

The Concept of Dimension in Homotopy Theory

We are fundamentally interested in the homotopy theory of CW complexes. CW complexes are not simply topological spaces: they have extra structure conferred upon them by virtue of their step-by-step construction. They are filtered by their skeleta, and it makes sense to talk about the dimension of a CW complex. In this section, we explore the uses of dimension in homotopy theory.

We have seen in Problem 4.92 that in order to decide whether or not a map $f : X \rightarrow Y$ is a homotopy equivalence, it is sufficient to show that for every space K , the induced map $f_* : [K, X] \rightarrow [K, Y]$ is bijective. But we may not be able to check this condition for all spaces K ; perhaps we can only check it for CW complexes, or for CW complexes of dimension at most n . This leads to the concept of *n-equivalence*.

There is a related notion which estimates how nearly contractible a space is. A space X is contractible if and only if $[K, X] = *$ for all K . If we only know that $[K, X] = *$ for all CW complexes of dimension at most n , then we say that X is *n-connected*.

In this chapter we explore the relation between *n-equivalence* and *n-connectivity*. We establish five different reformulations of the concept of *n-equivalence* and use them to prove the celebrated J. H. C. Whitehead theorem: a map which induces isomorphisms $f_* : \pi_k(X) \rightarrow \pi_k(Y)$ for all k induces bijections $f_* : [K, X] \rightarrow [K, Y]$ for all CW complexes K .

11.1. Induction Principles for CW Complexes

The step-by-step construction of CW complexes makes it possible to study them by induction on their skeleta. Straightforward induction gives results about all *finite-dimensional* CW complexes but does not provide information about the infinite-dimensional ones. This section contains some technical results that facilitate the jump from finite-dimensional to infinite-dimensional.

11.1.1. Attaching One More Cell. If X is a CW complex and $K \subseteq X$ is a subcomplex, then it may happen that $K_0 = X_0$, that is, that the 0-skeleta coincide. If they are the same, then it may be that $X_1 = K_1$, and so on. If K is a *proper* subcomplex, then there must be some n for which $K_n \neq X_n$. If n is the smallest dimension for which $X_n \neq K_n$, then we can build a new subcomplex $L \subseteq X$ by attaching to K one of the n -cells of X that it does not contain. This proves the following.

Lemma 11.1. *If X is a CW complex and $K \subseteq X$ is a proper subcomplex, then there is another subcomplex L such that*

- (a) $K \subseteq L \subseteq X$ and
- (b) $L = K \cup_{\lambda} D^n$ for some map $\lambda : S^{n-1} \rightarrow K$.

The proofs of many statements about CW complexes make essential use of Lemma 11.1, in the following way. We wish to show that some property is true of X , so we let $K \subseteq X$ be a subcomplex which is maximal with that property,¹ and we hope to show that $K = X$. So we assume that K is a proper subcomplex of X and find a slightly larger subcomplex L using Lemma 11.1. Using the close connection between K and L , we then prove that the property holds for L , contradicting the maximality of K , and thereby proving that X has the desired property. We'll refer to this kind of argument as a proof by **CW induction**.

11.1.2. Composing Infinitely Many Homotopies. Our second induction principle addresses the following question: suppose that for each n you have a homotopy $H_n : f_n \simeq f_{n+1}$. Can we piece these homotopies together to obtain a homotopy from f_1 to some other map, g ?

Proposition 11.2. *Let $f : X \rightarrow Y$, where X is a CW complex. Suppose there is an infinite sequence of maps $f_n : X \rightarrow Y$ (for $n \geq 0$) and homotopies $H_n : f_n \simeq f_{n+1}$ which satisfy the following conditions:*

- $f = f_1$,
- there is a function $z : \mathbb{N} \rightarrow \mathbb{N}$ such that for each $m \geq z(n)$, $H_m|_{X_n \times I}$ is the constant homotopy at a certain function g_n .

¹The main tool for proving the existence of maximal gadgets is Zorn's lemma.

Then the function $g : X \rightarrow Y$ defined by $g|_{X_n} = g_n$ is continuous and homotopic to f .

Problem 11.3. Reparametrize $H_0 : f \simeq f_0$ from $t = 0$ to $t = \frac{1}{2}$; call the reparametrized homotopy J_0 . More generally, let J_n be the reparametrization of H_n over the interval $[1 - \frac{1}{n+1}, 1 - \frac{1}{n+2}]$.

- Show that the map g defined in Proposition 11.2 is continuous.
- Show that the homotopies J_n for $n \geq 0$ glue together to give a continuous function $\tilde{J} : X \times [0, 1) \rightarrow Y$.
- Show that \tilde{J} can be extended to a homotopy $J : f \simeq g$.

As an application of infinite concatenation of homotopies we establish a criterion for the contractibility of a CW complex.

Proposition 11.4. *Let X be a connected pointed CW complex. If there is a pointed homotopy $H : \text{id}_X \simeq f$ where $f(X_n) \subseteq X_{n-1}$ for $n \geq 1$, then $X \simeq *$.*

Problem 11.5. Let X be a connected pointed CW complex.

- Show that id_X is homotopic in \mathcal{T}_* to a cellular map $g : X \rightarrow X$ such that $g(X_0) = *$.
- Show that under the hypotheses of Proposition 11.4, $\text{id}_X \simeq \phi$ in \mathcal{T}_* , where $\phi(X_n) \subseteq X_{n-1}$ for all $n \geq 1$ and $\phi(X_0) = *$.
- Show that $\phi \simeq *$ and derive Proposition 11.4.

11.2. n -Equivalences and Connectivity of Spaces

In this section, we introduce a system of approximations to the notion of homotopy equivalence, called n -equivalences. There is an analogous collection of measures of the triviality of spaces, called n -connectivity.

11.2.1. n -Equivalences. An unpointed map $f : X \rightarrow Y$ may be made pointed by choosing a basepoint $x \in X$ and considering it as a map $(X, x) \rightarrow (Y, f(x))$, which we denote by f_x . Thus each unpointed map f gives rise to a huge collection $\{f_x \mid x \in X\}$ of pointed maps. We say that an unpointed map f between *nonempty spaces* is an **n -equivalence** if for every $x \in X$ the induced map

$$(f_x)_* : [K, X] \longrightarrow [K, Y]$$

is an isomorphism for every pointed CW complex K with $\dim(K) < n$ and is a surjection if $\dim(K) \leq n$. A *pointed* map f is an n -equivalence if the unpointed map $f_- : X_- \rightarrow Y_-$ (that results from forgetting the basepoints) is an n -equivalence of unpointed spaces.

An ∞ -**equivalence** is a map which is an n -equivalence for each n . Such a map is also called a **weak homotopy equivalence** or simply a **weak equivalence**.

We would like to make weak equivalence into an equivalence relation between spaces, and it is tempting to say that X and Y are weakly equivalent if there is a weak equivalence $X \rightarrow Y$. But the existence of a weak equivalence $X \rightarrow Y$ does not imply the existence of a weak equivalence $Y \rightarrow X$, so this relation would not be symmetric. Instead, we let \sim be the smallest equivalence relation on spaces such that if there is a weak equivalence $X \rightarrow Y$, then $X \sim Y$.

Exercise 11.6.

- (a) Suppose we were to allow the empty space as a domain in our definition of n -equivalence. For each n , determine all spaces Y for which the unique map $f : \emptyset \rightarrow Y$ would be an n -equivalence.
- (b) Show that if $f : X \rightarrow Y$ is an n -equivalence, then the induced map $f_* : \pi_k(X, x) \rightarrow \pi_k(Y, f(x))$ is an isomorphism for $k < n$ and a surjection for $k = n$, no matter what basepoint $x \in X$ is chosen.
- (c) Criticize the following argument:
Because the functor $\langle K, X \rangle$ is naturally equivalent to $[K_+, X]$, a map $f : X \rightarrow Y$ is an n -equivalence if and only if the induced map $f_ : \langle K, X \rangle \rightarrow \langle K, Y \rangle$ is a bijection for $\dim(K) < n$ and a surjection for $\dim(K) = n$.*
- (d) Show that $X \sim Y$ if and only if there is a finite sequence of weak equivalences $X \rightarrow X_1 \leftarrow X_2 \rightarrow X_3 \leftarrow \cdots \rightarrow X_{n-1} \leftarrow X_n \rightarrow Y$.

This definition comes with a daunting list of conditions to verify: we need to check the infinitely many maps f_x and their induced maps on infinitely many functors $[K, ?]$. But this list can be drastically reduced and ultimately (for many spaces) made finite.

Problem 11.7. Let $f : X \rightarrow Y$.

- (a) Show that if X and Y are path-connected, then f is an n -equivalence if and only if, for your favorite basepoint $x \in X$, the induced maps $(f_x)_* : [K, X] \rightarrow [K, Y]$ in \mathcal{T}_* are isomorphisms for every pointed CW complex K with $\dim(K) < n$ and surjections for $\dim(K) \leq n$.

HINT. Use Problem 8.44.

- (b) Show that in general, f is an n -equivalence if and only if $\pi_0(f)$ is a bijection and f satisfies the condition in part (a) for your favorite basepoint in each path component of X .

Problem 11.8. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. Suppose two of the three maps f, g and $g \circ f$ are n -equivalences (with $n \leq \infty$). What can you say about the third map?

Exercise 11.9. Criticize the following argument:

If $f : X \rightarrow Y$ is a homotopy equivalence in \mathcal{T}_\circ , then by choosing a basepoint $x \in X$, we get a map $f_x : (X, x) \rightarrow (Y, f(x))$ which is a homotopy equivalence in \mathcal{T}_ . It follows that f_x induces isomorphisms on all functors $[K, ?]$ so f is a weak homotopy equivalence.*

Problem 11.10. Show that if f and g are pointwise equivalent in \mathbf{HT}_\circ , then f is an n -equivalence if and only if g is an n -equivalence.

11.2.2. Connectivity of Spaces. A space $X \in \mathcal{T}_\circ$ is **n -connected** if for every choice of basepoint $x \in X$, $[K, (X, x)] = *$ for all pointed CW complexes K with $\dim(K) \leq n$; we say that X is **∞ -connected**, or **weakly contractible**, if it is n -connected for all n . We define connectivity of pointed spaces by forgetting that they are pointed.

Exercise 11.11. Show that 0-connected is synonymous with path-connected.

A 1-connected space is also called **simply-connected**.

Here are some useful reformulations of the concepts of n -connectivity.

Problem 11.12. Show that the following are equivalent:

- (1) X is n -connected,
- (2) $X \rightarrow *$ is an $(n + 1)$ -equivalence,
- (3) $* \rightarrow X$ is an n -equivalence,
- (4) $\pi_k(X, x) = *$ for all $k \leq n$ and all $x \in X$.

HINT. For (4) \Rightarrow (1), show by induction on $m \leq n$ that a map $K \rightarrow X$ must factor through K/K_m .

Problem 11.13. Show that if $X \simeq Y$ in \mathcal{T}_\circ , then X is n -connected if and only if Y is n -connected.

The **connectivity** of a space X is the greatest n for which X is n -connected; we'll use the notation $\text{conn}(X) = n$ to mean that X is n -connected but not $(n + 1)$ -connected. Connectivity is a homotopy invariant of spaces that measures how close X is to being the trivial space $*$. Sometimes n -equivalences are referred to as *n -connected maps*.

Problem 11.14.

- (a) Suppose two of the three spaces in the fibration sequence $F \rightarrow E \rightarrow B$ are n -connected. What can you say about the third space?

- (b) Suppose X is n -connected and Y is m -connected. What is the connectivity of $X \times Y$?
- (c) How do the connectivities of X and ΩX compare?

Exercise 11.15. Criticize the following argument:

Since $[X, Y]$ is just a homotopy group of the space of maps $\text{map}_(X, Y)$, if X is n -dimensional and Y is n -connected, then $\text{map}_*(X, Y)$ is weakly contractible.*

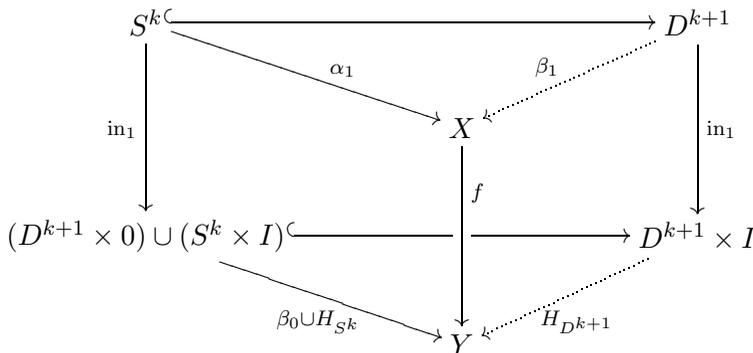
11.3. Reformulations of n -Equivalences

In this section, we establish five alternate characterizations of n -equivalences. The first of these reduces the burden from studying $[K, X] \rightarrow [K, Y]$ for all K with $\dim(K) \leq n$ to just checking this for $K = S^k$ for $k \leq n$. The importance of the other new statements is analogous to the importance of the Fundamental Lifting Property: they guarantee the existence of maps satisfying various properties.

Interestingly, the definition of n -equivalence, which is fundamentally a property of unpointed maps, makes use of homotopy functors of pointed spaces. Can the concept be defined entirely within \mathcal{T}_o ? As you read over Theorem 11.16, think about which parts, if any, do not involve pointed homotopies.

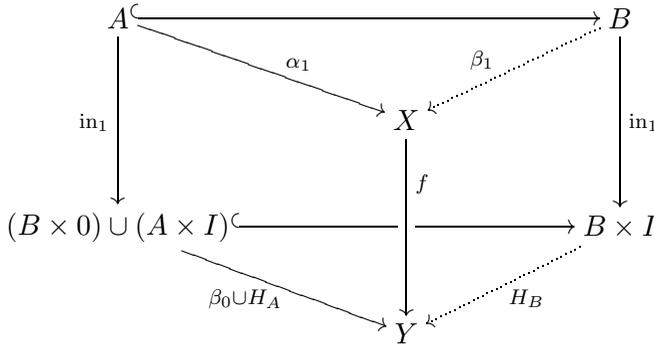
Theorem 11.16. *Let $f : X \rightarrow Y$ be a map in \mathcal{T}_o such that $\pi_0(f)$ is onto, and let $n \leq \infty$. The following six statements, in three groups of two, are equivalent:*

- (1) (a) For every $x \in X$, the map $(f_x)_* : \pi_k(X, x) \rightarrow \pi_k(Y, f(x))$ is an isomorphism for $k < n$ and a surjection for $k = n$.
- (b) f is an n -equivalence.
- (2) (a) In any strictly commutative diagram of the form



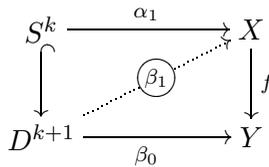
with $k < n$, the dotted arrows can be filled in to make the whole diagram strictly commutative.

- (b) If A is any space and B is obtained from A by attaching cells of dimension at most n ,² then in the strictly commutative diagram



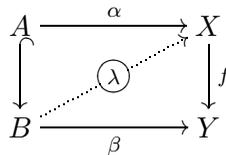
the dotted arrows can be filled in so that the entire diagram commutes.

- (3) (a) In any strictly commutative diagram of the form



with $k < n$, the dotted arrow can be filled in so that the upper triangle commutes on the nose and the lower triangle commutes up to a homotopy which is constant on S^k .

- (b) If A is any space and B is obtained from A by attaching cells of dimension at most n , then in the strictly commutative diagram



the dotted arrow can be filled in so that the upper triangle commutes on the nose, and the lower triangle commutes up to a homotopy which is constant on A .

One very useful consequence of the theorem is that the connectivity of a map can be determined by computing the connectivity of its fiber.

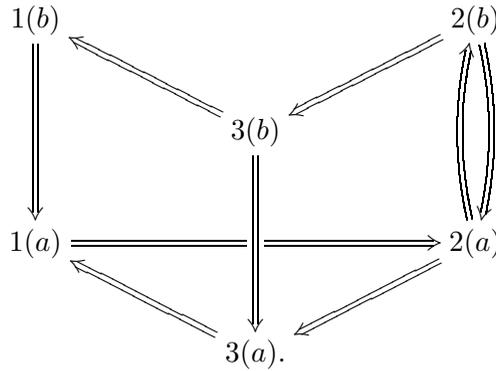
Corollary 11.17. *If $f : X \rightarrow Y$ is a map of path-connected pointed spaces, then f is an n -equivalence if and only if the homotopy fiber F is $(n - 1)$ -connected.*

²Thus B is a relative CW complex with dimension at most n .

Problem 11.18. Use Theorem 11.16 to prove Corollary 11.17.

Theorem 11.16(2)(b) is known as the **Homotopy Extension Lifting Property**, often abbreviated HELP.

Overview of the Proof of Theorem 11.16. The structure of this proof is a bit complicated; the plan is illustrated by the diagram



Since in each of the three parts, the ‘(a) statement’ is a special case of the ‘(b) statement’, the downward implications are trivially valid. We first prove that parts (1)(a), (2)(a) and (3)(a) imply one another cyclically. Then we show that (2)(a) and (2)(b) are equivalent. Finally we show that (2)(b) implies (3)(b) and (3)(b) implies (1)(b); since (1)(b) obviously implies (1)(a), this completes the proof.

11.3.1. Equivalence of the (a) Parts. To prove that Theorem 11.16(1)(a) implies Theorem 11.16(2)(a) we have to be able to recognize the inclusion of a sphere into a disk when we see it.

Problem 11.19.

(a) Show there is a homeomorphism $u : D^{k+1} \rightarrow (D^{k+1} \times 0) \cup (S^k \times I)$ making the diagram

$$\begin{array}{ccc}
 S^k & \xrightarrow{\text{id}} & S^k \\
 i \downarrow & & \downarrow \text{in}_1 \\
 D^{k+1} & \xrightarrow{u} & (D^{k+1} \times 0) \cup (S^k \times I)
 \end{array}$$

strictly commutative.

- (b) Show that there is a homeomorphism $v : D^{k+1} \rightarrow D^k \times I$ making the diagram

$$\begin{array}{ccc}
 S^k & \longrightarrow & (D^k \times 0) \cup (S^{k-1} \times I) \cup (D^k \times 1) \\
 \downarrow & & \downarrow \\
 D^{k+1} & \xrightarrow{v} & D^k \times I
 \end{array}$$

strictly commutative.

Now we are prepared to prove that part 1(a) implies part 2(a).

Problem 11.20. Take Theorem 11.16(1)(a) as known.

- (a) Show that α_1 extends to a map $\xi : D^{k+1} \rightarrow X$.
- (b) Explain how the maps β_0, H_{S^k} and ξ define a map $Q : S^{k+1} \rightarrow Y$.
- (c) Show that there is a map $R : S^{k+1} \rightarrow X$ such that $f \circ R \simeq Q$.
- (d) Let $\beta_1 : D^{k+1} \rightarrow X$ be the composition

$$\begin{array}{ccc}
 D^{k+1} & \xrightarrow{\beta_1} & Y \\
 \searrow c & & \nearrow (\xi, -R) \\
 & D^{k+1} \vee S^{k+1}, &
 \end{array}$$

where c is the map that collapses $\frac{1}{2}S^k$. Show that the map $S^{k+1} \rightarrow Y$ defined by the maps β_0, H_{S^k} and β_1 extends to a map $D^{k+2} \rightarrow Y$.

- (e) Why does this prove Theorem 11.16(1)(a) implies Theorem 11.16(2)(a)?

The other two implications are comparatively straightforward.

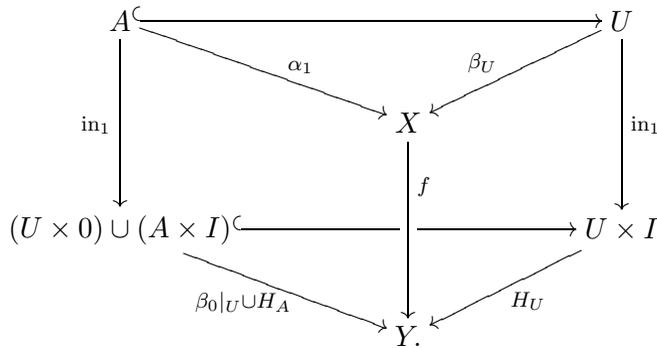
Problem 11.21.

- (a) Prove that Theorem 11.16(2)(a) implies Theorem 11.16(3)(a).
- (b) Prove that Theorem 11.16(3)(a) implies Theorem 11.16(1)(a).

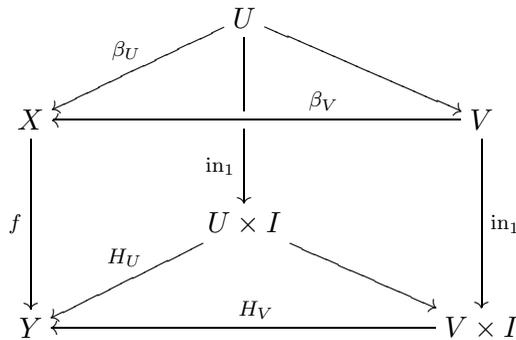
11.3.2. Equivalence of Parts (2)(a) and (2)(b). Since part (2)(b) obviously implies part (2)(a), we just need to show the reverse implication.

The proof makes use of our first induction principle for CW complexes. For the Zorn lemma portion of the proof, let \mathcal{P} be the set of all triples (U, β_U, H_U) where U is a (relative) subcomplex of L such that $A \subseteq U \subseteq B$

and the maps β_U and H_U fit into the strictly commutative diagram



Define a partial order on \mathcal{P} by setting $(U, \beta_U, H_U) \leq (V, \beta_V, H_V)$ if $U \subseteq V$ and the diagram

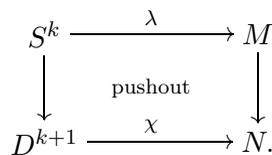


is strictly commutative.

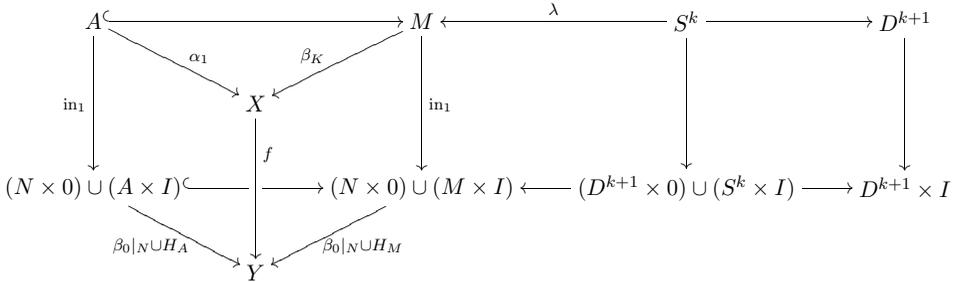
Problem 11.22.

- (a) Show that $\mathcal{P} \neq \emptyset$.
- (b) Show that every chain $\dots \leq (U, \beta_U, H_U) \leq (V, \beta_V, H_V) \leq \dots$ of elements of \mathcal{P} has an upper bound (Z, β_Z, H_Z) .
- (c) Use Zorn's lemma to show there is a complex maximal with the property that Theorem 11.16(2)(b) holds for the inclusion $j_M : A \hookrightarrow M$.

If $M = B$, we are done, so let us assume that M is a proper subcomplex. Then we can use Lemma 11.1 to find another subcomplex $N = M \cup_\lambda D^{k+1}$ (with $k < n$), which is the pushout in

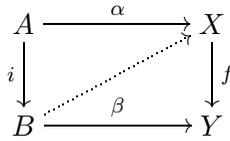


Problem 11.23. Use the diagram



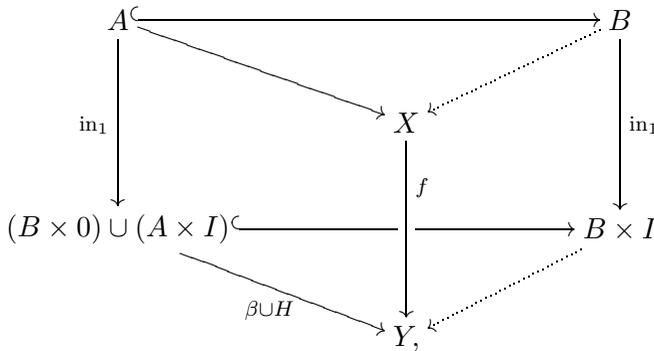
to show that there are maps β_N and H_N so that $(N, \beta_N, H_N) > (M, \beta_M, H_M)$. Deduce that Theorem 11.16(2)(a) implies Theorem 11.16(2)(b).

11.3.3. Proof that Part (2)(b) Implies Part (3)(b). We have to show that the dotted arrow in the square



can be filled in so that the upper triangle commutes and the lower triangle commutes up to homotopy under A .

Problem 11.24. Prove Theorem 11.16(3)(b) by applying part (2)(b) to the diagram



where $H = \boxed{f \circ \alpha}$ is the constant homotopy at $f \circ \alpha$.

11.3.4. Proof that Part (3)(b) Implies Part (1)(b). Now we apply 3(b) to show that f is an n -equivalence.

Problem 11.25.

- (a) Take $A = *$ and $B = K$ to prove surjectivity.
- (b) Take $A = (K \times 0) \cup (* \times I) \cup (K \times 1)$ and $B = K \times I$ to prove injectivity.

11.4. The J. H. C. Whitehead Theorem

One of the most important consequences of Theorem 12.34 is that a weak homotopy equivalence $f : X \rightarrow Y$ induces isomorphisms $f_* : [K, X] \rightarrow [K, Y]$ for *all* CW complexes, not just the finite ones. This is known as the **J. H. C. Whitehead theorem**.

Theorem 11.26 (J. H. C. Whitehead). *If $f : X \rightarrow Y$ is an ∞ -equivalence in \mathcal{T}_\circ , then for any $x \in X$, the induced map*

$$f_* : [K, (X, x)] \longrightarrow [K, (Y, f(x))]$$

is a bijection for all CW complexes K .

Problem 11.27. Prove the Whitehead theorem.

HINT. Look to Section 11.3.4 for inspiration.

This has an extremely important corollary: to determine whether a map of connected pointed CW complexes is a homotopy equivalence, it suffices to check that it induces isomorphisms on homotopy groups.

Corollary 11.28. *If $f : X \rightarrow Y$ is a map of connected pointed CW complexes, then the following are equivalent:*

- (1) $f_* : \pi_*(X) \rightarrow \pi_*(Y)$ is an isomorphism,
- (2) f is a homotopy equivalence in \mathcal{T}_* .

Problem 11.29. Prove Corollary 11.28.

Exercise 11.30.

- (a) Show that a weakly contractible CW complex is contractible.
- (b) Are there any spaces that are weakly contractible but not contractible?

11.5. Additional Problems

Problem 11.31. Let X be path-connected, and suppose that every map $S^k \rightarrow X$ with $k \leq n$ is freely homotopic to a constant. Show that X is n -connected.

Problem 11.32. Let $S^n \rightarrow S^m \rightarrow B$ be a fibration sequence with $n < m$. Prove that $\Omega B \simeq S^n \times \Omega S^m$.

Problem 11.33. Suppose $f : X \rightarrow Y$ is an n -equivalence in \mathcal{T}_* . What can you say about Ωf ? What can you say about $f_* : \text{map}_*(A, X) \rightarrow \text{map}_*(A, Y)$ if A is a finite-dimensional CW complex?

Problem 11.34. Suppose f and g are pointwise n -equivalent, in the sense that there is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ f \downarrow & & \downarrow g \\ B & \xrightarrow{\beta} & Y \end{array}$$

in \mathcal{T}_o in which α and β are n -equivalences. If f is an m -equivalence, then what can you say about g ?

Problem 11.35. Let $f : X \rightarrow Y$ in \mathcal{T}_* such that $f_* : \pi_k(X) \rightarrow \pi_k(Y)$ is an isomorphism for $k \leq n$ and Y is a connected CW complex.

- Show that if $\dim(Y) \leq n$, then Y is a retract, up to homotopy, of X .
- Show that if X is also a CW complex and both $\dim(X), \dim(Y) < n$, then f is a homotopy equivalence.

Problem 11.36. Let $f : X \rightarrow Y$ be a map of pointed path-connected spaces.

- Show that f is a weak homotopy equivalence if and only if its homotopy fiber F_f is weakly contractible.
- Show that if the cofiber C_f is contractible, then $\Sigma f : \Sigma X \rightarrow \Sigma Y$ is a homotopy equivalence.

We will see later (in Problem 19.38) that there are maps f which are not even 2-equivalences but whose cofibers are contractible.

Problem 11.37. Let X be a path-connected H-space. We have shown that $[A, X]$ has a multiplication whose unit is the trivial map. We have not shown that this multiplication is associative or that elements of $[A, X]$ have inverses.

- Show that, under the identification $\pi_*(X \times X) \cong \pi_*(X) \times \pi_*(X)$, the map $\mu_* : \pi_*(X) \times \pi_*(X) \rightarrow \pi_*(X)$ is given by $\mu_*(\alpha, \beta) = \alpha + \beta$.
- Show that the **shear map** $s : X \times X \rightarrow X \times X$ given by $s : (x_1, x_2) \mapsto (x_1, x_1x_2)$ is a weak homotopy equivalence.
- Show that for any CW complex A , if $\alpha, \beta \in [A, X]$, then there exist unique $\xi, \zeta \in [A, X]$ such that $\alpha \cdot \xi = \beta$ and $\zeta \cdot \alpha = \beta$.

In the terminology of abstract algebra, parts (c) and (d) assert that if A is a CW complex, then $[A, X]$ is an **(algebraic) loop**.

Problem 11.38.

- Show that if L is a loop, then every element $\alpha \in L$ has a left inverse and a right inverse. Show that if L is associative, then the left and right

inverses are equal, and give an example to show that they need not be equal in general.

- (b) Let X be a connected CW complex and an H-space. Show that X has a left inverse $\nu_L : X \rightarrow X$ and a right inverse $\nu_R : X \rightarrow X$. Show that if X is homotopy associative, then $\nu_R \simeq \nu_L$.³

³There are H-spaces for which $[A, X]$ is a nonassociative algebraic loop with distinct left and right inverses.

Understanding Suspension

We will show that the suspension map $\sigma : X \rightarrow \Omega\Sigma X$ adjoint to the identity $\Sigma X \rightarrow \Sigma X$ can be built by a domain-type procedure called the James construction, which produces the free topological monoid on X . The James construction enables us to understand the suspension operation $[X, Y] \rightarrow [\Sigma X, \Sigma Y]$; this understanding leads to the Freudenthal Suspension Theorem and a variety of computations.

We establish the existence and properties of Eilenberg-Mac Lane spaces and use them to settle the more delicate 1-dimensional case of the Freudenthal Suspension Theorem.

17.1. Moore Paths and Loops

Concatenation of paths endows the loop space ΩX with the structure of a homotopy-associative, but not strictly associative, H-space. In this section, we show that ΩX is homotopy equivalent, by an H-map, to the strictly associative H-space $\Omega_M X$ of measured loops. Later in this chapter, we'll use measured loops and paths to show that the James construction on X , which is extremely easy to compare to strictly associative H-spaces, is (weakly) homotopy equivalent to the loop space on ΣX .

17.1.1. Spaces of Measured Paths. The space of free **Moore paths** (also called **measured paths**) in a space X is the space

$$M(X) = \{(\alpha, a) \mid \alpha : [0, \infty) \rightarrow X \text{ and } \alpha|_{[a, \infty)} \text{ is constant}\},$$

with the topology it inherits as a subspace of $\text{map}_o([0, \infty), X) \times [0, \infty)$. The **initial point** of the Moore path (α, a) is $\alpha(0) \in X$, and its **endpoint** is $\alpha(a)$. If (α, a) and (β, b) are Moore paths in X and if the endpoint of (α, a) is equal to the initial point of (β, b) , then we may **concatenate** them by setting $(\alpha, a) \star (\beta, b) = (\alpha \star \beta, a + b)$, where

$$\alpha \star \beta(t) = \begin{cases} \alpha(t) & \text{if } t \leq a, \\ \beta(t - a) & \text{if } t \geq a. \end{cases}$$

Since this notion of concatenation does not rescale the paths, we'll refer to this as **rigid concatenation**.

The first thing to do is to show that the space of Moore paths is, homotopically speaking, equivalent to the ordinary path space. We'll use the evaluation map

$$\textcircled{0}_{0,\text{end}} : M(X) \rightarrow X \times X \quad \text{given by} \quad \textcircled{0}_{0,\text{end}}(\alpha, a) = (\alpha(0), \alpha(a)),$$

and $\textcircled{0}_{\text{end}} : M(X) \rightarrow X$ by $\textcircled{0}_{\text{end}}(\alpha, a) = \alpha(a)$.

Exercise 17.1. Show that $\textcircled{0}_{\text{end}}$ is continuous and deduce that $\textcircled{0}_{0,\text{end}}$ is continuous.

Proposition 17.2. The rule $(\alpha, a) \mapsto \boxed{t \mapsto \alpha(at)}$ defines a homotopy equivalence

$$\begin{array}{ccc} M(X) & \xrightarrow{\quad} & X^I \\ & \searrow \textcircled{0}_{0,\text{end}} & \swarrow \textcircled{0}_{0,1} \\ & X \times X & \end{array}$$

in the category $\mathcal{T} \downarrow X \times X$ of spaces over $X \times X$.

Corollary 17.3. The map $\textcircled{0}_{0,\text{end}}$ is a weak fibration.

Problem 17.4. Prove Proposition 17.2 and Corollary 17.3.

Exercise 17.5. Investigate the relationship between rigid concatenation in $M(X)$ and ordinary concatenation in X^I .

The space of **based Moore paths** in a pointed space X is the subspace of $M(X)$ comprising all Moore paths that end at the basepoint:

$$\mathcal{P}_M(X) = \{(\alpha, a) \mid \alpha([a, \infty)) = *\} \subseteq M(X).$$

The space of **Moore loops** on X , denoted $\Omega_M(X)$, is the fiber of the **evaluation map** $\textcircled{0}_0 : \mathcal{P}_M(X) \rightarrow X$ given by $(\alpha, a) \mapsto \alpha(0)$.

Problem 17.6.

(a) Show that $\textcircled{0}_0 : \mathcal{P}_M(X) \rightarrow X$ is a weak fibration.

(b) Show that the rule $(\alpha, a) \mapsto \boxed{t \mapsto \alpha(at)}$ defines homotopy equivalences

$$\mathcal{P}_M(X) \xrightarrow{\simeq} \mathcal{P}(X) \quad \text{and} \quad \Omega_M(X) \xrightarrow{\simeq} \Omega X.$$

Since the space of Moore loops on X is homotopy equivalent to ΩX , it can be given the structure of an H-space. But the advantage of the Moore loop space is that the multiplication is strictly associative.

Problem 17.7. Show that rigid concatenation of Moore loops gives $\Omega_M(X)$ the structure of a topological monoid.

It follows from Problem 17.6 that for any space A the induced map $[A, \Omega_M X] \rightarrow [A, \Omega X]$ is a bijection; but since these homotopy sets are groups, we'd really like the map to be a group isomorphism. To prove this, you simply need to verify that the map $\Omega_M X \rightarrow \Omega X$ is an H-map.

Problem 17.8. Show that the homotopy equivalence $\Omega_M X \rightarrow \Omega X$ is an H-map.

17.1.2. Composing Infinite Collections of Homotopies. A left homotopy $H : X \times I \rightarrow Y$ is adjoint to a right homotopy $\widehat{H} : X \rightarrow Y^I$. If we choose to index our homotopy on the interval $[0, a]$, then the adjoint is a map $X \rightarrow \text{map}_\circ([0, a], Y)$, which can be interpreted as an eventually constant right homotopy $\widehat{H} : X \rightarrow M(Y)$, given explicitly by

$$\widehat{H}(x) = \left(\boxed{t \mapsto H(x, \min\{t, a\})}, a \right).$$

The rigid concatenation of an infinite sequence of homotopies $H_0 : f_0 \simeq f_1$, $H_1 : f_1 \simeq f_2$ and so on defines a function $H : X \times [0, \infty) \rightarrow Y$. We can interpret the adjoint $X \rightarrow \text{map}([0, \infty), Y)$ as a map to $M(X)$ if for each $x \in X$, the (long) path $\widetilde{H}|_{x \times [0, \infty)}$ is eventually constant.

So let's suppose there is a continuous function $z : X \rightarrow [0, \infty)$ such that for each $x \in X$, the restriction $H|_{x \times [z(x), \infty)}$ is constant. Then the rule

$$\widehat{H}(x) = (H|_{x \times [0, \infty)}, z(x))$$

defines a lift in the diagram

$$\begin{array}{ccc} & & M(Y) \\ & \nearrow \widehat{H} & \downarrow \\ X & \xrightarrow{H} & \text{map}_\circ([0, \infty), Y). \end{array}$$

Exercise 17.9. Show that the map \widehat{H} is continuous.

Now we define equivalence relations on X^n : two points (x_1, \dots, x_n) and (y_1, \dots, y_n) are equivalent if and only if, after deleting all entries $*$, they are exactly the same list of elements in exactly the same order. For example, if $*, a, b \in X$, then

$$(*, a, *, *, b, *, *, a, *, b) \sim (*, *, *, a, *, b, a, *, b, *, *) \quad \text{in } X^n.$$

We let $J^n(X)$ denote the set of equivalence classes of points in X^n ; we of course have quotient maps $q_n : X^n \rightarrow J^n(X)$, which figure in the commutative ladder

$$\begin{array}{ccccccc} \dots & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \dots \\ & & \downarrow q_n & & \downarrow q_{n+1} & & \\ \dots & \longrightarrow & J^n(X) & \longrightarrow & J^{n+1}(X) & \longrightarrow & \dots \end{array}$$

The **James construction** on X , denoted $J(X)$, is the colimit of the bottom row; thus we have a natural map $q : X^\infty \rightarrow J(X)$. We'll write $[x_1, \dots, x_n]$ for the equivalence class of (x_1, \dots, x_n) in any of the spaces $J^m(X)$ for $n \leq m \leq \infty$.

Problem 17.13.

- (a) Show that q is a quotient map.
- (b) Show that this construction is functorial and that it respects homotopy.

HINT. First show that $X \mapsto X^n$ respects homotopy.

17.2.2. The Algebraic Structure of the James Construction. Next we show that the James construction builds the free topological monoid generated by X .

Problem 17.14. Show that the map $\mu_n : X \times J^n(X) \rightarrow J^{n+1}(X)$ given by $(x, [x_1, \dots, x_n]) \mapsto [x, x_1, \dots, x_n]$ is a quotient map.

We use the maps μ_n to fit the spaces $J^n(X)$ into a pushout square.

Problem 17.15.

- (a) Show that there is a categorical pushout square

$$\begin{array}{ccc} (X \times J^{n-1}(X)) \cup (* \times J^n(X)) & \xrightarrow{i} & X \times J^n(X) \\ \mu_{n-1} \cup \text{pr}_2 \downarrow & & \downarrow \mu_n \\ J^n(X) & \hookrightarrow & J^{n+1}(X). \end{array}$$

HINT. Let P be the pushout, and show that $P \rightarrow J^{n+1}(X)$ is a bijection.

- (b) Prove that if X is well-pointed, then each inclusion $J^n(X) \rightarrow J^{n+1}(X)$ is a cofibration, and conclude that the squares in part (a) are homotopy pushout squares.
- (c) Show that $J^{n+1}(X)/J^n(X) \cong X^{\wedge(n+1)}$.
- (d) Show that if X is a CW complex, then $J(X)$ is a generalized CW complex.

Exercise 17.16. What must be true in order for $J(X)$ to be a CW complex?

Finally, we justify the assertion that $J(X)$ is the free topological monoid generated by the set X . The multiplication is the map

$$\mu : J(X) \times J(X) \longrightarrow J(X)$$

given by the rule $\mu([x_1, \dots, x_n], [y_1, \dots, y_m]) = [x_1, \dots, x_n, y_1, \dots, y_m]$.

Proposition 17.17. *Let $X \in \mathcal{T}_*$.*

- (a) *Show that μ makes $J(X)$ into a topological monoid.*
- (b) *Show that a pointed map $f : X \rightarrow M$ from X to a topological monoid M extends to a unique monoid map $\phi : J(X) \rightarrow M$ and if f is continuous, then so is ϕ .*
- (c) *If $f \simeq g : X \rightarrow M$ in \mathcal{T}_* , then the extensions $\phi, \gamma : J(X) \rightarrow M$ are homotopic through homomorphisms.*

Problem 17.18. Prove Proposition 17.17.

HINT. Show that $((x_1, \dots, x_n), (y_1, \dots, y_m)) \mapsto (x_1, \dots, x_n, y_1, \dots, y_m)$ defines a continuous function $X^\infty \times X^\infty \rightarrow X^\infty$. For part (c), find a right homotopy that is a homomorphism.

Corollary 17.19. *If X is a topological monoid, then the inclusion $j : X \rightarrow J(X)$ has a left inverse τ .*

Problem 17.20. Prove Corollary 17.19.

The James Construction and Weak Equivalences. We will prove a powerful theorem identifying the homotopy type of the James construction $J(X)$, but it requires X to be a CW complex. To apply this result to other spaces, we need to know that the James construction carries weak equivalences to weak equivalences.

Problem 17.21. Show that if $f : X \rightarrow Y$ is an m -equivalence between well-pointed spaces, then $J^n(X) \rightarrow J^n(Y)$ is an m -equivalence for all $n \leq \infty$.

17.3. Identifying the Suspension Map

Suspension defines a natural transformation $\Sigma : [?, ?] \rightarrow [\Sigma?, \Sigma?]$. We also have a natural isomorphism $\alpha : [\Sigma A, \Sigma X] \xrightarrow{\cong} [A, \Omega\Sigma X]$ (coming from the exponential law). Writing $S = \alpha \circ \Sigma$, we have a commutative diagram of functors and natural transformations

$$\begin{array}{ccc} [A, X] & \xrightarrow{\Sigma} & [\Sigma A, \Sigma X] \\ \parallel & & \cong \downarrow \alpha \\ [A, X] & \xrightarrow{S} & [A, \Omega\Sigma X]. \end{array}$$

The diagram shows that the study of the natural transformation Σ is equivalent to the study of the transformation S .

Problem 17.22. Show that $S = \sigma_*$, where $\sigma(x) = \boxed{t \mapsto [x, t]}$.

The **suspension map** σ corresponds, via the inclusion $\Omega\Sigma X \hookrightarrow \Omega_M\Sigma X$, to the map

$$\sigma_M : X \longrightarrow \Omega_M X \quad \text{given by} \quad \sigma_M : x \mapsto \left(\boxed{t \mapsto [x, t]}, 1 \right),$$

which we call the **Moore suspension** map. By Proposition 17.17(b), there is a unique continuous homomorphism $e_X : J(X) \rightarrow \Omega_M(X)$ of topological monoids making the diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & J(X) \\ \parallel & & \downarrow e_X \\ X & \xrightarrow{\sigma_M} & \Omega_M \Sigma X \end{array}$$

strictly commutative.

Problem 17.23. Show that the rule $X \mapsto e_X$ defines a natural transformation $J(?) \rightarrow \Omega_M(?)$.

Theorem 17.24 (James). *If $X \in \mathcal{T}_*$ is a connected CW complex, the homomorphism*

$$e_X : J(X) \longrightarrow \Omega_M \Sigma X$$

is a homotopy equivalence.

Corollary 17.25. *If X is a well-pointed space, then e_X is a weak equivalence.*

Problem 17.26.

- (a) Show that if $f : X \rightarrow Y$ is a weak homotopy equivalence, then so is $\Omega_M \Sigma X \rightarrow \Omega_M \Sigma Y$.

(b) Use Theorem 17.24 to prove Corollary 17.25.

Now we fix a pointed connected CW complex X and write J and J^n for $J(X)$ and $J^n(X)$. The proof of Theorem 17.24 amounts to a detailed study of the strictly commutative cube

$$\begin{array}{ccccc}
 X \times J & \xrightarrow{\text{in}_0 \times \text{id}_J} & CX \times J & & \\
 \downarrow \text{pr}_1 & \searrow \mu & \downarrow & \searrow \nu & \\
 & J & \xrightarrow{\xi} & T & \\
 & \downarrow & \downarrow \text{pr}_1 & \downarrow q & \\
 X & \xrightarrow{\text{in}_0} & CX & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 * & \xrightarrow{\quad} & \Sigma X & &
 \end{array}$$

in which the top and bottom squares are both categorical and homotopy pushouts and q is the induced map of pushouts.

Problem 17.27. Let X be a connected pointed CW complex.

(a) Show that the squares

$$\begin{array}{ccccc}
 J & \xleftarrow{\mu} & X \times J & \xrightarrow{\text{in}_0 \times \text{id}_J} & CX \times J \\
 \downarrow & & \downarrow \text{pr}_1 & & \downarrow \text{pr}_1 \\
 * & \xleftarrow{\quad} & X & \xrightarrow{\text{in}_0} & CX
 \end{array}$$

are strong homotopy pullback squares.

(b) Show that the squares

$$\begin{array}{ccccc}
 J & \xrightarrow{\quad} & T & \xleftarrow{\nu} & CX \times J \\
 \downarrow & & \downarrow q & & \downarrow \text{pr}_1 \\
 * & \xrightarrow{\quad} & \Sigma X & \xleftarrow{\quad} & CX
 \end{array}$$

are strong homotopy pullback squares.

Is the connected hypothesis necessary in Problem 17.27? It is very instructive to work out an explicit example.

Exercise 17.28. Study this cube diagram for $X = S^0$. What is J ? What is T ? Explicitly describe the map $q : T \rightarrow S^1$. Is the square of Problem 17.27(a) a homotopy pullback square?

Now we study the space T and the map q in greater detail. Since the map μ is surjective, so is ν , and therefore each point of T is the equivalence class $\langle [x, t], [x_1, \dots, x_n] \rangle$ of a point $([x, t], [x_1, \dots, x_n]) \in J$.

Problem 17.29.

- (a) Show that each point in T , except $*$, has a unique expression in the form $\langle [x, t], [x_1, \dots, x_n] \rangle$ with $t < 1$.
- (b) Show that q is given by $q(\langle [x, t], [x_1, \dots, x_n] \rangle) = [x, t]$.
- (c) Show that $\xi : J \rightarrow T$ is given by $\xi([x_1, \dots, x_n]) = \langle *, [x_1, \dots, x_n] \rangle$.
- (d) Show that ν is a quotient map.
- (e) Show that T is the quotient space $(CX \times J)/\sim$, where \sim is the equivalence relation given by

$$([x, 0], [x_1, \dots, x_n]) \sim (*, [x, x_1, \dots, x_n]).$$

We will show that T is contractible by constructing a filtration $\dots \subseteq T_n \subseteq T_{n+1} \subseteq \dots \subseteq T$ and showing that the identity map is homotopic to a map $d : T \rightarrow T$ with $d(T_n) \subseteq T_{n-1}$ for each $n > 0$ and $d(T_0) = *$. Then a simple adaptation of the method of Problem 11.4 will show that T is contractible.

The filtration is defined by setting $T_n = \nu(CX \times J_n) \subseteq T$. Define a homotopy $H : T \times I \rightarrow T$ by the rule

$$H(\langle [x, t], [x_1, \dots, x_n] \rangle, s) = \langle [x, (1-s)t + s], [x_1, \dots, x_n] \rangle.$$

We write $d = H|_{T \times \{1\}}$; this map is given explicitly by the formula

$$d : ([x_0, t], [x_1, \dots, x_n]) \mapsto ([x_1, 0], [x_2, \dots, x_n]).$$

Problem 17.30.

- (a) Show that H is a continuous pointed homotopy $\text{id}_T \simeq d$.
- (b) Show that $H(T_n \times I) \subseteq T_n$ and that $d(T_n) \subseteq T_{n-1}$.
- (c) Show that T is the colimit of the diagram $\dots \rightarrow T_n \rightarrow T_{n+1} \rightarrow \dots$.
- (d) Show that d^∞ , the infinite composite of d with itself, is constant and homotopic to id_T .

Since T is contractible and the square

$$\begin{array}{ccc} J & \longrightarrow & T \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

is a strong homotopy pullback, we deduce that $J \simeq \Omega \Sigma X$. But we want to know even more: the homotopy equivalence should be none other than the map e_X . To show this, we study the front square of our cube in still more detail. Since $T \simeq *$, the map $q : T \rightarrow \Sigma X$ is nullhomotopic, so it factors

through the path fibration $\mathcal{P}_M \Sigma X \rightarrow \Sigma X$. Each choice of nullhomotopy $q \simeq *$ gives rise to a diagram of the form

$$\begin{array}{ccc}
 J & \xrightarrow{\xi} & T \\
 f \downarrow & & \downarrow u \\
 \Omega_M \Sigma X & \longrightarrow & \mathcal{P}_M(\Sigma X) \\
 \downarrow & \text{pullback} & \downarrow @_0 \\
 * & \longrightarrow & \Sigma X
 \end{array}$$

in which f is a homotopy equivalence. We will use the nullhomotopy that comes from the contraction of T we found in Problem 17.30.

Problem 17.31. Let $u : T \rightarrow \mathcal{P}_M(\Sigma X)$ be the lift of q corresponding to $H^{*\infty}$. In this problem, we'll use the notation $\omega_x : I \rightarrow \Sigma X$ for the loop $\omega_x(t) = [x, t]$.

- (a) Show that $u(\langle *, [x_1, \dots, x_n] \rangle) = \omega_{x_1} \star \dots \star \omega_{x_n}$.
- (b) Prove Theorem 17.24 by showing that the map $J \rightarrow \Omega_M \Sigma X$ is e_X .

Our analysis has shown that the suspension map σ is equivalent to both the Moore suspension σ_M and to the inclusion $j : X \hookrightarrow J(X)$. The situation is summarized in the strictly commutative diagram

$$\begin{array}{ccccc}
 & & X & & \\
 & \swarrow j & \downarrow \sigma_M & \searrow \sigma & \\
 J(X) & \xrightarrow[e \simeq]{} & \Omega_M \Sigma X & \xleftarrow[\simeq]{} & \Omega \Sigma X
 \end{array}$$

in which both of the horizontal maps are H-maps.

17.4. The Freudenthal Suspension Theorem

Our motivation in this chapter is to develop an understanding of the suspension transformation. We have shown that it is enough to understand the map $j : X \rightarrow J(X)$. The explicit topological construction of $J(X)$ gives valuable information about the connectivity of j .

Problem 17.32. Let $Y \in \mathcal{T}_*$ be an $(n - 1)$ -connected well-pointed space.

- (a) If Y is a CW complex, then $J(Y)$ inherits the structure of a generalized CW complex in which each $J^k(Y)$ is a subcomplex. Show that $J^k(Y)$ contains all cells in $J(Y)$ of dimension at most $n(k + 1) - 1$.
- (b) Show that the suspension map $\sigma : Y \hookrightarrow \Omega \Sigma Y$ is a $(2n - 1)$ -equivalence, whether Y is a CW complex or not.

The connectivity of σ determines the behavior of the suspension transformation for low-dimensional domains.

Theorem 17.33 (Freudenthal Suspension Theorem). *Let $Y \in \mathcal{T}_*$ be an $(n - 1)$ -connected well-pointed space and let X be a pointed CW complex. Then*

- (a) *if $\dim(X) < 2n - 1$, then $\Sigma : [X, Y] \rightarrow [\Sigma X, \Sigma Y]$ is an isomorphism, and*
- (b) *if $\dim(X) = 2n - 1$, then $\Sigma : [X, Y] \rightarrow [\Sigma X, \Sigma Y]$ is surjective.*

The special case in which X is a sphere is particularly important.

Corollary 17.34. *Let Y be an $(n - 1)$ -connected space. Then*

- (a) $\Sigma : \pi_k(Y) \rightarrow \pi_{k+1}(\Sigma Y)$ *is an isomorphism for $k < 2n - 1$, and*
- (b) $\Sigma : \pi_{2n-1}(Y) \rightarrow \pi_{2n}(\Sigma Y)$ *is onto.*

Problem 17.35. Prove Theorem 17.33 and Corollary 17.34.

The Connectivity of a Suspension. We have already shown, using cellular replacements, that $\text{conn}(\Sigma X) \geq \text{conn}(X) + 1$. Now we are prepared to show that this is an equality, at least for simply-connected spaces.

Problem 17.36. Show that if X is simply-connected, then $\text{conn}(\Sigma X) = \text{conn}(X) + 1$. Does your argument work for path-connected spaces X that are not simply-connected?

17.5. Homotopy Groups of Spheres and Wedges of Spheres

Using the Freudenthal Suspension Theorem, we can bootstrap up from the fundamental computation $\pi_1(S^1) \cong \mathbb{Z} \cdot \text{id}$ and determine the groups $\pi_k(S^n)$ for $k \leq n$.

Theorem 17.37. *For all $n \geq 1$, the suspension $\Sigma : \pi_n(S^n) \rightarrow \pi_{n+1}(S^{n+1})$ is an isomorphism, and hence*

$$\pi_n(S^n) = \begin{cases} \mathbb{Z} \cdot [\text{id}_{S^n}] & \text{if } k = n, \\ 0 & \text{if } k < n. \end{cases}$$

Problem 17.38.

- (a) Show that $\Sigma : \pi_1(S^1) \rightarrow \pi_2(S^2)$ is an isomorphism.
- (b) Prove Theorem 17.37.

HINT. $S^1 \subseteq \mathbb{C}$ is a topological group under multiplication.

Corollary 17.39. *For any $n > 1$ and any indexing set \mathcal{J} ,*

$$\pi_n(\bigvee_{\mathcal{J}} S^n) \cong \bigoplus_{\mathcal{J}} \mathbb{Z} \cdot [\text{in}_j].$$

Problem 17.40.

- (a) Show that any map $f : S^n \rightarrow \bigvee_{\mathcal{J}} S^n$ must factor through a finite sub-wedge.
- (b) Prove Corollary 17.39.

HINT. For a finite collection $\{X_j\}$, $\pi_n(\prod X_j) \cong \prod \pi_n(X_j) \cong \bigoplus \pi_n(X_j)$.

We have finally found homotopy functors that can distinguish between spheres and disks, and this is good news because it shows that our intuition that $S^n \not\cong *$ is accurate.

Exercise 17.41. Generalize and prove Problem 1.13.

Higher Homotopy Groups. Algebraic topology can be said to have begun when Poincaré introduced the ‘homology groups’ of a space in the 1890s. These are functors $\tilde{H}_n : \mathcal{T}_* \rightarrow \text{Ab } \mathcal{G}$, and they have the property that $\tilde{H}_k(X^n) = 0$ for $k \neq n$, and $\tilde{H}_n(S^n) \cong \mathbb{Z}$. Thus when the homotopy groups $\pi_*(X)$ were first introduced, and it was proved that $\pi_k(S^n) = 0$ for $k < n$ and $\pi_n(S^n) \cong \mathbb{Z} \cdot [\text{id}_{S^n}]$, most topologists thought that it would ultimately turn out that π_n and \tilde{H}_n were really the same functor.

This conventional wisdom was shattered when Hopf showed that in fact the ‘higher homotopy groups’ of spheres are not all trivial!

Problem 17.42 (Hopf). Show that $\pi_3(S^2) \cong \mathbb{Z}$, and exhibit a generator.

HINT. This is an instance where projective spaces offered deep insight into homotopy theory.

Now it is known that the homotopy groups of every simply-connected finite complex are nonzero in infinitely many dimensions; and there is no simply-connected finite complex all of whose homotopy groups are known. The structure of $\pi_*(S^n)$ is extremely complicated and messy, but there is structure. For example, the Whitehead product and composition of maps between spheres endow these groups with useful algebraic operations.

17.6. Eilenberg-Mac Lane Spaces

Let G be a group, and let $n \in \mathbb{N}$. A CW complex L is called an **Eilenberg-Mac Lane space of type (G, n)** if its homotopy groups are given by

$$\pi_k(L) \cong \begin{cases} G & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that we used \cong instead of $=$ in this definition. This is an important and somewhat subtle point. The group G is a certain set with a multiplication rule. For example, it might be a set of permutations (which are bijective

functions from a set to itself) or it might be a set of cosets, etc. The group $\pi_n(L)$ is a set of homotopy classes of maps from a sphere into L . Unless the stars align perfectly, it is essentially impossible for $\pi_n(L)$ to be *equal* to G . Instead, when we say that L is an Eilenberg-Mac Lane space of type (G, n) , we mean that the only nonzero homotopy group of L is $\pi_n(L)$ and we have in mind a *particular choice* of a group isomorphism $\theta : \pi_n(L) \rightarrow G$. Deciding on a choice of isomorphism is sometimes referred to as giving L the **structure** of an Eilenberg-Mac Lane space; different isomorphisms constitute different structures. Most of the time, these technical considerations can be safely ignored, but it is occasionally crucial to keep complete control of the situation.

Exercise 17.43.

- Show that S^1 is an Eilenberg-Mac Lane space of type $(\mathbb{Z}, 1)$. How many different such structures does it have?
- Show that $\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1)$ and $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$. Which Eilenberg-Mac Lane space is $\mathbb{H}P^\infty$?
- Show that for any discrete group G , the classifying space BG is an Eilenberg-Mac Lane space of type $(G, 1)$. How many ways can BG be structured as an Eilenberg-Mac Lane space of type $(G, 1)$?

17.6.1. Maps into Eilenberg-Mac Lane Spaces. Since the homotopy groups of Eilenberg-Mac Lane spaces are so simple, maps into them are especially amenable to construction and analysis by obstruction theory. In fact, maps from an $(n - 1)$ -connected domain into an Eilenberg-Mac Lane space are entirely determined by the algebra of their homotopy groups.

Theorem 17.44. *Let G be a group, and let Y be an Eilenberg-Mac Lane space of type (G, n) . Then for any $(n - 1)$ -connected CW complex X , the map*

$$\phi : [X, Y] \longrightarrow \text{Hom}(\pi_n(X), \pi_n(Y)) \quad \text{given by} \quad \phi(f) = f_*$$

is bijective.

Problem 17.45.

- Explain why it is enough to prove Theorem 17.44 in the special case that X has dimension at most $n + 1$.
- Show that an $(n - 1)$ -connected and $(n + 1)$ -dimensional CW complex X sits in a cofiber sequence $\bigvee_{i \in \mathcal{I}} S^n \xrightarrow{\alpha} \bigvee_{j \in \mathcal{J}} S^n \xrightarrow{i} X$.
- Show that the induced map $i_* : \pi_n(\bigvee_{\mathcal{J}} S^n) \rightarrow \pi_n(X)$ is surjective.
- Use Theorem 16.27 to show that ϕ is injective.

- (e) Let $\pi_n(X) = H$ and let $h : H \rightarrow G$ be any group homomorphism. Show that there is a map $\beta : \bigvee_{j \in \mathcal{J}} S^n \rightarrow K(G, n)$ whose induced map $\pi_n(\bigvee_{\mathcal{J}} S^n) \rightarrow G$ is $h \circ i_*$.
- (f) Show that ϕ is surjective.

Suppose X and Y are Eilenberg-MacLane spaces of type (G, n) . This means that we have in mind particular isomorphisms

$$\phi : \pi_n(X) \xrightarrow{\cong} G \quad \text{and} \quad \theta : \pi_n(Y) \xrightarrow{\cong} G.$$

According to Theorem 17.44, the composite $\theta^{-1} \circ \phi : \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$ is induced by a unique homotopy class $X \rightarrow Y$, which is manifestly a weak homotopy equivalence and hence a homotopy equivalence, since X and Y are CW complexes. Thus we can unambiguously write $K(G, n)$ to denote any Eilenberg-MacLane space of type (G, n) .

We have mentioned Milnor's theorem (Theorem 4.83), which has as a special case that if X is a CW complex, then ΩX is homotopy equivalent to a CW complex. This implies the following.

Corollary 17.46. *If $n > 0$, then $\Omega K(G, n) \simeq K(G, n - 1)$.*

Problem 17.47. Prove Corollary 17.46.

Problem 17.48. Show that the rule $G \mapsto K(G, n)$ is the object part of a functor $K(?, n) : \text{AB}\mathcal{G} \rightarrow \text{HT}_*$.

17.6.2. Existence of Eilenberg-MacLane Spaces. We have now shown that the Eilenberg-MacLane spaces that exist are unique up to canonical equivalence in HT_* . We have also seen that there are Eilenberg-MacLane spaces of the form $K(G, 1)$ for every group G . Since $\pi_n(X)$ is an abelian group for all $n > 1$, it remains to show that $K(G, n)$ exists for all abelian groups.

Theorem 17.49. *For every abelian group G and every $n \geq 1$, there is an Eilenberg-MacLane space $K(G, n)$ which is a topological monoid.*

Problem 17.50.

- (a) Show that if there is an $(n - 1)$ -connected space X with $\pi_n(X) \cong G$, then there is an Eilenberg-MacLane space of type (G, n) .
- (b) Show that if F is a free abelian group, then $K(F, n)$ exists.

Problem 17.51. Let $0 \rightarrow F_1 \xrightarrow{d} F_0 \rightarrow G \rightarrow 0$ be a free resolution of an abelian group G (see Section A.1).

- (a) Show that $K(F_1, n)$ and $K(F_0, n)$ exist and that there is a map $\delta : K(F_1, n) \rightarrow K(F_0, n)$ such that the square

$$\begin{array}{ccc} \pi_n(K(F_1, n)) & \xrightarrow{\delta_*} & \pi_n(K(F_0, n)) \\ \cong \downarrow & & \downarrow \cong \\ F_1 & \xrightarrow{d} & F_0 \end{array}$$

commutes, where the vertical isomorphisms are the structure maps for the Eilenberg-Mac Lane spaces.

- (b) Determine the fiber of δ and prove Theorem 17.49.

Proposition 17.52. *If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence of abelian groups, then the corresponding sequence*

$$K(A, n) \longrightarrow K(B, n) \longrightarrow K(C, n)$$

of Eilenberg-Mac Lane spaces is a fibration sequence.

Problem 17.53. Prove Proposition 17.52.

Exercise 17.54. Is it possible for $K(G, 1)$ to be an H-space if G is a non-abelian group?

Exercise 17.55. Criticize the following argument:

We know that for any space X , ΩX is an H-space that is H-equivalent to a topological monoid $\Omega_M X$; and we know that $\Omega^2 X$ is an abelian H-space. Thus $\Omega^2 X$ is H-equivalent to an abelian topological monoid.

It can be shown that if G is an abelian group, then there is an Eilenberg-Mac Lane space $K(G, n)$ which is a strictly associative abelian topological monoid (see Problem 20.64).

Problem 17.56. Show that if G is an abelian group and $f : X \rightarrow K(G, n)$ is nontrivial, then $\Sigma f : \Sigma X \rightarrow \Sigma K(G, n)$ is also nontrivial.

17.7. Suspension in Dimension 1

We were able to show that the suspension $\Sigma : \pi_1(S^1) \rightarrow \pi_2(S^2)$ is an isomorphism, but it was not a direct consequence of the Freudenthal Suspension Theorem. The Freudenthal Suspension Theorem only guarantees that the suspension $\Sigma : \pi_1(X) \rightarrow \pi_2(X)$ is *surjective* (if X is path-connected). To complete our understanding of the suspension map, we need to determine its kernel.

Exercise 17.57.

- (a) Is $\Sigma : \pi_1(X) \rightarrow \pi_2(X)$ surjective when X is not path-connected?

(b) Show that if $\pi_1(X)$ is nonabelian, then Σ has a kernel.

The **commutator** of two elements $x, y \in G$ is the element $[x, y] = x^{-1}y^{-1}xy \in G$. The **commutator subgroup** of a group G is the smallest subgroup of G that contains all the commutators in G ; this is frequently denoted G' , but we'll use the more suggestive (and also common) notation $[G, G]$. A surjective homomorphism $\phi : G \rightarrow H$ is called an **abelianization** of G if $\ker(\phi) = G'$. The quotient map $G \rightarrow G/G'$ may be called the 'standard abelianization' of G . A *nontrivial* group G is called **perfect** if $G' = G$; perfect groups are precisely those that have trivial abelianizations.

We'll frequently need to impose conditions on fundamental groups such as: $\pi_1(X)$ is not a nontrivial perfect group; or $\pi_1(X)$ has no nontrivial perfect subgroups. A group with the latter property is known as a **hypoabelian group**. If $N \triangleleft G$, then G is hypoabelian if and only if both N and G/N are hypoabelian. We'll generally use the phrase 'has no nontrivial perfect subgroups' instead of the more obscure term 'hypoabelian' in our statements.

Proposition 17.58. *If X is path-connected, then the suspension map*

$$\Sigma : \pi_1(X) \longrightarrow \pi_2(\Sigma X)$$

is an abelianization.

Problem 17.59. Prove Proposition 17.58.

HINT. Let A be an abelianization of $\pi_1(X)$ and consider maps $X \rightarrow K(A, 1)$.

We deduce that, even for spaces that are not simply-connected, suspension usually, *but not always*, increases connectivity by one.

Theorem 17.60. *The following are equivalent:*

- (1) $\pi_1(X)$ is not a nontrivial perfect group,
- (2) $\text{conn}(\Sigma X) = \text{conn}(X) + 1$.

Problem 17.61. Prove Theorem 17.60.

It is often useful to know that Theorem 17.60 applies to the homotopy fiber F_f of a map f .

Problem 17.62. Let $f : X \rightarrow Y$, where $\pi_1(X)$ has no nontrivial perfect subgroups. Show that $\pi_1(F_f)$ has no nontrivial perfect subgroups and, in particular, it is not a nontrivial perfect group.

Problem 17.63. Suppose $\pi_1(X)$ has no nontrivial perfect quotients, and suppose $f : X \rightarrow Y$ with $\pi_1(f)$ surjective and nontrivial. Show that $\Sigma f \not\cong *$.

17.8. Additional Topics and Problems

17.8.1. Stable Phenomena. We have seen several situations in which suspension makes a significant change in a topological problem. For example we have seen that $\Sigma(X \times Y) \simeq \Sigma(X \vee Y \vee (X \wedge Y))$, though we strongly suspect that this is not true without the suspension; and later (in Problem 19.38) we'll build a nontrivial space X such that $\Sigma X \simeq *$.

The Freudenthal Suspension Theorem implies that if we are studying CW complexes that are very highly connected in comparison to their dimension, many properties are *unchanged* by suspension. Such properties are called **stable phenomena**, and maps and spaces that satisfy the required dimension and connectivity conditions are said to be **in the stable range**.

Problem 17.64.

(a) Show that for any space X the suspension maps

$$\Sigma : \pi_{n+t}(\Sigma^t X) \longrightarrow \pi_{n+t+1}(\Sigma^{t+1} X)$$

are isomorphisms for all sufficiently large t .

(b) More generally, show that if X is a finite-dimensional CW complex, then the suspension maps

$$\Sigma : [\Sigma^t X, \Sigma^t Y] \longrightarrow [\Sigma^{t+1} X, \Sigma^{t+1} Y]$$

are bijections for sufficiently large t .

How large is 'sufficiently large'?

The group $\pi_{n+t}(\Sigma^t X)$ (for t sufficiently large) is called the n^{th} **stable homotopy group** of X and is denoted $\pi_n^S(X)$. Analogously, the stable group (it *is* a group) of homotopy classes $[X, Y]^S$ is the limiting value of $[\Sigma^t X, \Sigma^t Y]$.

17.8.2. The James Splitting. Using the James construction, we can give an explicit formula for $\Sigma\Omega\Sigma X$ in terms of the space X and using only basic operations. This formula is called the **James splitting**.

Problem 17.65.

(a) Show that the suspension of the inclusion $J^{n-1}(X) \hookrightarrow J^n(X)$ can be identified as in the diagram

$$\begin{array}{ccc} \Sigma J^{n-1}(X) & \longrightarrow & \Sigma J^n(X) \\ \parallel & & \downarrow \simeq \\ \Sigma J^{n-1}(X) & \xrightarrow{\text{in}_1} & \Sigma J^{n-1}(X) \vee \Sigma X^{\wedge n}. \end{array}$$

(b) Show that $\Sigma\Omega\Sigma X \simeq \Sigma \left(\bigvee_{n \geq 1} X^{\wedge n} \right)$.

17.8.3. The Hilton-Milnor Theorem. The wedge of two (or more) spaces is fundamentally a domain-type object, and so computing the homotopy groups of a wedge is a fundamentally difficult and interesting problem.

According to Problem 15.86, there is a decomposition

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega(X \rtimes \Omega Y)$$

for any well-pointed X and Y . This implies the Hilton-Milnor theorem, which concerns the loop space on a wedge of two suspensions. This topological approach to the Hilton-Milnor theorem is due to Brayton Gray [73].

Theorem 17.66 (Hilton-Milnor-Gray). *If A and B are well-pointed spaces, then*

$$\Omega(\Sigma A \vee \Sigma B) \simeq \Omega(\Sigma A) \times \Omega \Sigma \left(B \vee \left(\bigvee_{n=1}^{\infty} B \wedge A^{\wedge n} \right) \right).$$

Problem 17.67. Prove Theorem 17.66.

HINT. Use Problem 5.152 and the James splitting.

It is instructive to apply Theorem 17.66 to a wedge of spheres.

Problem 17.68. Let $m \leq n$

- Apply Theorem 17.66 to $S^m \vee S^n$.
- Repeat, always collapsing to a sphere of smallest dimension. What happens to the connectivity of the confusing loop space piece as you repeat the process?
- Argue that $\Omega(S^m \vee S^n)$ is homotopy equivalent to a big product of loop spaces of spheres of various dimensions.

Determining the list of spheres that show up in the splitting of $\Omega(S^m \vee S^n)$ is a complicated combinatorial problem, which has been worked out semi-explicitly by Peter Hilton. They may be put in bijective correspondence with a basis for a free graded Lie algebra.

Project 17.69. Work out the spheres that appear in the Hilton-Milnor splitting of $\Omega(S^n \vee S^m)$.

17.8.4. Problems.

Problem 17.70. Show that $\Sigma : \pi_*(S^3) \rightarrow \pi_*(S^4)$ is injective.

Problem 17.71. Let $f : A \rightarrow B$ be a homomorphism of abelian groups. Show that the homotopy fiber of the corresponding map $K(A, n) \rightarrow K(B, n)$ is $K(\ker(f), n) \times K(\operatorname{coker}(f), n - 1)$.

Problem 17.72. Write out the cellular structure of $J(S^n) \simeq \Omega S^{n+1}$. Is it possible that $\Omega S^3 \cong \mathbb{C}P^\infty$?

Exercise 17.73. Work out $J(X)$ and $\Omega\Sigma X$ for $X \in \mathcal{T}_*$ discrete and for $X = Y_+$ where $Y \in \mathcal{T}_o$. (The identification $J(X) \sim \Omega\Sigma X$ fails for these spaces.)

If $K(G, n)$ exists, then $K(G, n-1) \simeq \Omega K(G, n)$ is an associative H-space. But what about $K(G, n)$? Must it be an H-space? Here's a direct proof.

Problem 17.74. Let G be a group with multiplication $\mu : G \times G \rightarrow G$, and suppose $K(G, n)$ exists.

- Show that G is abelian if and only if μ is a homomorphism.
- Show that if G is abelian, then the homomorphism μ induces a map $m : K(G, n) \times K(G, n) \rightarrow K(G, n)$ which is an H-space multiplication.
- Show that in the situation of part (b), the multiplication m is homotopy associative.
- Show that if G is not abelian, then $K(G, 1)$ is *not* an H-space.

Problem 17.75.

- Show that if X is an n -fold loop space, the map $X \rightarrow \Omega^n \Sigma^n X$ has a left homotopy inverse. Compare with Problem 17.38.

HINT. This is an instance of an abstract result: if $L : \mathcal{C} \rightarrow \mathcal{D}$ and $R : \mathcal{D} \rightarrow \mathcal{C}$ are adjoint, then $RLX \rightarrow RLRX$ has a left inverse.

- Show that $\Sigma^n : [X, K(G, m)] \rightarrow [\Sigma^n X, \Sigma^n K(G, m)]$ is injective.

Problem 17.76. Show that if X is $(n-1)$ -connected and $(2n-1)$ -dimensional, then there is a space Y such that $X \simeq \Sigma Y$.

Problem 17.77. Establish a bijection $\phi : [\Omega S^2, \Omega S^2] \xrightarrow{\cong} \prod_{n=1}^{\infty} \pi_n(S^2)$. The domain and target are groups; is ϕ a homomorphism?