# Preface

The goal of this textbook is to provide an introduction to the language and methods of functional analysis, including chapters on Hilbert spaces, operator theory, basic theorems and methods of abstract functional analysis and a few applications of these methods to Banach algebras and the theory of unbounded self-adjoint operators.

The text represents notes for a series of two courses (12 to 14 weeks each; 3 hours weekly and 1 hour of exercises and discussions for both courses):

(i) Introduction to Hilbert spaces and operator theory (Chapters 1–7),
(ii) Introduction to functional analysis (Chapters 8–11).

I gave these courses for many years at Tel Aviv University and also, once in 1995, at the Ohio State University (OSU), Columbus. That one time at OSU was a very lucky time for me, because my then Ph.D. student Antonis Tsolomitis worked on my very rough notes of the lectures and suggested creating this book. He was indispensable in his rôle in our joint effort, and the book would not have come to publication without his agreement to join me in writing it.

Another stroke of luck came my way with the huge wave of emigration of Russian mathematicians in the 1990's. Among them was Dr. Yuli Eidelman, who at the start of his career here had time to assist me in the courses. He prepared exercises and material suitable for discussions. So, the final textbook is the result of the efforts of all three of us, Tony, Yuli and myself.

A few words about functional analysis and some necessary background: one very important goal of mathematics is to develop a language, a so-called "mathematical language". Firstly, it is needed for the very precise exchange of thoughts, and secondly—and not less importantly—we develop the terminology which fixes our understanding and catches new mathematical observations and laws. This continuously developing terminology "compresses" achievements of the previous stage of development of mathematics into "spoken language". Once deep theorems become a language and we no longer (need to) think of them as theorems, it helps us to "free" our minds in preparation for a new portion of mathematics.

(For example: we say "a linear space of dimension $n$", but a deep theorem stands behind this sentence. The notion of "dimension" is a theorem which students study in a linear algebra course.)

The most important rôle of functional analysis was to develop a mathematical language. Functional analysis became the language of twetnieth century mathematics (more precisely its part called analysis) and theoretical physics. Even articles on popular science and science fiction books use this language and talk about "operators" and their "spectrum".

To teach students to speak in this language is the main goal of this textbook. Numerous theorems (sometimes very short in their proofs) should help us in the end to feel comfortable with new notions, to get used to them, to speak new words without painful efforts to recall what they mean. They should become a part of the reader's mathematical culture.

We would like to emphasize that we took special care to be brief and not to overload the students (and other readers) with the enormous amount of information available on the subject. Over the years we have checked that the amount of mathematics presented in this course is absorbable in a year's study and provides the basis for future reading.

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# Introduction

As mentioned in the Preface, the text corresponds to two courses (12 to 14 weeks each; 3 hours weekly and 1 hour of exercises and discussions for both courses):

(i) Introduction to Hilbert spaces and operator theory (Chapters 1–7),
(ii) Introduction to functional analysis (Chapters 8–11).

These courses require only the knowledge from any first course in Linear Algebra. However, for the second course it would be useful if the reader had some knowledge of measure theory.

The book does not contain any "additional" chapters. Material, although important but which cannot be condensed into these two courses in the time available, without overloading the reader's ability to digest new notions and facts, is not included.

The reader may look at the bibliography for books that complement the material of this text. Moreover, one can see there other possibilities for presenting the same results.

In Chapter 1 we introduce linear spaces and normed spaces and we give some first, but important, examples. The spaces $L_p[a, b]$ are introduced through the completion of the continuous functions with the $L_p$-norm in order to avoid requiring the knowledge of measure theory from the reader.

In the second chapter, Hilbert spaces are introduced and we prove basic facts about them. Linear functionals are also introduced in this chapter, which closes with a natural example of a non-separable Hilbert space.

Chapter 3 discusses the notion of the dual Banach spaces. The Hahn-Banach theorem is stated here without proof and with the standard corollaries, needed for the rest of the course. The Hahn-Banach theorem is proved in the second course, in Chapter 9.

Chapter 4 introduces the bounded linear operators, the compact operators, the dual operators and the invertible operators. We also discuss a different kind of convergence in the space of bounded operators. Here we state the open mapping theorem and we postpone its proof until Chapter 9 (the Banach-Steinhaus theorem is not stated before Chapter 9 since it can certainly be avoided until then).

Chapter 5 is on spectral theory for the general operator. The classification of spectrum is discussed here as well as the development of Fredholm theory.

In Chapters 6 and 7 we focus our attention on the spectral theory of self-adjoint operators. The spectral integral is also given here.

Although Chapter 8, on the spectral theory of unitary operators, would fit more naturally into the first part of this course, we advise postponing dealing with it until Chapter 11, on unbounded symmetric operators. One reason is that the concepts of spectral theory and the spectral integral are not easy to absorb at first, and it is worthwhile returning later to basics to tackle them. Another reason is that the spectral theory of unbounded self-adjoint operators and the Cayley transform naturally begin with the understanding of unitary operators.

Chapter 9 contains the general, more classical results which form the base and the methods of functional analysis. Besides the main theorems of the theory and the central notions of weak and weak* topology, we selected a number of "branch" results, with a twofold goal in mind. We want to demonstrate how the method works and we want to get used to the notion and language of functional analysis; and we also want to use this opportunity to introduce additional important notions and enlarge the picture.

We selected the two remaining chapters of the course for the following reasons. First, we want to show that, by adding natural structure to the basic notion of Banach space, we quickly derive deep, rich and very concrete analytic consequences. So, Chapter 10 provides an introduction to Gelfand's beautiful theory of Banach algebras.

At this stage, the reader may get the feeling that the whole of functional analysis is about "good", "well-organized" objects. To remove this misapprehension and to show that the methods of functional analysis can deal with less "good" objects, e.g., unbounded operators, we consider unbounded symmetric operators and present the spectral theory of self-adjoint (unbounded) operators in Chapter 11.

There are many books written on this subject. Some of them are strictly textbooks, and some are more of a monograph type. We mention some of them in the bibliography at the end of the book. We would like to mention especially the books [GGK03] by I. Gohberg, S. Goldberg and M.A. Kaashoek and [AKR78] by A.B. Antonevich, P.N. Knyazev and Ya.V. Radyno, which were helpfull for us when preparing some of the exercises. We recommend these books for additional reading.

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