A Modern Theory of Integration

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Preface

It is hardly possible to overemphasize the importance of the theory of integration to mathematical analysis; indeed, it is one of the twin pillars on which analysis is built. Granting that, it is surprising that new developments continue to arise in this theory, which was originated by the great Newton and Leibniz over three centuries ago, made rigorous by Riemann in the middle of the nineteenth century, and extended by Lebesgue at the beginning of the twentieth century.

The purpose of this monograph is to present an exposition of a relatively new theory of the integral (variously called the "generalized Riemann integral", the "gauge integral", the "Henstock-Kurzweil integral", etc.) that corrects the defects in the classical Riemann theory and both simplifies and extends the Lebesgue theory of integration. Not wishing to tell only the easy part of the story, we give here a complete exposition of a theory of integration, initiated around 1960 by Jaroslav Kurzweil and Ralph Henstock. Although much of this theory is at the level of an undergraduate course in real analysis, we are aware that some of the more subtle aspects go slightly beyond that level. Hence this monograph is probably most suitable as a text in a first-year graduate course, although much of it can be readily mastered by less advanced students, or a teacher may simply skip over certain proofs.

The principal defects in the Riemann integral are several. The most serious one is that the class of Riemann integrable functions is too small. Even in calculus courses, one needs to extend the integral by defining "improper integrals", either because the integrand has a singularity, or because the interval of integration is infinite. In addition, by taking pointwise limits of Riemann integrable functions, one quickly encounters functions that are no longer Riemann integrable. Even when one requires uniform convergence, there are problems on infinite intervals.

Other difficulties center around the Fundamental Theorem(s) of Calculus. The Newton-Leibniz formula that we learn in calculus is that

$$\int_a^x f(t)\,dt = F(x) - F(a) \qquad \text{for all} \quad x \in [a,b],$$

when f and F are related by the formula $F'(x) = f(x)$ for all $x \in [a,b]$; that is, when F is a *primitive* (or *antiderivative*) of f on $[a,b]$. Unfortunately, this "theorem" is not always valid; or at least, it requires further hypotheses to be satisfied. The first disappointment a student encounters is that not every Riemann integrable function has a primitive — not only that he or she can't find one, but that such a primitive may not exist. The second potential disappointment (often not learned), is that even when a function has a primitive on $[a,b]$, the function may not be Riemann integrable. Thus, not only is the derivative of the integral not always the function in the integrand (which is perhaps not such a surprise if integration is to be a "smoothing process"), but the integral of the derivative does not always exist.

Towards the end of the nineteenth century, many mathematicians attempted to remedy some of these defects. The most successful was Henri Lebesgue, whose theory enabled one to remove the restriction that the integrand be bounded and the interval be compact. In addition his theory enlarged the class of integrable functions, and gave more satisfactory conditions under which one could take limits, or differentiate under the integral sign.

Unfortunately, Lebesgue's theory did little to simplify the Fundamental Theorem. Spurred by the desire to get an integral in which every derivative was integrable, in the early part of the twentieth century Arnaud Denjoy and Oskar Perron developed integrals that solved this problem — in two very different ways. Surprisingly, their integrals turned out to be equivalent! Moreover, the Denjoy-Perron integrable functions also include conditionally convergent integrals, such as the important Dirichlet integral

$$\int_0^\infty \frac{\sin x}{x}\,dx,$$

that are not included in the Lebesgue theory.

However, there is a price that had to be paid even for the Lebesgue integral: one must first construct a rather considerable theory of measure of sets in $\mathbb{R}$. Consequently, it has long been thought that an adequate theory of integration is necessarily based on notions that are beyond the undergraduate level of real analysis. (The demands imposed by the Denjoy or Perron theories are considerably greater!) However, Kurzweil and Henstock's integral, which is equivalent to the Denjoy-Perron integral, has a definition that is a slight modification of the definition of the classical Riemann integral. This new integral, which is still not well known, often comes as a surprise to mathematicians whose work is based on the Lebesgue theory.

One of the virtues of the presentation here is that *no measure theory* and virtually *no topology* is required. While some familiarity with the Riemann theory is anticipated as a background, we do not require a mastery of that theory. The only prerequisites are that the reader have good understanding of ε-δ arguments common in a first serious course in real analysis — at the level of the book by the author and D. R. Sherbert [B-S], for example. It will be seen that, by modifying very slightly the definition of the Riemann integral, one obtains an integral that (1) integrates all functions that have primitives, (2) integrates all Riemann integrable functions, (3) integrates all Lebesgue integrable functions, (4) integrates (without further limiting processes) all functions that can be obtained as "improper integrals", and (5) integrates all Denjoy-Perron integrable functions. In addition, this integral has theorems that generalize the Monotone Convergence Theorem and the Dominated Convergence Theorems associated with the Lebesgue integral; thus, it possesses satisfactory convergence theorems.

Although the author has long been familiar with the Riemann and Lebesgue integrals, he has become acquainted only recently with the theory presented here by reading the (relatively) few expositions of it. Most notable of these are: the monograph of McLeod [McL], the relevant chapters in the book of DePree and Swartz [DP-S], the booklets of Henstock [H-5] and P.-Y. Lee [Le-1] and the treatise of Mawhin [M]. In addition, some research articles have been found to be useful to the author. Since work on this monograph was started, the books of Gordon [G], Pfeffer [P], Schechter [Sch] and Lee and Výborný [L-V] have been published; we strongly recommend these books. The author makes few claims for originality, and will be satisfied if this monograph is successful in helping to make this theory better known to the mathematical world.

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In answer to questions about the title of the book, we chose the word "modern" to suggest that we think the theory given here is appropriate for present-day students who will need to combine important concepts from the past with their new ideas. It is not likely that these students will be able to make significant progress in analysis by successive abstraction or further axiomatization. It is our opinion that a student who thinks of the integral only as a linear functional on a class of functions, but who doesn't know what AC and BV mean has been deprived of fundamental tools from the past. We also think that those whose integration theory does not include the Dirichlet integral are doomed to miss some of the most interesting parts of analysis.

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A few words about the structure of this book are in order. We have chosen to develop rather fully the theory of the integral of functions defined on a compact interval in Part 1, since we think that is the case of greatest interest to the student. In addition, this case does not exhibit some of the technical problems that, in our opinion, only distract and impede the understanding of the reader. In Part 2, we show that this theory can be extended to functions defined on all of the real line. We then develop the theory of Lebesgue measure from the integral, and we make a connection with some of the traditional approaches to the Lebesgue integral.

We believe that the generalized Riemann integral provides a good background for integration theory, since the class of integrable functions is so inclusive. However, there is no doubt that the collection of Lebesgue integrable (i.e., absolutely integrable) functions remains of central importance for many applications. Therefore, we have taken pains to ensure that this class of functions is thoroughly discussed. We have developed the theory sufficiently far that, after reading this book, a reader should be able to continue a study of some of the more specialized (or more general) aspects of the theory of integration, or the applications of the integral to other parts of mathematical analysis.

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Since we believe that one learns best by doing, we have included a large collection of exercises; some are very easy and some are rather demanding. Partial solutions of almost one-third of these exercises are given in the back of the book. A pamphlet, designed for instructors, with partial solutions of all of the exercises can be obtained from the publisher.

In preparing this manuscript, we have obtained useful suggestions from a number of people; we wish to thank Professors Nicolae Dinculeanu, Ivan Dobrakov, Donald R. Sherbert and, especially, Eric Schechter. Two groups of students at Eastern Michigan University worked through the early stages of the initial material and made useful suggestions.

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