Preface

During their first year at the University of Chicago, graduate students in mathematics take classes in algebra, analysis, and geometry, one of each every quarter. The analysis courses typically cover real analysis and measure theory, functional analysis and applications, and complex analysis. This book grew out of the author's notes for the complex analysis classes which he taught during the Spring quarters of 2007 and 2008. These courses covered elementary aspects of complex analysis such as the Cauchy integral theorem, the residue theorem, Laurent series, and the Riemann mapping theorem, but also more advanced material selected from Riemann surface theory.

Needless to say, all of these topics have been covered in excellent textbooks as well as classic treatises. This book does not try to compete with the works of the old masters such as Ahlfors [**1, 2**], Hurwitz–Courant [**44**], Titchmarsh [**80**], Ahlfors–Sario [**3**], Nevanlinna [**67**], and Weyl [**88**]. Rather, it is intended as a fairly detailed, yet fast-paced introduction to those parts of the theory of one complex variable that seem most useful in other areas of mathematics (geometric group theory, dynamics, algebraic geometry, number theory, functional analysis).

There is no question that complex analysis is a cornerstone of a mathematics specialization at every university and each area of mathematics requires at least some knowledge of it. However, many mathematicians never take more than an introductory class in complex variables which ends up being awkward and slightly outmoded. Often this is due to the omission of Riemann surfaces and the assumption of a computational, rather than a geometric point of view.

The author has therefore tried to emphasize the intuitive geometric underpinnings of elementary complex analysis that naturally lead to Riemann surface theory. Today this is either not taught at all, given an algebraic slant, or is presented from a sophisticated analytical perspective, leaving the students without any foundation, intuition, historical understanding, let alone a working knowledge of the subject.

This book intends to develop the subject of Riemann surfaces as a natural continuation of the elementary theory without which basic complex analysis would indeed seem artificial and antiquated. At the same time, we do not overly emphasize the algebraic aspects such as applications to elliptic curves. The author feels that those students who wish to pursue this direction will be able to do so quite easily after mastering the material in this book. Because of this, as well as numerous other omissions (e.g., zeta, theta, and automorphic functions, Serre duality, Dolbeault cohomology) and the reasonably short length of the book, it is to be considered as an "intermediate introduction".

Partly due to the fact that the Chicago first year curriculum covers a fair amount of topology and geometry before complex analysis, this book assumes knowledge of basic notions such as homotopy, the fundamental group, differential forms, cohomology and homology, and from algebra we require knowledge of the notions of groups and fields, and some familiarity with the resultant of two polynomials (but the latter is needed only for the definition of the Riemann surfaces of an algebraic germ). However, for the most part merely the most elementary familiarity of these concepts is assumed and we collect the few facts that we do need in Appendix A. As far as analytical prerequisites are concerned, they are fairly low, not extending far beyond multi-variable calculus and basic Hilbert space theory (in Chapter 6 we use orthogonal projections). One exception to this occurs in Sections 3.3, 3.4, and 3.5, which use the weak and weak-$*$ topologies in L^p and the space of measures (Riesz representation theorem). Again, what we need is recalled in the appendix.

Let us now describe the contents of the individual chapters in more detail. Chapter 1 introduces the concept of differentiability over $\mathbb{C}$, the calculus of $\partial_z, \partial_{\bar{z}}$, the Cauchy-Riemann equations, power series, the Möbius (or fractional linear) transformations and the Riemann sphere. Applications of these transformations to hyperbolic geometry (the Poincaré disk and the upper half-plane models) are also discussed. In particular, we verify the Gauss-Bonnet theorem for this special case.

Next, we develop complex integration and Cauchy's theorem in various guises, as well as the Cauchy formula and estimates (with the fundamental theorem of algebra as an application), and then apply this to the study of analyticity, harmonicity, and the logarithm. We also prove Goursat's theorem, which shows that complex differentiability without continuity of the derivative already implies analyticity.

A somewhat unusual feature of this chapter is the order: integration theory and its basic theorems appear after Möbius transforms and applications in non-Euclidean geometry. The reason for this is that the latter can be considered to be more elementary, whereas it is hoped that the somewhat miraculous integration theory becomes more accessible to a student who has seen many examples of analytic functions. Finally, to the author it is essential that complex differentiability should not be viewed as an ad hoc extension of the "limit of difference quotients" definition from the real field to the complex field, but rather as a geometric property at the infinitesimal level: the linearization equals a rotation followed by a dilation, which are precisely the linear maps representing multiplication by a complex number. In other words, *conformality* (at least at non-degenerate points). If there is any one basic notion that appears in every chapter of this book, then it is that of a conformal transformation.

Chapter 2 begins with the winding number, and some brief comments about cohomology and the fundamental group. It then applies these concepts in the "global form" of the Cauchy theorem by extending the "curves that can be filled in without leaving the region of holomorphy" version of the Cauchy theorem, to zero homologous cycles, i.e., those cycles which do not wind around any point outside of the domain of holomorphy. We then classify isolated singularities, prove the Laurent expansion and the residue theorems with applications. More specifically, we derive the argument principle and Rouché's theorem from the residue theorem. After that, Chapter 2 studies analytic continuation—with a demonstration of how to proceed for the Γ-function—and presents the monodromy theorem. Then, we turn to convergence of analytic functions and normal families. This is applied to Mittag-Leffler's "partial fraction representation", and the Weierstrass product formula in the entire plane. The Riemann mapping theorem is proved, and the regularity at the boundary of Riemann maps is discussed. The chapter concludes with Runge's approximation theorem, as well as a demonstration of several equivalent forms of simple connectivity.

Chapter 3 studies harmonic functions in a wide sense, with particular emphasis on the Dirichlet problem on the unit disk. This means that we solve the boundary value problem for the Laplacian on the disk via the Poisson kernel. The Poisson kernel is also identified from its invariance properties under the automorphisms of the disk, and we sketch some basic probabilistic aspects as well. We then present the usual L^p-based Hardy classes of harmonic functions on the disk, and discuss the question of representing them via their boundary data both in the sense of L^p and the sense of "almost everywhere". A prominent role in this analysis is played by compactness ideas in functional analysis (weak-$*$ compactness of the unit ball, i.e., Alaoglu's theorem), as well as the observation that *positivity* can be substituted for compactness in many instances. This part therefore requires some analytical maturity, say on the level of Rudin's book [**73**]. However, up to the aforementioned basic tools from functional analysis, the presentation is self-contained.

We then sketch the more subtle theory of holomorphic functions in the Hardy class, or equivalently, of the boundedness properties of the conjugate harmonic functions, culminating in the classical F. & M. Riesz theorems.

The chapter also contains a discussion of the class of entire functions of exponential growth, the Jensen formula which relates zero counts to growth estimates, and the Hadamard product representation which refines the Weierstrass formula. We conclude with a gallery of conformal plots that will hopefully be both inspiring and illuminating.

The theory of Riemann surfaces begins with Chapter 4. This chapter covers the basic definition of such surfaces and of the analytic functions between them. Holomorphic and meromorphic functions are special cases where the target is either $\mathbb{C}$ or $\mathbb{C}P^1$ (the latter being conformally equivalent to the compactification of $\mathbb{C}$ obtained by "adding infinity"). The fairly long Section 4.2 introduces seven examples, or classes of examples, of Riemann surfaces. The first three are elementary and should be easily accessible even to a novice, but Examples 4)–7) are more involved and should perhaps only be attempted by a more experienced reader.

Example 4) shows that compact smooth orientable surfaces in $\mathbb{R}^3$ carry the structure of a Riemann surface, a fact of great historical importance to the subject. It means that we may carry out complex analysis on such surfaces rather than on the complex plane. The key idea here is that of *isothermal coordinates* on such a manifold, which reduces the metric to the one conformal to the standard metric. Example 5) discusses covering spaces, quotients etc., Example 6) is devoted to algebraic curves and how they are best viewed as Riemann surfaces. Example 7) presents Weierstrass' idea of looking for all possible analytic continuations of a power series and building a Riemann surface from this process.

After these examples, we investigate basic properties of functions on Riemann surfaces and how they relate to the topology of the surface as reflected, for example, by the genus in the compact case.

Elementary results such as the Riemann-Hurwitz formula relating the branch points to the genera of the surfaces are discussed. We then show how to define Riemann surfaces via discontinuous group actions and give examples of this procedure.

The chapter continues with a discussion of tori and some aspects of the classical theory of meromorphic functions on these tori. These functions are precisely the doubly periodic or elliptic functions. We develop the standard properties of the Weierstrass $\wp$ function, some of which foreshadow much more general facts which we will see in a much wider Riemann surface context in later chapters. We briefly discuss the connection between the Weierstrass function and the theory of integration of the square root of cubic polynomials (the so-called elliptic integrals).

In Section 4.7 the covering spaces of the doubly punctured plane are constructed and applied to Picard's small and big theorems, as well as the fundamental normality test of Montel. The chapter concludes with a discussion of groups of Möbius transforms, starting off with an analysis of the fixed points of maps in the automorphism group of the disk.

Then the *modular group* $\mathrm{PSL}(2,\mathbb{Z})$ is analyzed in some detail. We identify the fundamental region of that group, which implies, in particular, that the action of the group on the upper half-plane is discontinuous. As a particular example of an automorphic function, we introduce the basic modular function λ, which is constructed by means of the $\wp$ function. Remarkably, this function provides an explicit example of the covering map from Section 4.7.

Chapter 5 presents another way in which Riemann surfaces arise naturally, namely via analytic continuation. Historically, the desire to resolve unnatural issues related to "multi-valued functions" (most importantly for algebraic functions) led Riemann to introduce his surfaces. Even though the underlying ideas leading from a so-called analytic germ to its Riemann surface are geometric and intuitive, and closely related to covering spaces in topology, their rigorous rendition requires some patience as ideas such as "analytic germ", "branch point", "(un)ramified Riemann surface of an analytic germ", etc., need to be defined precisely. The chapter also develops some basic aspects of algebraic functions and their Riemann surfaces. At this point the reader will need to be familiar with basic algebraic constructions.

In particular, we observe that *every* compact Riemann surface is obtained through analytic continuation of some algebraic germ. This uses the machinery of Chapter 5 together with a potential-theoretic result that guarantees the *existence of a non-constant meromorphic function* on every Riemann surface. The reference to potential theory here means the we employ basic results on elliptic PDEs to obtain this (in fact, we will phrase the little we need in terms of harmonic functions and differentials).

This, as well as other fundamental existence results, is developed in Chapter 6. It turns out that differential forms are easier to work with on

Riemann surfaces than functions, and it is through forms that we construct functions. One of the reasons for this preference for forms over functions lies with the fact that it is meaningful to integrate 1-forms over curves, but not functions.

The chapter concludes with a discussion of ordinary differential equations with meromorphic coefficients. We introduce the concept of a *Fuchsian equation*, and illustrate this term by means of the example of the Bessel equation.

Chapter 6 introduces differential forms on Riemann surfaces and their integrals. Needless to say, the only really important class of linear forms are the 1-forms and we define harmonic, holomorphic and meromorphic forms and the residues in the latter case. Furthermore, the Hodge $*$ operator appears naturally (informally, it acts like a rotation by $\pi/2$). We then present some examples that lead up to the Hodge decomposition, which is established later in that chapter. This decomposition states that every 1-form can be decomposed additively into three components: a closed, co-closed, and a harmonic form (the latter being characterized as being simultaneously closed and co-closed). In this book, we follow the classical L^2-based derivation of this theorem. Thus, via Hilbert space methods one first derives this decomposition with L^2-valued forms and then uses Weyl's regularity lemma (weakly harmonic functions are smoothly harmonic) to upgrade to smooth forms.

Chapter 6 then applies the Hodge decomposition to establish some basic results on the existence of meromorphic differentials and functions on a general Riemann surface. In particular, we derive the striking fact that **every Riemann surface** carries a non-constant meromorphic function which is a key ingredient for the result on compact surfaces being algebraic in Chapter 5.

The chapter concludes with several examples of meromorphic functions and differentials on Riemann surfaces, mostly for the class of hyper-elliptic surfaces (compact surfaces that admit a meromorphic function of degree 2).

Chapter 7 presents the Riemann-Roch theorem which relates the dimension of certain spaces of meromorphic differentials with the dimension of a space of meromorphic functions, from properties of the underlying *divisor* and the genus of the compact Riemann surface. Before proving this theorem, which is of central importance both in historical terms as well as in applications, there are a number of prerequisites to be dealt with, such as a linear basis in the space of holomorphic differentials, the Riemann period relations, and the study of divisors.

Section 7.5 studies a diverse collection of applications of the Riemann-Roch theorem, such as the fact that every compact Riemann surface of genus g is a branched cover of S^2 with $g+1$ sheets, as well as the fact that surfaces of genus 2 only require 2 sheets (and are thus hyper-elliptic). Section 7.6 completes the identification of compact surfaces M as projective algebraic curves. Moreover, we show that every meromorphic function on such a surface M can be expressed by means of a *primitive pair* of meromorphic functions; see Theorem 7.24.

Section 7.7 discusses the *Abel and Jacobi theorems*. The former result identifies all possible divisors associated with meromorphic functions (the so-called *principal divisors*) on a compact Riemann surface by means of the vanishing of a certain function of the divisor modulo the period lattice. This implies, amongst other things, that every compact surface of genus 1 is a torus. For all genera $g \geq 1$ we obtain the surjectivity of the Jacobi map onto the Jacobian variety; in other words, we present the Jacobi inversion. In this chapter we omit the theta functions, which would require a chapter of their own.

Chapter 8 is devoted to the proof of the uniformization theorem. This theorem states that the only simply-connected Riemann surfaces (up to isomorphisms) are $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{C}P^1$. For the compact case, we deduce this from the Riemann-Roch theorem. But for the other two cases we use methods of potential theory which are motivated by the proof of the Riemann mapping theorem. In fact, we first reprove this result in the plane by means of a *Green function* associated with a domain.

The idea is then to generalize this proof strategy to Riemann surfaces. The natural question of when a Green function exists on a Riemann surface leads to the classification of non-compact surfaces as either hyperbolic (such as $\mathbb{D}$) or parabolic (such as $\mathbb{C}$); in the compact case a Green function cannot exist.

Via the Perron method, we prove the existence of a Green function for hyperbolic surfaces, thus establishing the conformal equivalence with the disk. For the parabolic case, a suitable substitute for the Green function needs to be found. We discuss this in detail for the simply-connected case, and also sketch some aspects of the non-simply-connected cases.

As in other key results in this text (equivalence between compact Riemann surfaces and algebraic curves, Riemann-Roch) the key here is to establish the existence of special types of functions on a given surface. In this context, the functions are harmonic (or meromorphic for the compact surfaces). Loosely speaking, the classification theorem then follows from the mapping properties of these functions.

Finally, Appendix A collects some of the material that arguably exceeds the usual undergraduate preparation which can be expected at the entry level to complex analysis. Naturally, this chapter is more expository and does not present many details. References are given to the relevant sources.

This text does perhaps assume more than other introductions to the subject. The author chose to present the material more like a landscape. Essential features that the reader encounters on his or her guided tour are pointed out as we go along. Since complex analysis does have to do with many basic features of mathematical analysis it is not surprising that examples can and should be drawn from different sources. The author hopes that students and teachers will find this to be an attractive feature.

How to use this book: On the largest scale, the structure is linear. This means that the material is presented progressively, with later chapters drawing on earlier ones. It is not advisable for a newcomer to this subject to "pick and choose". In the hands of an experienced teacher, though, such a strategy is to some extent possible. This will be also necessary with a class of varying backgrounds and preparation. For example, Sections 3.3–3.6 require previous exposure to basic functional analysis and measure theory, namely L^p spaces, their duals and the weak-$*$ compactness (Alaoglu's theorem). This is, however, the only instance where that particular background is required. If these sections are omitted, but Chapter 8 is taught, then the basic properties of subharmonic functions as presented in Section 3.5 will need to be discussed.

As far as functional analysis is concerned, of far greater importance to this text are rudiments of Hilbert spaces and L^2 spaces (but only some of the most basic facts such as completeness and orthogonal decompositions). These are essential for the Hodge theorem in Chapter 6.

As a general rule, all details are presented (with the exception of the appendix). On rare occasions, certain routine technical aspects are moved to the problem section which can be found at the end of each chapter. Some of the problems might be considered to be more difficult, but essentially all of them are to be viewed as an integral part of this text. As always in mathematics courses, working through at least some of the exercises is essential to mastering this material. References are not given in the main text since they disturb the flow, but rather collected at the end in the "Notes". This is the same format employed in the author's books with Camil Muscalu [**65**]. By design, this text should be suitable for both independent—but preferably guided—study and the traditional classroom setting. A well-prepared student will hopefully be able to read the eight main chapters in linear succession, occasionally glancing at the appendix if needed.

The main motivation for writing this book was to bridge a gap in the literature, namely between the introductory complex analysis literature such as Lang [**55**], and to a lesser extent perhaps Ahlfors [**1**] on the one hand, and on the other hand, well-established pure Riemann surface texts such as Forster [**29**], Farkas, Kra [**23**]. Ideally, this book could serve as a stepping stone into more advanced texts such as [**23**], as well as the recent ones by Donaldson [**18**] and Varolin [**84**]. The author hopes that the somewhat higher-level machinery that is used in the latter two books (complex line bundles, Serre duality, etc.) will become more natural as well as more easily accessible after the classical approach, which we employ here, has been understood.

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