Preface

From the earliest days of measure theory, invariant measures have held the interests of geometers and analysts alike. With Hausdorff's introduction of the measures that bear his name and the subsequent cementing of the relationships between measure theory and geometry, those interests attained a degree of permanency. Simultaneously, efforts at solving Hilbert's fifth problem (on recognizing Lie groups by their locally Euclidean structure) naturally found invariant measures an ally in analyzing the structure of topological groups, particularly compact and locally compact groups. Existence, uniqueness, and applications of invariant measures attracted the attention of many of the strongest mathematical minds. We hope in this volume to detail some of the highlights of those developments.

This book is aimed at an audience of people who have been exposed to a basic course in real variables, although we do review the development of Lebesgue's measure. As is usually the case, a certain amount of mathematical maturity is critical to the understanding of many of the topics discussed. We have included a few exercises; their occurrence is planned to coincide with sections where material somewhat divorced from typical experience is presented.

In the first chapter we develop Lebesgue measure in Euclidean spaces from a topological perspective. Roughly speaking, we start with knowledge of how big an open set is, pass from there to measuring the size of compact sets, and then, using regularity as a guide, determine which sets are measurable. Naturally, the details are more technical but the result—Lebesgue measure—is worth the effort.

We next discuss measures on metric spaces with special attention paid to Borel measures. We encounter the aforementioned Hausdorff measures and find that Hausdorff n-measure on Euclidean n-space is a multiple of Lebesgue measure on the same space. What's more, the constant of multiplicity is an apt rate of exchange between rectilinear measurements (Lebesgue measure) and spherical ones (Hausdorff's n-measure). Along the way we encounter and embrace Carathéodory's fundamental method of outer measures, a method we will return to throughout these deliberations.

We turn then to topological groups and give a brief introduction to this intriguing topic. A highlight of this chapter is the often surprising consequences one can draw about topological groups: the mixture of algebra and topology produces a sum in excess of what the summands hint. For instance, every (Hausdorff) topological group is completely regular, and if it satisfies the first axiom of countability (having a countable basis for the open sets about each point), then it is metrizable, with a left invariant metric moreover. These follow from the beautiful theorem of Birkhoff and Kakutani. We also show that if the group is locally compact then it is paracompact.

Next, in the chapter on Banach and measure theory, we present Banach's proof on the existence of an invariant measure on a compact metrizable topological group. Banach's proof, which is plainly of geometric flavor, is more general than showing "just" that compact metrizable groups have invariant measures; indeed, his proof asserts the existence of a Borel measure on any compact metric space that is invariant under the action of a transitive group of homeomorphisms. As is to be expected, Banach's proof relies on the methods of functional analysis, a subject he was deeply active in developing—most particularly in his use of "Banach limits", the existence of which relies on the Hahn–Banach theorem. To put Banach's result in context, it's important to know that we have a Borel probability in hand, one that allows every continuous function to be integrated, and for this we present Saks' proof that positive linear functionals of norm-1 $C(K)$'s, where K is a compact metric space, correspond to Borel probabilities. We follow the presentation of Saks' proof with Banach's approach to the Lebesgue integral. This appeared as an appendix in Saks' classical monograph *Theory of the Integral* [**111, 113**]. It contains Banach's proof of the Riesz Representation of $C(K)^*$ for a compact metric space K.

Having discussed the situation of compact metrizable topological groups, we next present von Neumann's proof of the existence and uniqueness of normalized Haar measure on any compact topological group. The importance to this proof of the uniform continuity of continuous real-valued functions defined an a compact group and the classical theorem of Arzelá and Ascoli should be plain and clear. Von Neumann's proof shows, in quite a natural way, that the normalized Haar measure is simultaneously left and right invariant. We include several other proofs of the existence of a Haar measure in the *Notes and Remarks* to this chapter.

An all-too-short chapter on applications of Haar measure on compact groups follows. Homogeneous spaces are shown to have unique invariant measures, invariance being with respect to a transitive group of homeomorphisms. This is followed by a presentation of the Peter–Weyl theorem on the existence of a complete system of irreducible finite-dimensional unitary representations of the group. We then broach the topic of absolutely p-summing operators on Banach spaces; after showing the existence of a "Pietsch measure" for any absolutely p-summing operator, we use the uniqueness of Haar measure to show that under appropriate mild invariance assumptions on a p-summing operator on a space that has an invariant norm that Haar measure serves as a Pietsch measure.

A chapter detailing the existence and uniqueness of Haar measure on a general locally compact topological group is next. There appears to be no clever trick to pass from the compact case to the locally compact situation; only hard work will suffice. The measure theory is more delicate and the proofs of existence and uniqueness of Haar measure follow suit. We present Weil's proof of existence, followed by H. Cartan's simultaneous proof of existence and uniqueness. Our *Notes and Remarks* in Chapter 6 complement this with the more commonly known proof via the Fubini theorem.

The special character of Haar measure is the topic of our next chapter with a gorgeous theorem of Bandt center stage. The theorem calls on an ingenious use of Hausdorff-like measures in tandem with the uniqueness aspects of Haar measure to show that if we encounter a locally compact metrizable topological group with a left invariant metric in place, then subsets that are isometric with this metric have the same Haar (outer) measure.

Just when we feel that we've done all that can be done with regard to Haar measure in a locally compact setting, we present Steinlage's remarkable description of necessary and sufficient conditions that a G-invariant Borel content exists on a locally compact Hausdorff space, where G is a suitable group of homeomorphisms of the space onto itself. The proof of existence is reminiscent of Banach's proof with a touch of Weil thrown in.

We finish with an all-too-brief description of Oxtoby's work on invariant Borel measures on nonlocally compact Polish groups.

We have two appendices. In one we discuss Haar's original proof of the existence of Haar measure in the case where the group is a compact metric group. The other appendix discusses the remarkable result of Kakutani and

Oxtoby in which they show that Haar measure on an infinite compact metric group can be extended to an amazingly large sigma field in a countably additive, translation invariant manner.

Our presentation of this material was greatly influenced by the experiences of talking about the material in a classroom setting, either in seminars or graduate classes. We found often that presenting material at a slightly less general level aided in conveying the essential ideas without any serious sacrifice. This also had the beneficial effect of inspiring questions, leading to deeper understanding, for both the students and us.

In any undertaking like this, many friends and colleagues have contributed through discussions, lectures, and reading attempts at exposition in varying states of preparation. We rush to thank all who have helped. We extend particular thanks to (the late) Diomedes Barcenas, Floyd Barger, Jonathan Borwein, Geraldo Botelho, Bruno Braga, John Buoni, Antonia Cardwell, Neal Carothers, Charlotte Crowther, John Dalbec, Geoff Diestel, Rocco Duvenhage, (the late) Doug Faires, Paul Fishback, Ralph Howard, Jozsi Jalics, Hans Jarchow, Livia Karetka, Jay Kerns, Darci Kracht, Charles Maepa, (the late) Roy Mimna, Daniel Pellegrino, Zbginiew Piotrowski, David Pollack, Zach Riel, Nathan Ritchey, Sarah Ritchey, Stephen Rodabaugh, Pilar Rueda, Dima Ryabogin, Juan Seoane, Brailey Sims, Anton Stroh, Johan Swart, Jamal Tartir, Padraic Taylor, Andrew Tonge, Thomas Wakefield, Matt Ward, Eric Wingler, George Yates, and Artem Zvavitch.

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