \xi_t\big|_{\varphi_t(p)}$$

for all points $p \in M$ and all $t \in [0,T)$.

Proof. Using Lemma 2.3, we obtain

$$\frac{\partial}{\partial t}\varphi_t(p) = (\Delta_{g(t),h}\,\varphi_t)\big|_p = (\Delta_{\varphi_t^*(\tilde{g}(t)),h}\,\varphi_t)\big|_p = (\Delta_{\tilde{g}(t),h}\,\mathrm{id})\big|_{\varphi_t(p)} = \xi_t\big|_{\varphi_t(p)}$$

for all points $p \in M$ and all $t \in [0,T)$. Since $\varphi_t^*(\tilde{g}(t)) = g(t)$, it follows that

$$\varphi_t^*\Big(\frac{\partial}{\partial t}\tilde{g}(t) + \mathscr{L}_{\xi_t}\tilde{g}(t)\Big) = \frac{\partial}{\partial t}g(t).$$

By assumption, the metrics $g(t)$ form a solution to the Ricci flow. Thus, we conclude that

$$\varphi_t^*\Big(\frac{\partial}{\partial t}\tilde{g}(t) + 2\,\mathrm{Ric}_{\tilde{g}(t)} + \mathscr{L}_{\xi_t}\tilde{g}(t)\Big) = \frac{\partial}{\partial t}g(t) + 2\,\mathrm{Ric}_{g(t)} = 0.$$

This implies

$$\frac{\partial}{\partial t}\tilde{g}(t) = -2\,\mathrm{Ric}_{\tilde{g}(t)} - \mathscr{L}_{\xi_t}\tilde{g}(t),$$

as claimed. □

Theorem 2.8 (R. Hamilton [**44**]). *Let M be a compact manifold and let g_0 be a smooth metric on M. Then there exist a real number $T > 0$ and a smooth one-parameter family of metrics $g(t)$, $t \in [0,T)$, such that $g(t)$ is a solution of the Ricci flow and $g(0) = g_0$. Moreover, the solution $g(t)$ is unique.*

Proof. We first prove the existence statement. By Proposition 2.5, there exists a solution $\tilde{g}(t)$ of the Ricci-DeTurck flow which is defined on some time interval $[0,T)$ and satisfies $\tilde{g}(0) = g_0$. Consequently, we have

$$\frac{\partial}{\partial t}\tilde{g}(t) = -2\,\mathrm{Ric}_{\tilde{g}(t)} - \mathscr{L}_{\xi_t}\tilde{g}(t),$$

where $\xi_t = \Delta_{\tilde{g}(t),h}\,\mathrm{id}$. For each point $p \in M$, we denote by $\varphi_t(p)$ the solution of the ODE

$$\frac{\partial}{\partial t}\varphi_t(p) = \xi_t\big|_{\varphi_t(p)}$$

with initial condition $\varphi_0(p) = p$. By Proposition 2.6, the metrics $g(t) = \varphi_t^*(\tilde{g}(t))$, $t \in [0,T)$, form a solution of the Ricci flow with $g(0) = g_0$.

We now describe the proof of the uniqueness statement. Suppose that $g^1(t)$ and $g^2(t)$ are two solutions to the Ricci flow which are defined on some time interval $[0,T)$ and satisfy $g^1(0) = g^2(0)$. We claim that $g^1(t) = g^2(t)$ for all $t \in [0,T)$. In order to prove this, we argue by contradiction. Suppose that that $g^1(t) \neq g^2(t)$ for some $t \in [0,T)$. We define a real number $\tau \in [0,T)$ by

$$\tau = \inf\{t \in [0,T) : g^1(t) \neq g^2(t)\}.$$

Clearly, $g^1(\tau) = g^2(\tau)$. Let φ_t^1 be the solution of the harmonic map heat flow

$$\frac{\partial}{\partial t}\varphi_t^1 = \Delta_{g^1(t),h}\,\varphi_t^1$$

with initial condition $\varphi_\tau^1 = \mathrm{id}$. Similarly, we denote by φ_t^2 the solution of the harmonic map heat flow

$$\frac{\partial}{\partial t}\varphi_t^2 = \Delta_{g^2(t),h}\,\varphi_t^2$$

with initial condition $\varphi_\tau^2 = \mathrm{id}$. It follows from standard parabolic theory that φ_t^1 and φ_t^2 are defined on some time interval $[\tau, \tau + \varepsilon)$, where ε is a positive real number. Moreover, if we choose $\varepsilon > 0$ small enough, then $\varphi_t^1 : M \to M$ and $\varphi_t^2 : M \to M$ are diffeomorphisms for all $t \in [\tau, \tau + \varepsilon)$.

For each $t \in [\tau, \tau + \varepsilon)$, we define two Riemannian metrics $\tilde{g}^1(t)$ and $\tilde{g}^2(t)$ on M by $(\varphi_t^1)^*(\tilde{g}^1(t)) = g^1(t)$ and $(\varphi_t^2)^*(\tilde{g}^2(t)) = g^2(t)$. It follows from Proposition 2.7 that $\tilde{g}^1(t)$ and $\tilde{g}^2(t)$ are solutions of the Ricci-DeTurck flow. Since $\tilde{g}^1(\tau) = \tilde{g}^2(\tau)$, the uniqueness statement in Proposition 2.5 implies that $\tilde{g}^1(t) = \tilde{g}^2(t)$ for all $t \in [\tau, \tau + \varepsilon)$. For each $t \in [\tau, \tau + \varepsilon)$, we define a vector field ξ_t on M by

$$\xi_t = \Delta_{\tilde{g}^1(t),h}\,\mathrm{id} = \Delta_{\tilde{g}^2(t),h}\,\mathrm{id}.$$

By Proposition 2.7, we have

$$\frac{\partial}{\partial t}\varphi_t^1(p) = \xi_t\big|_{\varphi_t^1(p)}$$

and

$$\frac{\partial}{\partial t}\varphi_t^2(p) = \xi_t\big|_{\varphi_t^2(p)}$$

for all points $p \in M$ and all $t \in [\tau, \tau + \varepsilon)$. Since $\varphi_\tau^1 = \varphi_\tau^2 = \mathrm{id}$, it follows that $\varphi_t^1 = \varphi_t^2$ for all $t \in [\tau, \tau + \varepsilon)$. Putting these facts together, we conclude that

$$g^1(t) = (\varphi_t^1)^*(\tilde{g}^1(t)) = (\varphi_t^2)^*(\tilde{g}^2(t)) = g^2(t)$$

for all $t \in [\tau, \tau + \varepsilon)$. This contradicts the definition of τ. □

2.3. Evolution of the Riemann curvature tensor

In this section, we derive evolution equations for the Levi-Civita connection and the curvature tensor along the Ricci flow. These evolution equations were first derived in [**44**].

Let X, Y be fixed vector fields on M (that is, X, Y are independent of t). We define

$$A(X, Y) = \frac{\partial}{\partial t}(D_X Y).$$

Observe that the difference of two connections is always a tensor; consequently, A is a tensor.

Proposition 2.9. *Let X, Y, Z be fixed vector fields on M. Then*

$$g(A(X, Y), Z) = -(D_X \mathrm{Ric})(Y, Z) - (D_Y \mathrm{Ric})(X, Z) + (D_Z \mathrm{Ric})(X, Y).$$

Proof. By definition of the Levi-Civita connection, we have

$$\begin{aligned} 2\,g(D_X Y, Z) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \end{aligned}$$

(cf. Section 1.1). We now differentiate this identity with respect to t. This yields

$$\begin{aligned} g(A(X,Y),Z) &= \frac{\partial}{\partial t}\big(g(D_X Y, Z)\big) + 2\,\mathrm{Ric}(D_X Y, Z) \\ &= -X(\mathrm{Ric}(Y,Z)) - Y(\mathrm{Ric}(X,Z)) + Z(\mathrm{Ric}(X,Y)) \\ &\quad - \mathrm{Ric}([X,Y],Z) + \mathrm{Ric}([X,Z],Y) + \mathrm{Ric}([Y,Z],X) \\ &\quad + 2\,\mathrm{Ric}(D_X Y, Z). \end{aligned}$$

Since A is a tensor, we conclude that

$$g(A(X,Y),Z) = -(D_X \mathrm{Ric})(Y,Z) - (D_Y \mathrm{Ric})(X,Z) + (D_Z \mathrm{Ric})(X,Y),$$

as claimed. □

We next compute the evolution equation for the curvature tensor:

Proposition 2.10. *Let X, Y, Z, W be fixed vector fields on M. Then*

$$\begin{aligned} &\frac{\partial}{\partial t} R(X,Y,Z,W) \\ &= (D^2_{X,Z}\mathrm{Ric})(Y,W) - (D^2_{X,W}\mathrm{Ric})(Y,Z) \\ &\quad - (D^2_{Y,Z}\mathrm{Ric})(X,W) + (D^2_{Y,W}\mathrm{Ric})(X,Z) \\ &\quad - \sum_{k=1}^n \mathrm{Ric}(Z,e_k)\,R(X,Y,e_k,W) - \sum_{k=1}^n \mathrm{Ric}(W,e_k)\,R(X,Y,Z,e_k). \end{aligned}$$

Proof. We have

$$\begin{aligned} &\frac{\partial}{\partial t}(D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z) \\ &= D_X(A(Y,Z)) - D_Y(A(X,Z)) \\ &\quad + A(X, D_Y Z) - A(Y, D_X Z) - A([X,Y],Z) \\ &= (D_X A)(Y,Z) - (D_Y A)(X,Z). \end{aligned}$$

This implies

$$\begin{aligned} \frac{\partial}{\partial t} R(X,Y,Z,W) &= -g\big((D_X A)(Y,Z),W\big) + g\big((D_Y A)(X,Z),W\big) \\ &\quad - 2\sum_{k=1}^n R(X,Y,Z,e_k)\,\mathrm{Ric}(e_k,W). \end{aligned}$$

Using Proposition 2.9, we obtain

$$\begin{aligned} &g\big((D_X A)(Y,Z),W\big) \\ &= -(D^2_{X,Y}\mathrm{Ric})(Z,W) - (D^2_{X,Z}\mathrm{Ric})(Y,W) + (D^2_{X,W}\mathrm{Ric})(Y,Z). \end{aligned}$$

Interchanging the roles of X and Y yields

$$\begin{aligned} &g\big((D_Y A)(X,Z),W\big) \\ &= -(D^2_{Y,X}\mathrm{Ric})(Z,W) - (D^2_{Y,Z}\mathrm{Ric})(X,W) + (D^2_{Y,W}\mathrm{Ric})(X,Z). \end{aligned}$$

Moreover, we have

$$\begin{aligned} &(D^2_{X,Y}\mathrm{Ric})(Z,W) - (D^2_{Y,X}\mathrm{Ric})(Z,W) \\ &= \sum_{k=1}^n R(X,Y,Z,e_k)\,\mathrm{Ric}(e_k,W) + \sum_{k=1}^n R(X,Y,W,e_k)\,\mathrm{Ric}(Z,e_k). \end{aligned}$$

Putting these facts together, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}R(X,Y,Z,W) \\ &= (D^2_{X,Y}\mathrm{Ric})(Z,W) + (D^2_{X,Z}\mathrm{Ric})(Y,W) - (D^2_{X,W}\mathrm{Ric})(Y,Z) \\ &- (D^2_{Y,X}\mathrm{Ric})(Z,W) - (D^2_{Y,Z}\mathrm{Ric})(X,W) + (D^2_{Y,W}\mathrm{Ric})(X,Z) \\ &- 2\sum_{k=1}^n R(X,Y,Z,e_k)\,\mathrm{Ric}(e_k,W) \\ &= (D^2_{X,Z}\mathrm{Ric})(Y,W) - (D^2_{X,W}\mathrm{Ric})(Y,Z) \\ &- (D^2_{Y,Z}\mathrm{Ric})(X,W) + (D^2_{Y,W}\mathrm{Ric})(X,Z) \\ &+ \sum_{k=1}^n R(X,Y,W,e_k)\,\mathrm{Ric}(Z,e_k) - \sum_{k=1}^n R(X,Y,Z,e_k)\,\mathrm{Ric}(e_k,W), \end{aligned}$$

as claimed. □

We claim that the right-hand side in the evolution equation for the curvature tensor equals the Laplacian of the curvature tensor, up to lower order terms. To show this, we define a tensor $Q(R)$ by

$$\begin{aligned} Q(R)(X,Y,Z,W) &= \sum_{p,q=1}^n R(X,Y,e_p,e_q)\,R(Z,W,e_p,e_q) \\ &+ 2\sum_{p,q=1}^n R(X,e_p,Z,e_q)\,R(Y,e_p,W,e_q) \\ &- 2\sum_{p,q=1}^n R(X,e_p,W,e_q)\,R(Y,e_p,Z,e_q). \end{aligned} \tag{5}$$

Then we have the following identity, which is independent of any evolution equation:

Proposition 2.11. *Let X, Y, Z, W be arbitrary vector fields on M. Then*

$$\begin{aligned}
&(D^2_{X,Z}\mathrm{Ric})(Y,W) - (D^2_{X,W}\mathrm{Ric})(Y,Z) \\
&- (D^2_{Y,Z}\mathrm{Ric})(X,W) + (D^2_{Y,W}\mathrm{Ric})(X,Z) \\
&= (\Delta R)(X,Y,Z,W) + Q(R)(X,Y,Z,W) \\
&- \sum_{k=1}^n \mathrm{Ric}(X,e_k)\, R(e_k,Y,Z,W) - \sum_{k=1}^n \mathrm{Ric}(Y,e_k)\, R(X,e_k,Z,W).
\end{aligned}$$

Proof. Note that

$$\begin{aligned}
&\sum_{k=1}^n (D^2_{X,e_k}R)(e_k,Y,Z,W) - \sum_{k=1}^n (D^2_{e_k,X}R)(e_k,Y,Z,W) \\
&= \sum_{k,l=1}^n R(X,e_k,e_k,e_l)\, R(e_l,Y,Z,W) + \sum_{k,l=1}^n R(X,e_k,Y,e_l)\, R(e_k,e_l,Z,W) \\
&+ \sum_{k,l=1}^n R(X,e_k,Z,e_l)\, R(e_k,Y,e_l,W) + \sum_{k,l=1}^n R(X,e_k,W,e_l)\, R(e_k,Y,Z,e_l).
\end{aligned}$$

Interchanging the roles of X and Y yields

$$\begin{aligned}
&\sum_{k=1}^n (D^2_{Y,e_k}R)(e_k,X,Z,W) - \sum_{k=1}^n (D^2_{e_k,Y}R)(e_k,X,Z,W) \\
&= \sum_{k,l=1}^n R(Y,e_k,e_k,e_l)\, R(e_l,X,Z,W) + \sum_{k,l=1}^n R(Y,e_k,X,e_l)\, R(e_k,e_l,Z,W) \\
&+ \sum_{k,l=1}^n R(Y,e_k,Z,e_l)\, R(e_k,X,e_l,W) + \sum_{k,l=1}^n R(Y,e_k,W,e_l)\, R(e_k,X,Z,e_l).
\end{aligned}$$

We now subtract the second identity from the first. This implies

$$\begin{aligned}
&\sum_{k=1}^n (D^2_{X,e_k} R)(e_k, Y, Z, W) - \sum_{k=1}^n (D^2_{Y,e_k} R)(e_k, X, Z, W) \\
&- \sum_{k=1}^n (D^2_{e_k,X} R)(e_k, Y, Z, W) + \sum_{k=1}^n (D^2_{e_k,Y} R)(e_k, X, Z, W) \\
&= \sum_{k,l=1}^n [R(X, e_k, Y, e_l) - R(Y, e_k, X, e_l)]\, R(e_k, e_l, Z, W) \\
&+ 2 \sum_{k,l=1}^n R(X, e_k, Z, e_l)\, R(Y, e_k, W, e_l) - 2 \sum_{k,l=1}^n R(X, e_k, W, e_l)\, R(Y, e_k, Z, e_l) \\
&- \sum_{l=1}^n \mathrm{Ric}(X, e_l)\, R(e_l, Y, Z, W) + \sum_{l=1}^n \mathrm{Ric}(Y, e_l)\, R(e_l, X, Z, W).
\end{aligned}$$

It follows from the first Bianchi identity that

$$R(X, e_k, Y, e_l) - R(Y, e_k, X, e_l) = R(X, Y, e_k, e_l).$$

Hence, we obtain

$$\begin{aligned}
&\sum_{k=1}^n (D^2_{X,e_k} R)(e_k, Y, Z, W) - \sum_{k=1}^n (D^2_{Y,e_k} R)(e_k, X, Z, W) \\
&- \sum_{k=1}^n (D^2_{e_k,X} R)(e_k, Y, Z, W) + \sum_{k=1}^n (D^2_{e_k,Y} R)(e_k, X, Z, W) \\
&= Q(R)(X, Y, Z, W) \\
&- \sum_{l=1}^n \mathrm{Ric}(X, e_l)\, R(e_l, Y, Z, W) - \sum_{l=1}^n \mathrm{Ric}(Y, e_l)\, R(X, e_l, Z, W).
\end{aligned}$$

Using the second Bianchi identity, we obtain

$$\begin{aligned}
&\sum_{k=1}^n (D^2_{X,e_k} R)(e_k, Y, Z, W) \\
&= \sum_{k=1}^n (D^2_{X,Z} R)(e_k, Y, e_k, W) - \sum_{k=1}^n (D^2_{X,W} R)(e_k, Y, e_k, Z) \\
&= (D^2_{X,Z} \mathrm{Ric})(Y, W) - (D^2_{X,W} \mathrm{Ric})(Y, Z).
\end{aligned}$$

Interchanging the roles of X and Y yields

$$\begin{aligned}
&\sum_{k=1}^n (D^2_{Y,e_k} R)(e_k, X, Z, W) \\
&= \sum_{k=1}^n (D^2_{Y,Z} R)(e_k, X, e_k, W) - \sum_{k=1}^n (D^2_{Y,W} R)(e_k, X, e_k, Z) \\
&= (D^2_{Y,Z} \mathrm{Ric})(X, W) - (D^2_{Y,W} \mathrm{Ric})(X, Z).
\end{aligned}$$

Moreover, the second Bianchi identity implies that

$$\begin{aligned}
&\sum_{k=1}^n (D^2_{e_k,X} R)(e_k, Y, Z, W) - \sum_{k=1}^n (D^2_{e_k,Y} R)(e_k, X, Z, W) \\
&= \sum_{k=1}^n (D^2_{e_k,e_k} R)(X, Y, Z, W) = (\Delta R)(X, Y, Z, W).
\end{aligned}$$

Putting these facts together, the assertion follows. □

As a consequence, we obtain the following reaction-diffusion equation for the curvature tensor:

Corollary 2.12. *Let X, Y, Z, W be fixed vector fields on M. Then*

$$\begin{aligned}
&\frac{\partial}{\partial t} R(X, Y, Z, W) \\
&= (\Delta R)(X, Y, Z, W) + Q(R)(X, Y, Z, W) \\
&- \sum_{k=1}^n \mathrm{Ric}(X, e_k)\, R(e_k, Y, Z, W) - \sum_{k=1}^n \mathrm{Ric}(Y, e_k)\, R(X, e_k, Z, W) \\
&- \sum_{k=1}^n \mathrm{Ric}(Z, e_k)\, R(X, Y, e_k, W) - \sum_{k=1}^n \mathrm{Ric}(W, e_k)\, R(X, Y, Z, e_k).
\end{aligned}$$

Let E be the pull-back of the tangent bundle TM under the projection $M \times (0, T) \to M$, $(p, t) \mapsto p$. In other words, the fiber of E over a point $(p, t) \in M \times (0, T)$ is given by $E_{(p,t)} = T_p M$.

There is a natural connection D on E, which extends the Levi-Civita connection on TM. In order to define this connection, we need to specify the covariant time derivative $D_{\frac{\partial}{\partial t}}$. Given any section X of the vector bundle E, we define

$$D_{\frac{\partial}{\partial t}} X = \frac{\partial}{\partial t} X - \sum_{k=1}^n \mathrm{Ric}(X, e_k)\, e_k, \tag{6}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame with respect to the metric $g(t)$.

Proposition 2.13. *The connection D is compatible with the natural bundle metric on E. More precisely, we have*

$$(D_{\frac{\partial}{\partial t}} g)(X, Y) = 0$$

for all vector fields X, Y.

Proof. Without loss of generality, we may assume that X, Y are constant in time. In this case, we have

$$D_{\frac{\partial}{\partial t}} X = -\sum_{k=1}^{n} \mathrm{Ric}(X, e_k)\, e_k, \qquad D_{\frac{\partial}{\partial t}} Y = -\sum_{k=1}^{n} \mathrm{Ric}(Y, e_k)\, e_k.$$

This implies

$$\begin{aligned} (D_{\frac{\partial}{\partial t}} g)(X, Y) &= \frac{\partial}{\partial t}(g(X, Y)) - g\big(D_{\frac{\partial}{\partial t}} X, Y\big) - g\big(X, D_{\frac{\partial}{\partial t}} Y\big) \\ &= \frac{\partial}{\partial t}(g(X, Y)) + 2\,\mathrm{Ric}(X, Y) \\ &= 0, \end{aligned}$$

as claimed. □

The evolution equation for the Riemann curvature tensor simplifies if we replace the ordinary time derivative $\frac{\partial}{\partial t}$ by the covariant time derivative $D_{\frac{\partial}{\partial t}}$. This is known as Uhlenbeck's trick (cf. [**45**]).

Proposition 2.14. *We have*

$$(D_{\frac{\partial}{\partial t}} R)(X, Y, Z, W) = (\Delta R)(X, Y, Z, W) + Q(R)(X, Y, Z, W)$$

for all vector fields X, Y, Z, W.

Proof. Without loss of generality, we may assume that the vector fields X, Y, Z, W are constant in time. In this case, we have

$$\begin{aligned} D_{\frac{\partial}{\partial t}} X &= -\sum_{k=1}^{n} \mathrm{Ric}(X, e_k)\, e_k, \qquad & D_{\frac{\partial}{\partial t}} Y &= -\sum_{k=1}^{n} \mathrm{Ric}(Y, e_k)\, e_k, \\ D_{\frac{\partial}{\partial t}} Z &= -\sum_{k=1}^{n} \mathrm{Ric}(Z, e_k)\, e_k, \qquad & D_{\frac{\partial}{\partial t}} W &= -\sum_{k=1}^{n} \mathrm{Ric}(W, e_k)\, e_k. \end{aligned}$$

This implies

$$\begin{aligned}
&(D_{\frac{\partial}{\partial t}} R)(X,Y,Z,W) - \frac{\partial}{\partial t} R(X,Y,Z,W) \\
&= -R\big(D_{\frac{\partial}{\partial t}} X,Y,Z,W\big) - R\big(X,D_{\frac{\partial}{\partial t}} Y,Z,W\big) \\
&- R\big(X,Y,D_{\frac{\partial}{\partial t}} Z,W\big) - R\big(X,Y,Z,D_{\frac{\partial}{\partial t}} W\big) \\
&= \sum_{k=1}^n \mathrm{Ric}(X,e_k)\, R(e_k,Y,Z,W) + \sum_{k=1}^n \mathrm{Ric}(Y,e_k)\, R(X,e_k,Z,W) \\
&+ \sum_{k=1}^n \mathrm{Ric}(Z,e_k)\, R(X,Y,e_k,W) + \sum_{k=1}^n \mathrm{Ric}(W,e_k)\, R(X,Y,Z,e_k).
\end{aligned}$$

Hence, the assertion follows from Corollary 2.12. □

2.4. Evolution of the Ricci and scalar curvature

We next compute the evolution equations for the Ricci and scalar curvature. As above, we assume that M is a compact manifold, and $g(t)$, $t \in [0,T)$, is a solution to the Ricci flow on M.

Proposition 2.15. *The Ricci tensor of $g(t)$ satisfies the evolution equation*

$$(D_{\frac{\partial}{\partial t}} \mathrm{Ric})(X,Y) = (\Delta \mathrm{Ric})(X,Y) + 2 \sum_{p,q=1}^n R(X,e_p,Y,e_q)\, \mathrm{Ric}(e_p,q_q).$$

Proof. Recall that $D_{\frac{\partial}{\partial t}} g = 0$. Hence, it follows from Proposition 2.14 that

$$(D_{\frac{\partial}{\partial t}} \mathrm{Ric})(X,Y) = (\Delta \mathrm{Ric})(X,Y) + \sum_{k=1}^n Q(R)(X,e_k,Y,e_k).$$

Moreover, we have

$$\begin{aligned}
\sum_{k=1}^n Q(R)(X,e_k,Y,e_k) &= \sum_{k,p,q=1}^n R(X,e_k,e_p,e_q)\, R(Y,e_k,e_p,e_q) \\
&+ 2 \sum_{k,p,q=1}^n R(X,e_p,Y,e_q)\, R(e_k,e_p,e_k,e_q) \qquad (7) \\
&- 2 \sum_{k,p,q=1}^n R(X,e_p,e_k,e_q)\, R(Y,e_q,e_k,e_p)
\end{aligned}$$

by definition of $Q(R)$. Using the first Bianchi identity, we obtain

$$\begin{aligned}
&2 \sum_{k,p,q=1}^{n} R(X, e_p, e_k, e_q)\, R(Y, e_q, e_k, e_p) \\
&= \sum_{k,p,q=1}^{n} R(X, e_p, e_k, e_q)\, [R(Y, e_q, e_k, e_p) - R(Y, e_k, e_q, e_p)] \\
&= \sum_{k,p,q=1}^{n} R(X, e_p, e_k, e_q)\, R(Y, e_p, e_k, e_q).
\end{aligned}$$

Hence, the identity (7) can be rewritten as

$$\begin{aligned}
\sum_{k=1}^{n} Q(R)(X, e_k, Y, e_k) &= 2 \sum_{k,p,q=1}^{n} R(X, e_p, Y, e_q)\, R(e_k, e_p, e_k, e_q) \\
&= 2 \sum_{p,q=1}^{n} R(X, e_p, Y, e_q)\, \mathrm{Ric}(e_p, e_q).
\end{aligned}$$

Putting these facts together, the assertion follows. □

Corollary 2.16. *The scalar curvature of $g(t)$ satisfies*

$$\frac{\partial}{\partial t}\mathrm{scal} = \Delta \mathrm{scal} + 2\, |\mathrm{Ric}|^2.$$

Proof. This follows from Proposition 2.15 by taking the trace over X and Y. □

Corollary 2.17. *The trace-free Ricci tensor of $g(t)$ satisfies the evolution equation*

$$\begin{aligned}
(D_{\frac{\partial}{\partial t}} \overset{\circ}{\mathrm{Ric}})(X, Y) &= (\Delta \overset{\circ}{\mathrm{Ric}})(X, Y) + 2 \sum_{p,q=1}^{n} R(X, e_p, Y, e_q)\, \overset{\circ}{\mathrm{Ric}}(e_p, q_q) \\
&\quad + \frac{2}{n}\, \mathrm{scal}\, \overset{\circ}{\mathrm{Ric}}(X, Y) - \frac{2}{n}\, |\overset{\circ}{\mathrm{Ric}}|^2\, g(X, Y).
\end{aligned}$$

Proof. This is an immediate consequence of Proposition 2.15 and Corollary 2.16. □

In the remainder of this section, we discuss some implications of Corollary 2.16. Note that the reaction term in the evolution equation for the scalar curvature is always nonnegative. Consequently, the minimum of the scalar curvature of $g(t)$ is monotone increasing. In particular, we obtain:

Proposition 2.18. *Suppose that $(M, g(0))$ has nonnegative scalar curvature. Then $(M, g(t))$ has nonnegative scalar curvature for all $t \in [0, T)$.*

Moreover, if $\mathrm{scal}_{g(t_0)}(p_0) = 0$ *for some point* $p_0 \in M$ *and some time* $t_0 \in (0, T)$*, then the metrics* $g(t)$ *are Ricci flat for all* $t \in [0, T)$*.*

Proof. The first statement follows immediately from the maximum principle. To prove the second statement, suppose that $\mathrm{scal}_{g(t_0)}(p_0) = 0$ for some point $p_0 \in M$ and some time $t_0 \in (0, T)$. The strict maximum principle implies that $\mathrm{scal}_{g(t)}(p) = 0$ for all points $p \in M$ and all times $t \times [0, t_0)$. Substituting this into the evolution equation for the scalar curvature, we conclude that the metric $g(t)$ is Ricci flat for all $t \in [0, t_0)$. Hence, the uniqueness statement in Theorem 2.8 implies that $g(t) = g(0)$ for all $t \in [0, T)$. □

We can prove a stronger result if we take advantage of the reaction term:

Proposition 2.19. *Suppose that* $\inf_M \mathrm{scal}_{g(0)} = \alpha > 0$*. Then* $T \leq \frac{n}{2\alpha}$ *and* $\inf_M \mathrm{scal}_{g(t)} \geq \frac{n\alpha}{n-2\alpha t}$ *for all* $t \in [0, T)$*.*

Proof. Let $\tau = \min\{T, \frac{n}{2\alpha}\}$. We define a function $h : M \times [0, \tau) \to \mathbb{R}$ by

$$h = \mathrm{scal} - \frac{n\alpha}{n - 2\alpha t}.$$

Using Corollary 2.16, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} h &= \Delta h + 2\,|\mathrm{Ric}|^2 - \frac{2}{n}\Big(\frac{n\alpha}{n - 2\alpha t}\Big)^2 \\
&\geq \Delta h + \frac{2}{n}\,\mathrm{scal}^2 - \frac{2}{n}\Big(\frac{n\alpha}{n - 2\alpha t}\Big)^2 \\
&= \Delta h + \frac{2}{n}\Big(\mathrm{scal} + \frac{n\alpha}{n - 2\alpha t}\Big)\,h
\end{aligned}$$

on $M \times [0, \tau)$. By definition of α, we have $h(p, 0) \geq 0$ for all $p \in M$. Hence, the maximum principle implies that $h(p, t) \geq 0$ for all $p \in M$ and all $t \in [0, \tau)$. Therefore, we have $\inf_M \mathrm{scal}_{g(t)} \geq \frac{n\alpha}{n-2\alpha t}$ for all $t \in [0, \tau)$. From this, we deduce that $T \leq \frac{n}{2\alpha}$. □