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# Preface

This book is intended to introduce the reader into the world of differential forms and, at the same time, to cover those topics from analysis, differential geometry, and mathematical physics to which forms are particularly relevant. It is based on several graduate courses on analysis and differential geometry given by the second author at Humboldt University in Berlin since the beginning of the eighties. From 1998 to 2000 the authors taught both courses jointly for students of mathematics and physics, and seized the opportunity to work out a self-contained exposition of the foundations of differential forms and their applications. In the classes accompanying the course, special emphasis was put on the exercises, a selection of which the reader will find at the end of each chapter. Approximately the first half of the book covers material which would be compulsory for any mathematics student finishing the first part of his/her university education in Germany. The book can either accompany a course or be used in the preparation of seminars.

We only suppose as much knowledge of mathematics as the reader would acquire in one year studying mathematics or any other natural science. From linear algebra, basic facts on multilinear forms are needed, which we briefly recall in the first chapter. The reader is supposed to have a more extensive knowledge of calculus. Here, the reader should be familiar with differential calculus for functions of several variables in euclidean space  $\mathbb{R}^n$ , the Riemann integral and, in particular, the transformation rule for the integral, as well as the existence and uniqueness theorem for solutions of ordinary differential equations. It is a reader with these prerequisites that we have in mind and whom we will accompany into the world of vector analysis, Pfaffian

systems, the differential geometry of curves and surfaces in euclidean space, Lie groups and homogeneous spaces, symplectic geometry and mechanics, statistical mechanics and thermodynamics and, eventually, electrodynamics.

In Chapter 2 we develop the differential and integral calculus for differential forms defined on open sets in euclidean space. The central result is Stokes' formula turning the integral of the exterior derivative of a differential form over a singular chain into an integral of the form itself over the boundary of the chain. This is in fact a far-reaching generalization of the main theorem of differential and integral calculus: differentiation and integration are mutually inverse operations. At the end of a long historical development mathematicians reached the insight that a series of important integral formulas in vector analysis can be obtained by specialization from Stokes' formula. We will show this in the second chapter and derive in this way Green's first and second formula, Stokes' classical formula, and Cauchy's integral formula for complex differentiable functions. Furthermore, we will deduce Brouwer's fixed point theorem from Stokes' formula and the Weierstrass approximation theorem.

In Chapter 3 we restrict the possible integration domains to "smooth" chains. On these objects, called manifolds, a differential calculus for functions and forms can be developed. Though we only treat submanifolds of euclidean space, this section is formulated in a way to hold for every Riemannian manifold. We discuss the concept of orientation of a manifold, its volume form, the divergence of a vector field as well as the gradient and the Laplacian for functions. We then deduce from Stokes' formula the remaining classical integral formulas of Riemannian geometry (Gauss-Ostrogradski formula, Green's first and second formula) as well as the Hairy Sphere theorem, for which we decided to stick to its more vivid German name, "Hedgehog theorem". A separate section on the Lie derivative of a differential form leads us to the interpretation of the divergence of a vector field as a measure for the volume distortion of its flow. We use the integral formulas to solve the Dirichlet problem for the Laplace equation on the ball in euclidean space and to study the properties of harmonic functions on  $\mathbb{R}^n$ . For these we prove, among other things, the maximum principle and Liouville's theorem. Finally we discuss the Laplacian acting on forms over a Riemannian manifold, as well as the Hodge decomposition of a differential form. This is a generalization of the splitting of a vector field with compact support in  $\mathbb{R}^n$  into the sum of a gradient field and a divergence-free vector field, going back to Helmholtz. In the final chapter we prove Helmholtz' theorem within the framework of electrodynamics.

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Apart from Stokes' theorem, the integrability criterion of Frobenius is one of the fundamental results in the theory of differential forms. A geometric distribution (Pfaffian system) is defined by choosing a  $k$ -dimensional subspace in each tangent space of an  $n$ -dimensional manifold in a smooth way. A geometric distribution can alternatively be described as the zero set of a set of linearly independent 1-forms. What one is looking for then is an answer to the question of whether there exists a  $k$ -dimensional submanifold such that, at each point, the tangent space coincides with the value of the given geometric distribution. Frobenius' theorem gives a complete solution to this question and provides a basic tool for the integration of certain systems of first order partial differential equations. Therefore, Chapter 4 is devoted to a self-contained and purely analytical proof of this key result, which, moreover, will be needed in the sections on surfaces, symplectic geometry, and completely integrable systems.

Chapter 5 treats the differential geometry of curves and surfaces in euclidean space. We discuss the curvature and the torsion of a curve, Frenet's formulas, and prove the fundamental theorem of the theory of curves. We then turn to some special types of curves and conclude this section by a proof of Fenchel's inequality. This states that the total curvature of a closed space curve is at least  $2\pi$ . Surface theory is treated in Cartan's language of moving frames. First we describe the structural equations of a surface, and then we prove the fundamental theorem of surface theory by applying Frobenius' theorem. The latter is formulated with respect to a frame adapted to the surface and the resulting 1-forms. Next we start the tensorial description of surfaces. The first and second fundamental forms of a surface as well as the relations between them as expressed in the Gauss and the Codazzi-Mainardi equations are the central concepts here. We reformulate the fundamental theorem in this tensorial description of surface theory. Numerous examples (surfaces of revolution, general graphs and, in particular, reliefs, i. e. the graph of the modulus of an analytic function, as well as the graphs of their real and imaginary parts) illustrate the differential-geometric treatment of surfaces in euclidean space. We study the normal map of a surface and are thus lead to its Gaussian curvature, which by Gauss' Theorema Egregium belongs to the inner geometry. Using Stokes' theorem, we prove the Gauss-Bonnet formula and an analogous integral formula for the mean curvature of a compact oriented surface, going back to Steiner and Minkowski. An important class of surfaces are minimal surfaces. Their normal map is always conformal, and this observation leads to the so-called Weierstrass formulas. These describe the minimal surface locally by a pair of holomorphic functions. Then we turn to the study of geodesic curves on surfaces, the integration of the geodesic flow using first integrals as well as the investigation

of maps between surfaces. Chapter 5 closes with an outlook on the geometry of pseudo-Riemannian manifolds of higher dimension. In particular, we look at Einstein spaces, as well as spaces of constant curvature.

Symmetries play a fundamental role in geometry and physics. Chapter 6 contains an introduction into the theory of Lie groups and homogeneous spaces. We discuss the basic properties of a Lie group, its Lie algebra, and the exponential map. Then we concentrate on proving the fact that every closed subgroup of a Lie group is a Lie group itself, and define the structure of a manifold on the quotient space. Many known manifolds arise as homogeneous spaces in this way. With regard to later applications in mechanics, we study the adjoint representation of a Lie group.

Apart from Riemannian geometry, symplectic geometry is one of the essential pillars of differential geometry, and it is particularly relevant to the Hamiltonian formulation of mechanics. Examples of symplectic manifolds arise as cotangent bundles of arbitrary manifolds or as orbits of the coadjoint representation of a Lie group. We study this topic in Chapter 7. First we prove the Darboux theorem stating that all symplectic manifolds are locally equivalent. Then we turn to Noether's theorem and interpret it in terms of the moment map for Hamiltonian actions of Lie groups on symplectic manifolds. Completely integrable Hamiltonian systems are carefully discussed. Using Frobenius' theorem, we demonstrate an algorithm for finding the action and angle coordinates directly from the first integrals of the Hamilton function. In §7.5, we sketch the formulations of mechanics according to Newton, Lagrange, and Hamilton. In particular, we once again return to Noether's theorem within the framework of Lagrangian mechanics, which will be applied, among others, to integrate the geodesic flow of a pseudo-Riemannian manifold. Among the exercises of Chapter 7, the reader will find some of the best known mechanical systems.

In statistical mechanics, particles are described by their position probability in space. Therefore one is interested in the evolution of statistical states of a Hamiltonian system. In Chapter 8 we introduce the energy and information entropy for statistical equilibrium states. Then we characterize Gibbs states as those of maximal information entropy for fixed energy, and prove that the microcanonical ensemble realizes the maximum entropy among all states with fixed support. By means of the Gibbs states, we assign a thermodynamical system in equilibrium to a Hamiltonian system with auxiliary parameters satisfying the postulates of thermodynamics. We discuss the

role of pressure and free energy. A series of examples, like the ideal gas, solid bodies, and cycles, conclude Chapter 8.

Chapter 9 is devoted to electrodynamics. Starting from the Maxwell equations, formulated both for the electromagnetic field strengths and for the dual 1-forms, we first deal with the static electromagnetic field. We prove the solution formula for the inhomogeneous Laplace equation in three-space and obtain, apart from a description of the electric and the magnetic field in the static case, at the same time a proof for Helmholtz' theorem as mentioned before. Next we turn to the vacuum electromagnetic field. Here we prove the solution formula for the Cauchy problem of the wave equation in dimensions two and three. The chapter ends with a relativistic formulation of the Maxwell equations in Minkowski space, a discussion of the Lorentz group, the Maxwell stress tensor and a thorough treatment of the Lorentz force.

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The English version at hand does not differ by much from the original German edition. Besides small corrections and additions, we included a detailed discussion of the Lorentz force and related topics in Chapter 9. Finally, we thank Dr. Andreas Nestke for his careful translation.

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