Chapter 1

An Introductory Problem

1.1. Introduction and heuristic considerations

In this part we consider free boundary problems of the following type.

In the ball $B_1 = B_1(0)$ we have a continuous function u satisfying

(a) $\Delta u = 0$ in $\Omega^+(u) := \{u > 0\}$ and in $\Omega^-(u) := \{u \le 0\}^0$,

(b) the flux balance

$$G(u_\nu^+, u_\nu^-) = 0 \tag{1.1}$$

across $F(u) := \partial\Omega^+(u) \cap B_1$, the free boundary.

In (1.1), u_ν^+ and u_ν^- denote the normal derivatives in the inward direction to $\Omega^+(u)$ and $\Omega^-(u)$, respectively, so that $u_\nu^\pm$ are both nonnegative.

The simplest example of this type of problems arises in the minimization of the variational integral

$$J_0(u) = \int_{B_1} \left(|\nabla u|^2 + \chi_{\{u>0\}}\right) \, dx \tag{1.2}$$

that appears in many applications (e.g., in jet flows, see [AC], [ACF1], [ACF2], [ACF3]).

Suppose u is a local minimizer and assume that the free boundary is differentiable (say) at the origin.

Since

$$u_\lambda(x) = \frac{1}{\lambda} u(\lambda x)$$

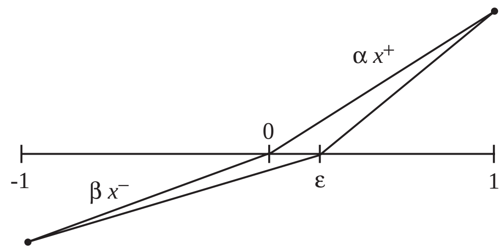

Figure 1.1

is again a local minimizer, that is, the problem is invariant under "elliptic" dilations if we let λ go to zero, we could expect to "guess" the free boundary condition by examining one-dimensional, linear solutions of the type

$$u(x) = \alpha x^+ - \beta x^- \qquad (\alpha, \beta > 0)$$

in $[-1, 1]$.

If we perturb the origin by ε ($\varepsilon \gtrless 0$), the Dirichlet integral

$$\int_{-1}^{1} (u')^2 \, dx$$

changes from $\alpha^2 + \beta^2$ to

$$\frac{\alpha^2}{(1-\varepsilon)^2}(1-\varepsilon) + \frac{\beta^2}{(1+\varepsilon)^2}(1+\varepsilon) = \frac{\alpha^2}{1-\varepsilon} + \frac{\beta^2}{1+\varepsilon} \approx \alpha^2 + \beta^2 + \varepsilon(\alpha^2 - \beta^2)$$

while the volume integral $\int_{-1}^{1} \chi_{\{u>0\}} \, dx$ changes from 1 to $1 - \varepsilon$. The "Euler equation" at the origin is therefore

$$\alpha^2 - \beta^2 - 1 = 0;$$

i.e., in this case we expect

$$G(u_\nu^+, u_\nu^-) = (u_\nu^+)^2 - (u_\nu^-)^2 = 1 \ . \tag{1.3}$$

Another way to recover, at least formally, the free boundary condition (1.3) is to use the classical Hadamard formula. Indeed, if we assume that $F(u)$ is smooth and perturb it inward w.r.t. $\Omega^+(u)$ around a point $x \in F(u)$, so that $|\Omega^+(u)|$ decreases by $\delta(\text{Vol})$, then, from Hadamard's formula we get that $\int_{B_1} |\nabla u^+|^2$ increases by an amount $(u_\nu^+)\delta(\text{Vol})$ while $\int_{B_1} |\nabla u^-|^2$ decreases by $(u_\nu^-)^2\delta(\text{Vol})$.

Thus, the minimization condition implies

$$(u_\nu^+)^2 - (u_\nu^-)^2 \geq 1 \ .$$

An inward perturbation w.r.t. $\Omega^-(u)$ would give the opposite inequality so that, on $F(u)$, the "Euler equation" for J_0 is exactly (1.3).

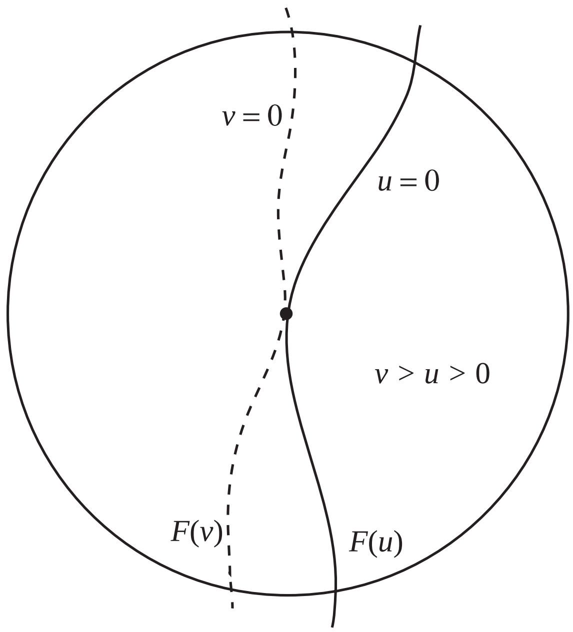

Figure 1.2

The above considerations show a sort of "stability" property of the free boundary relation: if we perturb it towards the positive (negative) region, u_ν^+ tends to increase (decrease), so that u deviates from being a solution.

Referring to the general free boundary condition (1.1), this behavior is reflected in an "ellipticity" condition on G that can be motivated in terms of a comparison principle, like maximum principles.

Consider, for instance, the equation

$$H(D^2u) = 0 \ .$$

One way of requiring its ellipticity is by asking for a *strict comparison principle*: what is the natural condition on H that prevents two (smooth) solutions u and v ($H(D^2u) = H(D^2v) = 0$) from becoming into contact, i.e.,

$$u \geq v \quad \text{and} \quad u(x_0) = v(x_0) \ ?$$

At x_0 we have $D^2u(x_0) \geq D^2v(x_0)$, but, suppose (poetic license...)

$$D^2u(x_0) > D^2v(x_0) \ .$$

Thus, if we ask that H be strictly increasing as a function of symmetric matrices, we conclude

$$H(D^2u(x_0)) > H(D^2v(x_0)) \ ,$$

a contradiction. In other words, monotonicity in H with respect to D^2u is "an ellipticity condition".

Suppose now that u and v are solutions of the same free boundary problem, with $u \geq v$. Since away from their free boundaries $F(u)$ and $F(v)$, respectively, u and v are harmonic and hence cannot touch, the analog of the question above is: what will exclude the possibility that $F(u)$ and $F(v)$ can touch at a free boundary point (say $x_0 = 0$)?

The Hopf maximum principle gives, at $x_0 = 0$,

$$u_\nu^+ > v_\nu^+ \quad \text{and} \quad u_\nu^- < v_\nu^- \ .$$

Therefore, if $G = G(a, b)$ is strictly increasing w.r.t. a and strictly decreasing w.r.t. b, the possibility that both

$$G(u_\nu^+, u_\nu^-) = G(v_\nu^+, v_\nu^-) = 0$$

is excluded.

Definition 1.1. The free boundary relation (1.1) is *elliptic* if $G = G(a, b)$ is strictly increasing w.r.t. a and strictly decreasing w.r.t. b.

The one-dimensional computation done above indicates also an important difference between one-phase and two-phase problems. In one-phase problems it is possible to get universal interior bounds, in the sense that, if u is a solution in a ball $B_1(0)$ and $0 \in F(u)$, then $|\nabla u|$ or even D^2u (as in the case of the obstacle and the one-phase Stefan problem) is bounded in $B_{1/2}(0)$ by a universal constant, no matter what the boundary data are.

In two-phase problems this is, in general, impossible. For instance, in the one-dimensional minimization problem, by raising the boundary data, one can have a large gradient of the solution near the origin.

1.2. A one-phase singular perturbation problem

Let us go back to the minimization of the functional (1.2). If we put boundary data

$$u_{|\partial B_1} = f \geq 0,$$

then the solution u will be nonnegative and we are dealing with a one-phase problem. We will discuss this problem in full detail in order to introduce some of the main ideas and techniques, useful also in a more general context.

We shall consider minimizers constructed as limits of singular perturbations because in this case the technique can be somewhat simplified. Observe that the problem has no uniqueness so there could exist other types of solutions (see [AC]). All the theory can be developed anyway for any minimizer of J_0 ([AC]).

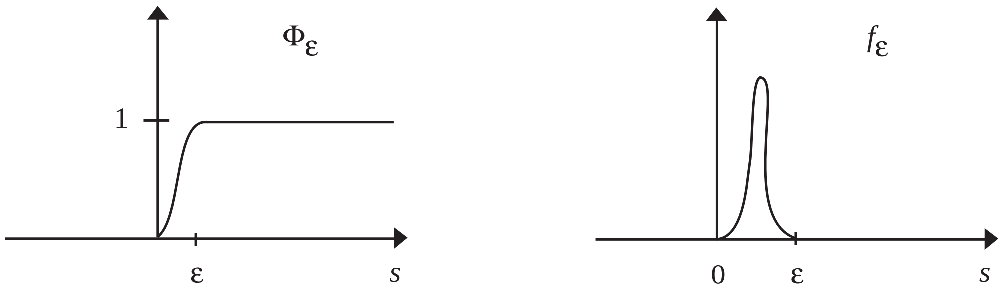

Figure 1.3

So we start studying the minimizers v_ε of the variational integral

$$J_\varepsilon(u) = \int_{B_1} \{|\nabla u|^2 + \Phi_\varepsilon(u)\}\, dx$$

where Φ_1 is a smooth nondecreasing function on the real line with $\Phi_1(s) = 0$ for $s \le 0$, $\Phi_1(s) = 1$ for $s \ge 1$, and $\Phi_\varepsilon(s) = \Phi_1(s/\varepsilon)$. Therefore,

$$f_\varepsilon(s) = \Phi_\varepsilon'(s) = \frac{1}{\varepsilon}\Phi_1'\Big(\frac{s}{\varepsilon}\Big) = \frac{1}{\varepsilon} f_1\Big(\frac{s}{\varepsilon}\Big)$$

is an approximation of the Dirac measure.

Proposition 1.1. *Given $g \in H^1(B_1)$, there exists a minimizer $u \in H^1(B_1)$ with $u - g \in H_0^1(B_1)$.*

A minimizer u_ε satisfies the Euler equation

$$2\Delta u = f_\varepsilon(u) \tag{1.4}$$

and so, since f_ε is smooth, it is a smooth solution. Since $g_{|\partial B_1} \ge 0$ and f_ε is supported in $0 < u < \varepsilon$, by the maximum principle, u_ε is nonnegative.

We start now with optimal regularity and nondegeneracy. Given the jump conditions along the free boundary, the optimal global regularity we can expect from the limiting solution is Lipschitz continuity and this is what we prove for u_ε.

Theorem 1.2. *Let $\varepsilon < 1/3$ and suppose $u_\varepsilon(0) = \varepsilon$. Then there exists a (universal) constant C_0 such that*

$$\|\nabla u_\varepsilon\|_{L^\infty(B_{1/2}(0))} \le C_0.$$

At this stage it is important to emphasize a renormalization property of the problem. If u is a solution of equation (1.4) in $B_R(x_0)$, then the renormalization of u given by

$$w(y) = \frac{1}{\varepsilon} u(x_0 + \varepsilon y) \tag{1.5}$$

satisfies the equation

$$\Delta w = \frac{1}{2} f_1(w) \quad \text{in} \quad B_{R/\varepsilon}(0)$$

where, notice, $0 \le f_1(w) \le 1$.

Moreover, observe that $\nabla w(0) = \nabla u(x_0)$. Thus, proving that u is Lipschitz in $B_1(x_0)$ is equivalent to proving that w is Lipschitz in $B_{R/\varepsilon}(0)$. When a property is invariant under the rescaling (1.5), we say that it is a "renormalization property". So, Lipschitz continuity is a renormalization property in this sense.

The next two lemmas show two of these renormalization properties. Since they express general facts, useful in many other situations, we state them in a renormalized form. Theorem 1.2 is then an immediate consequence.

Lemma 1.3. *Let $w \ge 0$ be a solution of*

$$|\Delta w| \le C \quad in \quad B_2(0).$$

If $w(0) \le 1$, then

$$|\nabla w(0)| \le C_0.$$

Proof. Interior Shauder estimates and Harnack's inequality give

$$|\nabla w(0)| \le \bar{C}(C + \|w\|_{L^\infty(B_1(0))}) \le \bar{C}(C+1) \equiv C_0. \qquad \square$$

Corollary 1.4. *If $x \in B_{1/2}$ and $0 \le u_\varepsilon(x) \le \varepsilon$, then*

$$|\nabla u_\varepsilon(x)| \le C\ .$$

Corollary 1.4 shows Lipschitz continuity in the region where $0 \le u_\varepsilon \le \varepsilon$. Applying the next lemma to $v = u - \varepsilon$, we take care of the region Ω_ε where $u > \varepsilon$.

Lemma 1.5. *Let Ω be an open set with $0 \in \partial\Omega$ and let v be harmonic and nonnegative in $B_1 \cap \Omega$. Suppose that on $\Gamma \equiv \partial\Omega \cap B_1$, $v = 0$ and $|\nabla v|$ is bounded. Then for every $x \in B_{1/2}$*

(a) $v(x) \le C\, d(x, \partial\Omega) \sup_\Gamma |\nabla v|$,

(b) $\|\nabla v\|_{L^\infty(B_{1/2} \cap \Omega)} \le C \sup_\Gamma |\nabla v|$.

Proof. Let $x_0 \in B_{1/2} \cap \Omega$, $h = d(x_0, \partial\Omega)$ and $A = \sup_\Gamma |\nabla v|$. Suppose

$$v(x_0) = \lambda h\ .$$

We want to show that $\lambda \le CA$. Rescale v in $B_h(x_0)$ by putting

$$w(y) = \frac{1}{h} v(x_0 + hy)\ .$$

Then w is harmonic and nonnegative in $B_1(0)$ with $w(0) = \lambda$. Since $h = d(x_0, \partial\Omega)$, there exists $y_1 \in \partial B_1(0)$ at which $w(y_1) = 0$. Moreover, since $\nabla w(y) = \nabla v(x_0 + hy)$, we have $|\nabla w(y_1)| \le A$. Now, by Harnack's inequality, in $B_{1/2}(0)$,

$$w \ge c\lambda \ .$$

In the ring $B_1 \setminus \bar{B}_{1/2}$, compare w with the harmonic function

$$z(y) = \frac{c\lambda}{2^{n-2} - 1}(|y|^{2-n} - 1) \ .$$

z vanishes on ∂B_1 and equals $c\lambda$ on $\partial B_{1/2}$; therefore, by the maximum principle

$$w \ge z \ \text{ in } \ B_1 \setminus B_{1/2} \ .$$

If ν is the inward unit normal to ∂B_1 at y_1, we have

$$A \ge |\nabla w(y_1)| \ge w_\nu \ge \frac{n-2}{2^{n-2} - 1} c\lambda \ .$$

This proves (a).

For (b), use interior Shauder estimates and Harnack's inequality to get for any point $y \in B_{1/2} \cap \Omega$

$$|\nabla w(y)| \le C \frac{w(y)}{d(y, \Gamma)}$$

and using (a), we end the proof. □

Remark. In this proof we only use the behavior of v_ν at those points with an inner tangent ball.

The next theorem is a linear growth result: if we stay a fixed amount away from $\partial\Omega_\varepsilon$, then u_ε starts growing linearly.

Theorem 1.6. *There exist (universal) constants c_1, c_2 such that if $x_0 \in B_{1/2}$ and $u_\varepsilon(x_0) = \lambda \ge c_1\varepsilon$ ($\varepsilon < \frac{1}{3}$), then*

$$d(x_0, \partial\Omega_\varepsilon) \le c_2\lambda \ .$$

From Theorems 1.2 and 1.6 we immediately obtain

Corollary 1.7. *In $\Omega_{c_1\varepsilon} = \{u > c_1\varepsilon\}$,*

$$C^{-1} \operatorname{dist}(x, \partial\Omega_\varepsilon) \le u_\varepsilon(x) \le C \operatorname{dist}(x, \partial\Omega_\varepsilon) \ . \tag{1.6}$$

Remark. Observe that also (1.6) is a renormalization property.

Proof of Theorem 1.6. Put $d_0 = d(x_0, \partial\Omega_\varepsilon)$ and suppose $\lambda = \alpha d_0$. We want to show that $\alpha \ge c > 0$. Rescale u in $B_{d_0}(x_0)$ by setting

$$w(y) = \frac{1}{d_0} u_\varepsilon(d_0 y + x_0) \ .$$

Then $\Delta w = 0$ and $w \geq 0$ in B_1 with $w(0) = \alpha$. By Harnack's inequality, in $B_{1/2}$, $\underline{c}\alpha \leq w \leq \overline{c}\alpha$. Now let ψ be a radial cut-off function, $\psi \equiv 0$ in $B_{1/4}$, $\psi \equiv 1$ outside $B_{1/2}$ and define

$$z = \begin{cases} \min\{w, \overline{c}\alpha\psi\} & \text{in } B_{1/2}, \\ w & \text{in } B_1 \setminus B_{1/2}. \end{cases}$$

Then since w is a minimizer of the functional

$$\tilde{J}(v) = \int_{B_1} \left\{ |\nabla v|^2 + \Phi_\varepsilon(d_0 v) \right\} \, dy$$

among all $v \in w + H_0^1(B_1)$ and since z is an admissible function, we must have

$$\tilde{J}(w) \leq J(z).$$

On the other hand,

$$\int_{B_1} |\nabla z|^2 - \int_{B_1} |\nabla w|^2 \leq c\alpha$$

while

$$\int_{B_1} \Phi_\varepsilon(d_0 v) \, dy - \int_{B_1} \Phi_\varepsilon(d_0 v) \, dy \geq c > 0$$

since, in $B_{1/4}$, $z = 0$ and $\frac{d_0}{\varepsilon} w \geq \frac{d_0}{\varepsilon} \underline{c}\alpha \geq \underline{c}c_1$. The assertion follows easily. $\square$

Remark. Regularity uses only the "weak equation", while linear growth needs the minimization property.

Our next purpose is to estimate the Hausdorff measure of the sets $\partial\Omega_{c\varepsilon}$. To this aim we need the strong nondegeneracy expressed in the following theorem.

Theorem 1.8. *There exist two (universal) constants c, c_1 such that if $x_0 \in B_{1/2}$ and $u_\varepsilon(x_0) \geq c_1\varepsilon$, then*

$$\sup_{B_\rho(x_0)} u_\varepsilon \geq c\rho \ .$$

If $w_\varepsilon(y) = \frac{1}{\rho} u_\varepsilon(x_0 + \rho y)$, this is equivalent to saying that if $\delta = \varepsilon/\rho$ and $w_\varepsilon(0) \geq c_1\delta$, then

$$\sup_{B_1(0)} w_\varepsilon \geq c \ .$$

Once again, this is a consequence of a renormalization result expressed in the following

Theorem 1.9. *Let Ω be an open set with $0 \in \partial\Omega$ and $w \geq 0$, Lipschitz in $B_2(0)$ and harmonic in $\Omega \cap B_2$. Suppose*

(i) $w(x_0) = \delta > 0$,

(ii) *in the region* $\{w \geq \delta/c_1\}$, $c_1 > 1$,

$$w(x) \sim \operatorname{dist}(x, \partial\Omega) \ .$$

Then

$$\sup_{B_1(x_0)} w \geq c \ .$$

First, we give a lemma.

Lemma 1.10. *Under the condition of Theorem 1.9 there exist positive constants* c, C, γ *such that*

$$C\delta \geq \sup_{B_{c\delta}(x_0)} w \geq (1+\gamma)\delta \ .$$

Proof. Let $B_\rho(x_0)$ be the largest ball contained in $\{w > \delta/c_1\}$. From (ii) in Theorem 1.9, $\rho \sim c\delta$. Let $y_0 \in \partial B_\rho(x_0)$ with $w(y_0) = \delta/c_1$. By Lipschitz continuity, for a suitable positive h, $w \leq \delta/2c_1$ in $B_{h\rho}(y_0)$ and therefore also in a fixed fraction of $\partial B_\rho(x_0)$. Since w is harmonic in $B_\rho(x_0)$,

$$\delta \leq w(x_0) = \fint_{\partial B_\rho(x_0)} u$$

so that there must be a point $x_1 \in \partial B_\rho(x_0)$ with $w(x_1) \geq (1+\gamma)\delta$. This gives the second inequality. The first one follows from Lipschitz continuity. □

Proof of Theorem 1.9. We construct a polygonal along which w grows linearly, starting from x_0. From the proof of Lemma 1.10, there exists a point x_1 such that $|x_1 - x_0| = \operatorname{dist}(x_0, \partial\Omega_\varepsilon)$ and

$$cw(x_0) \geq w(x_1) \geq (1+\gamma)w(x_0) \geq w(x_0) + \gamma'|x_1 - x_0| \ .$$

Starting now from x_1 and iterating the application of Lemma 1.10, we construct a sequence $\{x_j\}_{j\geq 1}$ such that $|x_j - x_{j-1}| = \operatorname{dist}(x_{j-1}, \partial\Omega_\varepsilon)$ and

$$cw(x_{j-1}) \geq w(x_j) \geq (1+\gamma)w(x_{j-1}) \geq w(x_{j-1}) + \gamma'|x_j - x_{j-1}| \ .$$

Therefore

(a) $c^j w(x_0) \geq w(x_j) \geq (1+\gamma)^j w(x_0)$,

(b) $w(x_j) - w(x_{j-1}) \geq \gamma'|x_j - x_{j-1}|$,

and in particular,

$$w(x_j) - w(x_0) \geq \gamma' \sum_{k=1}^{j} |x_k - x_{k-1}| \geq \gamma'|x_j - x_0| \ .$$

From (a) we deduce that after a finite number of steps, x_j exits from $B_1(x_0)$. Let k be such that $x_k \in B_1(x_0)$ and $x_{k+1} \notin B_1(x_0)$. Then

$$|x_k - x_0| \geq c > 0 \ .$$

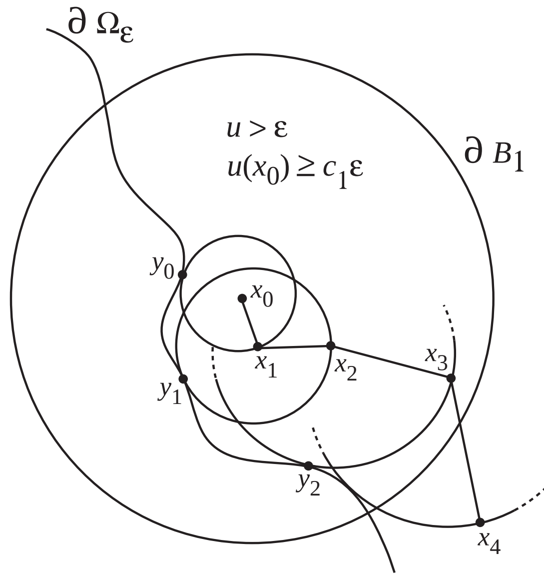

Figure 1.4. The polygonal constructed in the proof of Theorem 1.9

In fact, if $|x_k - x_0| = \alpha$, from (b) and Lipschitz continuity,

$$\gamma'|x_k - x_{k-1}| \leq \gamma' \sum_{j=1}^{k} |x_j - x_{j-1}| \leq w(x_k) - w(x_0) \leq c_0|x_k - x_0| = c_0\alpha$$

so that

$$\operatorname{dist}(x_k, \partial\Omega_\varepsilon) \leq 2|x_k - x_{k-1}| \leq \frac{2c_0}{\gamma'}\alpha \ .$$

Thus

$$\begin{aligned} 1 \leq |x_{k+1} - x_0| &\leq |x_{k+1} - x_k| + |x_k - x_0| \leq \operatorname{dist}(x_k, \partial\Omega_\varepsilon) + \alpha \\ &\leq \left(2\frac{c_0}{\gamma'} + 1\right)\alpha \end{aligned}$$

or $|x_k - x_0| \geq (2\frac{c_0}{\gamma'} + 1)^{-1}$.

From (b)

$$\sup_{B_1(x_0)} w \geq w(x_k) \geq w(x_0) + \gamma'|x_k - x_0| \geq c \ . \quad \square$$

A first consequence of Theorem 1.8 is the following result, which we call the *uniform positive density* of $\Omega_{c\varepsilon}$ along $\partial\Omega_{c\varepsilon}$, $c \gg 1$.

Corollary 1.11. *Let $x_0 \in B_{1/2}$. There exist (universal) constants c_1, c_2, c_3 such that if $x_0 \in B_{1/2}$, $u(x_0) = \lambda \geq c_1\varepsilon$ and $\rho \geq c_2\lambda$, then*

$$|B_\rho(x_0) \cap \{u_\varepsilon > \lambda\}| \geq c_3\rho^n \ .$$

Proof. Let $u_\varepsilon(y) = \sup_{B_{\rho/2}(x_0)} u_\varepsilon = c\rho$ (Theorem 1.8). Then $d(y, \partial\Omega_\varepsilon) \geq c_1\rho$, by Lipschitz continuity (or Corollary 1.7). By Harnack's inequality, for $\bar{c}$ small and c_2 large,

$$u_\varepsilon(x) \geq \frac{c\rho}{2} \geq \frac{cc_2\lambda}{2} > \lambda \quad \text{in} \quad B_{\bar{c}\rho}(y) \ .$$

Thus, $B_{\bar{c}\rho}(y) \subset B_\rho(x_0) \cap \{u_\varepsilon > \lambda\}$. □

Remark. The proof of Theorem 1.9 uses the conclusions of Theorem 1.6 as its hypotheses but we make no use of the variational properties of w in its proof.

We are now in a position to estimate the Hausdorff measure at the λ scale of the level sets $\Omega_{c\varepsilon}$, for $\lambda > 3c\varepsilon$, where c is a large universal constant. This is a consequence of the following theorem, where $\mathcal{N}_\delta(E)$ denotes a δ-neighborhood of the set E.

Theorem 1.12. *If $x_0 \in \partial\Omega_{c\varepsilon}$, $\lambda > 3c\varepsilon$, $R \geq c_1\lambda$, $B_R = B_R(x_0)$, then*

$$|\mathcal{N}_\lambda(\partial\Omega_{c\varepsilon}) \cap B_R| \leq c_3\lambda R^{n-1}$$

where all the constants are universal.

We split the proof into several lemmas.

Lemma 1.13. *Under the hypotheses of Theorem 1.12,*

$$|\mathcal{N}_\lambda(\partial\Omega_{c\varepsilon}) \cap B_R| \sim |\mathcal{N}_\lambda(\partial\Omega_{c\varepsilon}) \cap \Omega_{c\varepsilon} \cap B_R|$$

and

$$\begin{aligned} \left[\{c\varepsilon < u_\varepsilon < c^{-1}\lambda\} \cap B_R\right] &\subset [\mathcal{N}_\lambda(\partial\Omega_{c\varepsilon}) \cap \Omega_{c\varepsilon} \cap B_R] \\ &\subset [\{c\varepsilon < u_\varepsilon < C\lambda\} \cap B_R] \ . \end{aligned}$$

Proof. It follows from Lipschitz continuity and nondegeneracy. □

Lemma 1.14.

$$\int\limits_{\{c\varepsilon < u_\varepsilon < \lambda\} \cap B_R} |\nabla u_\varepsilon|^2 \leq c\lambda R^{n-1}.$$

Proof. By the Gauss formula in $\Omega_{c\varepsilon} \cap B_R$, if $w = \min\{(u_\varepsilon - c\varepsilon)^+, \lambda - c\varepsilon\}$, we have

$$\int\limits_{\{c\varepsilon < u_\varepsilon < \lambda\} \cap B_R} \nabla(u_\varepsilon - c\varepsilon)\nabla w + \int\limits_{\{c\varepsilon < u_\varepsilon < \lambda\} \cap B_R} w\Delta(u_\varepsilon - c\varepsilon) = \int\limits_{\partial[\Omega_{c\varepsilon} \cap B_R]} w(u_\varepsilon - c\varepsilon)^+_\nu \, d\sigma$$

or

$$\int\limits_{\{c\varepsilon < u_\varepsilon < \lambda\} \cap B_R} |\nabla u_\varepsilon|^2 \leq c_0\lambda R^{n-1} \ . \quad \square$$

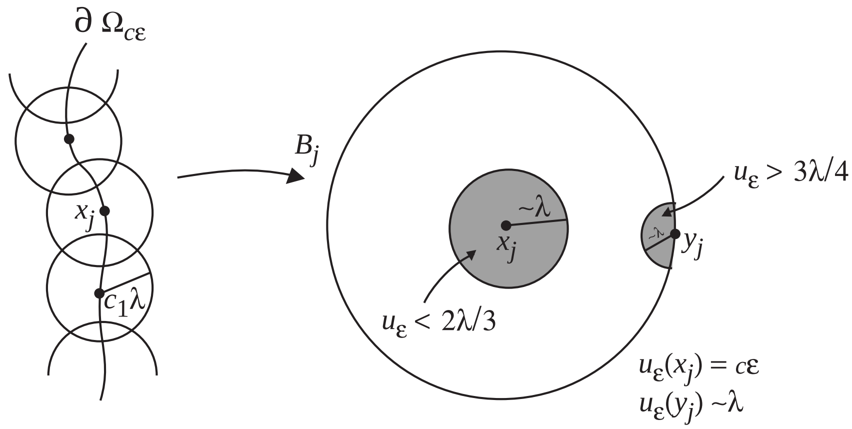

Figure 1.5

It remains to relate the Dirichlet integral of u_ε with the measure of the set $\{c\varepsilon < u_\varepsilon < \lambda\} \cap B_R$. The next lemma completes the proof of Theorem 1.12.

Lemma 1.15. *If* $x_0 \in \partial\Omega_{c\varepsilon}$, $\lambda > 3c\varepsilon$, $R \geq c_1\lambda$, $B_R = B_R(x_0)$, *then*

$$\int_{\{c\varepsilon<u_\varepsilon<\lambda\}\cap B_R} |\nabla u_\varepsilon|^2 \sim |\{c\varepsilon < u_\varepsilon < \lambda\} \cap B_R|.$$

Proof. From Lipschitz continuity the "$\leq$" inequality follows immediately. On the other hand, let $\{B_j\}$ be a finite overlapping covering of $\partial\Omega_{c\varepsilon}$ by balls of radius $c_1\lambda$ and center on $\partial\Omega_{c\varepsilon}$. In every B_j there are subballs B_j^1 and B_j^2 of radius r_j of order λ, where

$$u_\varepsilon \geq \frac{3}{4}\lambda \quad \text{and} \quad u_\varepsilon \leq \frac{2}{3}\lambda,$$

respectively. Therefore, if $m_j = \fint_{B_j} u_\varepsilon$, then $|u - m_j| \geq c\lambda$ on at least one of the two subballs (c universal). Thus, by Poincaré inequality,

$$c\lambda^2 \leq \fint_{B_j} (u_\varepsilon - m_j)^2 \leq \bar{c}r_j^2 \fint_{B_j} |\nabla_\varepsilon u|^2$$

so that

$$\int_{B_j} |\nabla_\varepsilon u|^2 \geq c|B_j|.$$

For c_1 large enough, $\{c\varepsilon < u_\varepsilon < \lambda\} \subset UB_j$ and the proof is complete. □

We are now ready to pass to the limit as $\varepsilon \to 0$. Since $\{u_\varepsilon\}$ is a bounded set in $H^1(B_1)$ and uniformly locally Lipschitz, there exists a sequence $u_k = u_{\varepsilon_k}$, converging to u_0, strongly in $L^2(B_1)$, weakly in $H^1(B_1)$ and uniformly in every compact subset of B_1, as $\varepsilon_k \to 0$.

Theorem 1.16. *u_0 is a local minimizer of J_0.*

Let us first record the main properties of u_0, $\Omega_0 = \{u_0 > 0\}$ and $F(u_0) = \partial\Omega_0 \cap B_1$ in the following

Lemma 1.17. *If $\Omega_0 = \{u_0 > 0\}$, then the following hold.*

(a) *u_0 is locally Lipschitz in B_1, harmonic in Ω_0 and nondegenerate away from $\partial\Omega_0 \cap B_{1/2}$, that is, $\sup_{B_\rho(x)} u_0 \geq c\rho$.*

(b) *Ω_0 is the limit in the Hausdorff distance of $\Omega_k = \{u_k > c\varepsilon_k\}$; i.e., given $\delta > 0$, for c and k large enough,*

$$B_{1/2} \cap \Omega_k \subset N_\delta(\Omega_0) \cap B_{1/2}$$

and

$$B_{1/2} \cap \Omega_0 \subset N_\delta(\Omega_k) \cap B_{1/2}.$$

(c) *$|\mathcal{N}_\delta(\partial\Omega_0) \cap B_R| \leq c\delta R^{n-1}$ for every $\delta > 0$, in particular*

$$|H_{n-1}(\partial\Omega_0) \cap B_R| \leq cR^{n-1} \; . \tag{1.7}$$

Proof. (a) It is clear that u_0 is locally Lipschitz and harmonic in Ω_0. To see it is also nondegenerate, let $x_0 \in \Omega_0 \cap B_{1/2}$. Then there must exist a sequence $\{x_j\}$, with $x_j \to x_0$, $x_j \in \Omega_j \cap B_{2/3}$. By nondegeneracy, $u_j(x_j) \geq c\varepsilon_j$ implies, for any $\rho > 0$ small,

$$u_j(y_j) = \sup_{B_{\rho/4}(x_j)} u_j \geq c\rho \qquad \left(y_j \in \partial B_{\rho/4}(x_j)\right) .$$

When $|x_j - x_0| \leq \rho/4$, $B_{\rho/4}(x_j) \subset B_\rho(x_0)$ and y_j (or a subsequence) converges to some $y^* \in B_\rho(x_0)$. Since $c\rho \leq u_j(y_j) \to u_0(y^*)$, we conclude that

$$\sup_{B_\rho(x_0)} u_0 \geq c\rho \; .$$

(b) Suppose the first inclusion is false. Then there exists a sequence $\{x_k\}$ such that, for some δ,

- $\operatorname{dist}\{x_k, \Omega_0\} \geq \delta$,
- $x_k \in \Omega_k \cap B_{1/2}$,
- $x_k \to x_0$ with $\operatorname{dist}(x_0, \Omega_0) \geq \delta$.

Therefore, $u_0(x_0) = 0$ while $u_k(x_k) \geq c\varepsilon_k$. By nondegeneracy,

$$u_k(y_k) = \sup_{B_\rho(x_k)} u_k \geq c\rho \; .$$

When $|x_k - x_0| < \delta/8$, for $\rho = \delta/8$, $B_\rho(x_k) \subset B_{\delta/2}(x_0)$ and (a subsequence of) $y_k \to y^* \in B_\delta(x_0)$. Since $u_k(y_k) \to u_0(y^*)$, we conclude that

$$0 = \sup_{B_{\delta/2}(x_0)} u_0 \geq c\delta,$$

a contradiction.

If the second inclusion is false, there exists a sequence $\{x_k\} \subset \Omega_0 \cap B_{1/2}$ such that $\operatorname{dist}(x_k, \Omega_k \cap B_{1/2}) \geq \delta$. This means that $u_k(x) \leq c\varepsilon_k$ for $x \in B_{\delta/2}(x_k)$. Suppose $x_k \to x^*$; then when $|x_k - x^*| < \delta/8$, $B_{\delta/8}(x^*) \subset B_{\delta/2}(x_k)$ and therefore $B_{\delta/8}(x^*) \subset B_{1/2} \setminus \Omega_0$. Contradiction.

(c) It is a consequence of (b) and Theorem 1.12. □

End of the proof of Theorem 1.16. We want to show that u_0 is a local minimizer. Assume not. Then in some ball $B_r = B_r(x_0) \subset\subset B_1$, there exist $a > 0$ and $v \in H^1(B_1)$ such that $v = u_0$ on ∂B_r and

$$\int_{B_r} \{|\nabla v|^2 + \chi_{\{v>0\}}\} \leq \int_{B_r} \{|\nabla u_0|^2 + \chi_{\{u_0>0\}}\} - a \ .$$

Fix $h > 0$ small, and radially interpolate in a linear fashion between u_0 and u_k in the ring $B_{r+h} \setminus B_r$; i.e., define

$$v_{h,k} = \begin{cases} u_0 + \dfrac{|x| - r}{h}(u_k - u_0) & \text{in } B_{r+h} \setminus B_r, \\ v & \text{in } B_r. \end{cases}$$

Then, on $B_{r+h} \subset\subset B_1$,

$$\begin{aligned} J_{r+h,k}(v_{h,k}) &= \int_{B_{r+h}} \{|\nabla v_{h,k}|^2 + F_{\varepsilon_k}(v_{h,k})\} \\ &\leq chr^{n-1} + \frac{2}{h^2} \int_{B_{r+h} \setminus B_r} (u_k - u_0)^2 + J_{r,0}(v) \end{aligned}$$

where

$$J_{r,0}(v) = \int_{B_r} \left\{|\nabla v|^2 + F(v)\right\}$$

since

$$\int_{B_r} F_{\varepsilon_k}(v_{h,k}) = \int_{B_r} F_{\varepsilon_k}(v) \leq |\chi_{\{v>0\}}| \ .$$

Thus

$$\overline{\lim_{k\to\infty}} \, J_{r+h,k}(v_{h,k}) \leq chr^{n-1} + J_{r,0}(v) \ .$$

On the other hand,

$$J_{r+h,k}(v_{h,k}) \geq J_{r,k}(v_{h,k}) \geq J_{\varepsilon_k}(u_k)$$

and

$$J_{r,0}(u_0) \leq \underline{\lim_{k\to\infty}} \, J_{\varepsilon_k}(u_k) \ .$$

This follows from

$$\int_{B_r} |\nabla v_0|^2 \leq \underline{\lim_{k\to\infty}} \int_{B_r} |\nabla u_k|^2$$

by the weak convergence of u_k to u_0 and (by Lemma 1.17 applied to B_r instead of $B_{1/2}$ and Theorem 1.12) from

$$|\Omega_0 \cap B_r| \le |\mathcal{N}_h(\Omega_k) \cap B_r| \le chr^{n-1} + |\Omega_k \cap B_r| \qquad (h \gg \varepsilon_k) .$$

Thus

$$J_{r,0}(u_0) \le chr^{n-1} + J_{r,0}(v) \le J_{r,0}(v) + chr^{n-1} - a,$$

a contradiction for $h < \frac{a}{2c} r^{1-n}$. □

1.3. The free boundary condition

At this point we have constructed a local minimizer u_0 of the functional J_0; u_0 is Lipschitz and nondegenerate inside B_1 and for any ball $B_R(x_0) \subset\subset B_1$ and any $\delta > 0$,

$$|\mathcal{N}_\delta(\partial\Omega_0) \cap B_R| \le c\delta R^{n-1} \tag{1.8}$$

which in particular means that the $(n-1)$-dimensional Hausdorff measure of $\partial\Omega_0$ is locally finite.

Differing from the obstacle problem, the structure of the free boundary $\partial\Omega_0 \cap B_1 = F(u_0)$ is somewhat nicer since cusps cannot occur. This is expressed in the next theorem.

Theorem 1.18. *Let $x_0 \in \partial\Omega_0 \cap B_{1/2}$, $r \le 1/4$. Then both*

$$|\mathcal{C}\Omega_0 \cap B_r(x_0)| \sim cr^n \quad \text{and} \quad |\Omega_0 \cap B_r(x_0)| \sim r^n. \tag{1.9}$$

As a consequence, from the isoperimetric inequality and (1.7)

$$H_{n-1}(\partial\Omega_0 \cap B_r(x_0)) \sim r^{n-1} . \tag{1.10}$$

Proof. $|\Omega_0 \cap B_r(x_0)| \sim r^n$ follows from Lemma 1.17(a). We have to prove that

$$|\{u_0 = 0\} \cap B_r(x_0)| \ge cr^n .$$

Let v be the harmonic function in $B_r(x_0)$ with $u_0 = v$ on $\partial B_r(x_0)$. Then since $v > 0$ in $B_r(x_0)$,

$$\int_{B_r(x_0)} \left(|\nabla u_0|^2 + \chi_{\{u_0>0\}}\right) \le \int_{B_r(x_0)} \left(|\nabla v|^2 + \chi_{\{v>0\}}\right)$$
$$= \int_{B_r(x_0)} |\nabla v|^2 + |B_r(x_0)| .$$

Therefore

$$\int_{B_r} \left(|\nabla u_0|^2 - |\nabla v|^2\right) \le |\{u = 0\} \cap B_r(x_0)| .$$

On the other hand, by Poincaré inequality

$$\int_{B_r(x_0)} \left(|\nabla u_0|^2 - |\nabla v|^2\right) = \int_{B_r(x_0)} |\nabla(u_0 - v)|^2 \ge \frac{c}{r^2} \int_{B_r(x_0)} (u_0 - v)^2$$

so that

$$\int_{B_r} (u_0 - v)^2 \le cr^2 |\{u_0 = 0\} \cap B_r(x_0)| \ .$$

Now,

$$v(x_0) = \fint_{\partial B_r(x_0)} u \ge cr$$

by nondegeneracy, and

$$v(y) \ge cr$$

on $B_{r/2}(x_0)$. Since, by Lipschitz continuity,

$$u(y) \le chr$$

in $B_{hr}(x_0)$, we conclude that

$$v - u_0 \ge cr \quad \text{on} \quad B_{hr}(x_0)$$

if h is small enough.

Therefore

$$|\{u = 0\} \cap B_r(x_0)| \ge \frac{c}{r^2} \int_{B_r(x_0)} (u_0 - v)^2 \ge \frac{c}{r^2} \int_{B_{hr}(x_0)} (u_0 - v)^2 \ge cr^2 \ . \quad \square$$

Theorem 1.18 says that both Ω_0 and its complement $\mathcal{C}\Omega_0$ have uniform density along the free boundary $F(u_0)$ and that Ω_0 is a set of finite perimeter. But the main and most challenging mathematical question is the regularity of $F(u_0)$.

Before addressing this problem, we must ask another basic question: precisely, in which sense are the free boundary conditions satisfied by u_0? Recall that if we know that $F(u_0)$ is smooth, Hadamard's classical formula gives

$$\partial_\nu u_0^+ = 1 \ . \tag{1.11}$$

There are several ways to interpret in a weak sense the condition (1.11). One way is suggested by (1.10) and by the fact that Δu_0 is a nonnegative measure whose total mass in a ball B_r centered on $F(u_0)$ is equivalent to r^{n-1}. Precisely, we have

Theorem 1.19. *Let $x_0 \in F(u_0)$ and put $\mu = \Delta u_0$. Then μ is a nonnegative measure supported on $F(u_0)$ and for any $r > 0$*

$$\int_{\partial B_r(x_0)} \partial_\nu u_0^+ \, dH_{n-1} = \int_{B_r(x_0)} d\mu \sim r^{n-1} \ . \tag{1.12}$$

Proof. Since (1.12) expresses a renormalization property, it is enough to check it for $r = 1$. The inequality

$$\int_{\partial B_1(x_0)} \partial_\nu u_0^+ \, dH_{n-1} \le c$$

follows just from Lipschitz continuity.

To prove the opposite inequality, let w be harmonic in $B_1(x_0)$ with $w = u_0$ on $\partial B_1(x_0)$. Then $\Delta(w - u_0) = -\mu$, $w \ge u_0$ and

$$w(y) - u_0(y) = \int_{B_1(x_0)} G(y, z) \, d\mu(z) \quad \text{in} \quad B_1$$

where G is the Green's function for B_1. By nondegeneracy and Lipschitz continuity there exists a point $y \in B_h(x_0)$, h small, with $u_0(y) \sim ch$ and consequently

$$u_0 > 0 \quad \text{in} \quad B_{ch}(y) \; .$$

Thus $d\mu = 0$ on $B_{ch}(y)$ and

$$w(y) - u_0(y) = \int_{B_1(x_0) \setminus B_{ch}(y)} G(y, z) \, d\mu(z) \le c \int_{B_1(x_0)} d\mu \; . \tag{1.13}$$

On the other hand, from nondegeneracy, if $p > 1$,

$$\fint_{B_{1/2}(x_0)} w^p \ge \fint_{B_{1/2}(x_0)} u_0^p \ge c$$

and, from Harnack's inequality,

$$w(y) \ge \left(\fint_{B_{1/2}(x_0)} w^p \right)^{1/p} \ge c$$

so that, if h is small enough

$$w(y) - u_0(y) \ge c - ch \ge \frac{c}{2}$$

and we conclude the proof from (1.13). □

From Theorem 1.19, we obtain immediately the following representation theorem.

Theorem 1.20. *There exists a H_{n-1}-measurable function g on $F(u_0) \cap B_{1/2}$ such that, in $B_{1/2}$*

(i) $0 < c \le g \le C$,

(ii) $\Delta u_0 = g H_{n-1} \lfloor_{F(u_0)}$.

Remark. $g = \frac{d\mu}{dH_{n-1}}$ in the Radon-Nikodym sense.

Since, heuristically, $d\mu$ represents $\partial_\nu u_0^+ \, dH_{n-1}$ on $F(u_0)$, we expect $g \equiv 1$ so that, in conclusion, the free boundary condition $\partial_\nu u_0^+ = 1$ should be interpreted as

$$\Delta u_0 = H_{n-1} \quad \text{on} \quad F(u_0) \tag{1.14}$$

in the sense of measure. We can better understand the free boundary condition if we replace $F(u_0)$ with its *reduced* part $F^*(u_0)$, that is, the set of points x at which a generalized (interior to $\Omega_0(u_0)$) normal

$$\nu(x) = \lim_{r\to 0} \frac{\nabla \chi_{\Omega_0}(B_r)}{|\nabla \chi_{\Omega_0}(B_r)|} \tag{1.15}$$

exists with $|\nu(x)| = 1$. In (1.15), $|\nabla \chi_{\Omega_0}(B_r)|$ denotes the total variation in B_r of the measure $\nabla \chi_{\Omega_0}$.

Thus, let $0 \in F^*(u_0)$ and consider the "blow-up" sequence

$$u_k(x) = k u_0\left(\frac{x}{k}\right) \qquad \text{in} \quad B_k(0) \ .$$

Now let $k \to +\infty$. Let us record the main properties of "blow-up limits".

First of all, since u_0 is locally Lipschitz continuous, for a subsequence

$$u_k \to u_\infty \quad \text{in} \quad C^\alpha_{\text{loc}}(\mathbb{R}^2) \quad \text{(for every } \alpha < 1\text{)},$$
$$\nabla u_k \to \nabla u_\infty \quad \text{weakly star in} \quad L^\infty_{\text{loc}}(\mathbb{R}^n) \ .$$

Clearly u_∞ is nonnegative, harmonic in $\{u_\infty > 0\}$ and globally Lipschitz. Moreover,

Lemma 1.21.

(a) $F(u_k) \to F(u_\infty)$ *locally, in the Hausdorff distance,*
(b) $\chi_{\{u_k>0\}} \to \chi_{\{u_\infty>0\}}$ *in* $L^1_{\text{loc}}(\mathbb{R}^n)$,
(c) $\nabla u_k \to \nabla u_\infty$ *a.e. in* $\mathbb{R}^n$.

Proof. (a) It follows from the uniform convergence of u_k to u_∞ and the uniform nondegeneracy of u_k.

(b) u_∞ is nondegenerate along $F(u_\infty)$. Indeed, if $x \in F(u_\infty)$, then there exists a sequence $y_k \in F(u_k)$ such that $y_k \to x$, with

$$\fint_{\partial B_r(y_k)} u_k \geq cr$$

and therefore, also

$$\fint_{\partial B_r(x)} u_\infty \geq cr \ .$$

This implies that

$$|\{u_\infty > 0\} \cap B_r(x)| \geq cr^n$$

and therefore that $|F(u_\infty)| = 0$. Using (a), (b) follows.

(c) It is enough to show that $|\nabla u_k| \to |\nabla u_\infty|$ a.e. in $\{u_\infty = 0\}$. Now, a.e. points in $\{u_\infty = 0\}$ are 1-density point. Let S denote the set of such points and let x_0 be one of these. We claim that

$$u_\infty(x) = o(|x - x_0|) \quad \text{near} \quad x_0 \ . \tag{1.16}$$

Suppose not. Then there exists a sequence $x_m \to x_0$ such that

$$\frac{u_\infty(x_m)}{|x_m - x_0|} \geq c > 0 \ .$$

By Lipschitz continuity,

$$u_\infty(x) \geq c|x_m - x_0| \equiv cr_m$$

in $B_{hr_m}(x_m)$, h small. Therefore, for m large enough and $r \gg r_m$, $B_r(x_0)$ contains the ball $B_{hr_m}(x_m)$, which is a contradiction to the 1-density of x_0.

From (1.16) we get $\nabla u_\infty(x_0) = 0$ and, in particular, that, given $\varepsilon > 0$,

$$\frac{u_k(x)}{\delta} < 2\varepsilon \quad \text{in} \quad B_\delta(x_0)$$

provided k is large enough, say $k \geq k_0(\varepsilon, \delta)$. Then, by nondegeneracy, $u_k \equiv 0$ in $B_{\delta_{\varepsilon/2}}(x_0)$ and consequently, $u_0 = 0$ in a neighborhood of x_0 and S is open. The above argument shows that $u_k = u_0$ in any compact subset of S if k is large enough. This completes the proof of (c). □

We now identify u_∞.

Theorem 1.22. *Let $0 \in F^*(u_0) \cap B_{1/2}$. Then u_∞ is a local minimizer in $\mathbb{R}^n$ and*

$$u_\infty(x) = \langle x, \nu(0)\rangle^+ .$$

Corollary 1.23.

$$\Delta u_0 = H_{n-1}\lfloor_{F^*(u_0)} \ . \tag{1.17}$$

Proof. One can prove that u_∞ is a local minimizer in $\mathbb{R}^n$ by using the same technique of Theorem 1.16.

We may suppose $\nu(0) = e_n$. A well-known property of sets of finite perimeter says that the blow-up limits of Ω_0 and of $\mathcal{C}\Omega_0$ are, respectively, the half planes

$$x_n > 0 \quad \text{and} \quad x_n < 0 \ .$$

Thus $u_\infty(x)$ is positive if $x_n > 0$ and equal to zero if $x_n \leq 0$ and in particular on the hyperplane $x_n = 0$. Reflecting u_∞ in an odd way with respect to this hyperplane, we get a function $\tilde{u}_\infty$, harmonic in $\mathbb{R}^n$. Since u_∞ is globally Lipschitz, it follows (Liouville theorem) that $\tilde{u}_\infty$ is a linear function and therefore $u_\infty(x) = \alpha x_n^+$ for some positive α.

We want now to show that $\alpha = 1$; as a consequence we obtain $g \equiv 1$ proving Corollary 1.23:

$$\Delta u_0 = H_{n-1} \quad \text{on} \quad F^*(u_0) \ . \tag{1.18}$$

This gives a possible interpretation of the free boundary condition.

Suppose the problem is 1-dimensional. Then

$$u_\infty(x) = \alpha x^+ \ .$$

Make a perturbation inside $(-1,1)$ by taking

$$w(x) = \frac{\alpha}{1-b}(x_n - b) \qquad (|b| < 1) \ .$$

Then the minimization condition gives, as in Section 1.1,

$$\alpha^2(b + o(b)) - b \geq 0$$

from which $\alpha = 1$.

In the n-dimensional case we make a similar perturbation inside a strip $|x_n| < 1$, $|x'| = |(x_1, \ldots, x_{n-1})| < M$ with a large M. Let $\psi = \psi(x')$ be a cut-off function such that $\psi \equiv 1$ for $|x'| \geq M+1$, $0 \leq \psi \leq 1$, and $\psi \equiv 0$ in $B'_M = \{|x'| < M\}$. Define

$$w(x) = \max\left\{\frac{\alpha}{1-b}(x_n - b)^+\big(1 - \psi(x')\big) + \alpha x_n^+ \psi(x'), 0\right\} .$$

Then it is not difficult to check that

$$\int_{B_M} \left(|\nabla w|^2 - |\nabla u_\infty|^2\right) \leq \alpha^2\big(b + o(b)\big)|B'_{M+1}| + C|b|\,|B'_{M+1} \setminus B'_M|$$

and

$$|\{w > 0\}| - |\{u_\infty > 0\}| \leq b|B'_{M+1}| + c|b|\,|B'_{M+1} \setminus B'_M| \ .$$

The minimization condition gives

$$(\alpha^2 - 1)\big(b + o(b)\big) + c|b|/M \geq 0$$

and letting $M \to \infty$, we get again $\alpha = 1$. □

There are other ways to interpret the free boundary condition. Let us go back to one of the key lemmas, Lemma 1.5, where the Lipschitz continuity of the minimizers u_ε (and consequently of u_0) is proved.

In that lemma, to the nonnegative harmonic function v we required having bounded gradient along its zero level set. In fact, we only used that v had some bounded linear behavior at points of the free boundary where there exists a touching ball from inside $\{v > 0\}$, i.e., a ball $B \subset \{v > 0\}$ such that $B \cap \{v > 0\} = \{y\}$.

This would suggest taking into consideration, as far as the free boundary condition is concerned, only the points at which there exists a touching ball from one side or the other of $F(u)$. (We will call these points *regular points*.)

This leads naturally towards notions of the free boundary condition in a viscosity sense, which we will formalize in the next chapter.

We point out that these notions of solutions are considerably weaker than the one in the measure-theoretical sense, described above: H_{n-1}-a.e. points on the free boundary have a generalized normal, since $H_{n-1}(F(u) \setminus F^*(u)) = 0$, while, in principle, the set of regular points can be very small.

Clearly, a careful balance is required in constructing the definitions if one looks for both existence of a solution and regularity of the free boundary. For minimizers in variational problems this need is less essential, since the minimization process conveys some "stability" both to the solution and its free boundary. It is not so, for instance, in evolution free boundary problems, where the two requirements could strongly compete.

We end this section by briefly examining what happens to the free boundary conditions $u_\nu^+ = 1$ in the one-phase minimization problem if we adopt this new point of view. Suppose 0 is a regular point of $F(u_0)$ with a touching ball B (from either one of the two sides of $F(u_0)$), whose inward normal at 0 is e_n.

Then from Lemma 11.17 and the remark after it, u_0 has the linear behavior

$$u_0(x) = \alpha x_n^+ + o(|x|) \tag{1.19}$$

near zero if $B \subset \mathcal{C}\Omega_0$, near zero and $x_n \geq 0$ if $B \subset \Omega_0$. In (1.19), $0 < c_1 \leq \alpha \leq c_2$, by Lipschitz continuity and nondegeneracy.

Look again at the blow-up sequence $u_k = ku_0(x/k)$ and at its limit u_∞. From (1.19) we get

$$u_\infty(x) = \alpha x_n^+ \tag{1.20}$$

for $x_n \geq 0$ if $B \subset \Omega_0$ or for all x if $B \subset \mathcal{C}\Omega_0$.

From the monotonicity formula (Theorem 12.3) and the uniform density estimate of $\mathcal{C}\Omega_0$ (or from Lemma 12.8) we deduce that (1.19) holds for every $x \in \mathbb{R}^n$, also if $B \subset \Omega_0$.

We can now proceed as before to prove that $\alpha = 1$.

The free boundary condition is therefore to be interpreted in the following sense: near any regular point $x_0 \in F(u_0)$, u_0 has the linear behavior

$$u_0(x) = \langle x - x_0, \nu(x_0)\rangle^+ + o(|x - x_0|)$$

where $\nu(x_0)$ is the normal to ∂B at x_0, inward to Ω_0.

Let us summarize. In this chapter we have shown the main general properties of a solution to our introductory free boundary problem:

optimal regularity (Lipschitz),

nondegeneracy and linear growth,

uniform density of Ω and $\mathcal{C}\Omega$,

finite Hausdorff measure of the free boundary.

As we will see, this will provide us with the model approach to studying more general free boundary problems.

Chapter 2

Viscosity Solutions and Their Asymptotic Developments

2.1. The notion of viscosity solution

The notion of viscosity solution was introduced by M. Crandall and P. L. Lions (see [CL]) in the context of Hamilton-Jacobi equations and in the last two decades has become the central notion in the theory of fully nonlinear parabolic and elliptic equations.

Let us see, for instance, how viscosity harmonic functions are defined. The key idea is to switch the action of the Laplace operator to smooth test functions, using the comparison principle.

Suppose D is a domain in $\mathbb{R}^n$ and $\varphi \in C^2(D)$. If u is a classical subharmonic function in D and φ touches u (locally) from above at x_0, i.e.,

$$\varphi \geq u \text{ near } x_0 \text{ and } \varphi(x_0) = u(x_0), \tag{2.1}$$

then clearly

$$\Delta\varphi(x_0) \geq 0. \tag{2.2}$$

Analogously, if u is a classical superharmonic function in D,

$$\varphi \leq u \text{ near } x_0 \text{ and } \varphi(x_0) = u(x_0), \tag{2.1$'$}$$

then

$$\Delta\varphi(x_0) \leq 0\ . \tag{2.2$'$}$$

Notice that conditions (2.2) and (2.2′) do not require derivatives of u, while (2.1) and (2.1′) make sense with u continuous.

Definition 2.1. A function $u \in C(D)$ is a viscosity subharmonic (superharmonic) function if for every $\varphi \in C^2(D)$ satisfying (2.1) (resp. (2.1′)), (2.2) (resp. (2.2′)) holds; u is a viscosity harmonic function if it is both a viscosity subharmonic and superharmonic function.

In other words u is subharmonic if the only way a smooth function may touch it from above is by being itself subharmonic.

Definition 2.1 is perfectly consistent with the classical definition: u is a viscosity harmonic function if and only if it is harmonic in the classical sense. Indeed, it is easy to prove that if $u \in C(D)$ is a viscosity harmonic function, then $u \in C^2(D)$ and $\Delta u = 0$ in D.

Observe that we could equivalently have defined u subharmonic (superharmonic) in the viscosity sense by requiring that if $\varphi \in C^2(\Omega)$ and $\Delta\varphi \leq 0$ (resp. $\Delta\varphi \geq 0$) in Ω, then $\varphi - u$ cannot have a local minimum (resp. maximum). We leave the proof of the equivalence of the two definitions as an exercise.

As we see, the "leitmotiv" of the definition is to prevent a subsolution (a supersolution) from touching a solution from below (above). This is exactly what we require for a viscosity solution of a free boundary problem. First, define a C^k-classical subsolution and supersolution ($1 \leq k \leq \infty$) as follows:

Definition 2.2. A function $v \in C(D)$ is called a C^k-classical subsolution of our free boundary problem if

(i) $v \in C^k(\bar{\Omega}^+(v)) \cap C^k(\bar{\Omega}^-(v))$,

(ii) $\Delta v \geq 0$ in $\Omega^+(v) = \{v > 0\} \cap D$ and $\Omega^-(v) = \{v \leq 0\}^0 \cap D$,

(iii) the free boundary $F(v) = \partial\Omega^+(v) \cap D$ is a C^k-surface, $|\nabla u^+| > 0$ on $F(v)$ and

$$G(v_\nu^+, v_\nu^-) \geq 0 \quad \text{on} \quad F(v) \tag{2.3}$$

where $\nu = \frac{\nabla u^+}{|\nabla u^+|}$.

If the inequality in (2.3) is strict, we call v the *strict subsolution*. A C^k-classical supersolution is defined by reversing the inequalities in Definition 2.2, while a C^k-classical solution is both a C^k-classical subsolution and supersolution.

Notice that in the one-phase minimization problem of Section 1.1, a subsolution of the problem satisfies the condition $u_\nu^+ \geq 1$.

We now use classical strict subsolutions and supersolutions as test functions. From now on we choose $k = 2$ to fix the ideas, but other choices may be more convenient.

Definition 2.3. A function $u \in C(D)$ is a viscosity subsolution if for every classical strict supersolution v, v cannot touch u from above at a free boundary point.

Notice that if $v \in C^2(\bar{\Omega}(v))$ is a superharmonic function in $\Omega^{\pm}(v)$ touching u from above at a free boundary point x_0, i.e., $v(x) \geq u(x)$ near x_0 and $x_0 \in F(v) \cap F(u)$, then necessarily

$$G\big(v_\nu^+(x_0), v_\nu^-(x_0)\big) \geq 0 \ . \tag{2.4}$$

Viscosity supersolutions are defined reversing the inequalities in Definition 2.3 and interchanging sub with super. A viscosity solution is both a viscosity subsolution and supersolution. So, no classical strict subsolution (supersolution) can touch a solution from below (above).

It is not difficult to prove that a C^2-classical subsolution (supersolution, solution) is also a viscosity subsolution (supersolution, solution) and that a viscosity solution (subsolution, supersolution) of class C^2, with its free boundary, in $\bar{\Omega}^+(w)$ and $\bar{\Omega}^-(w)$ is a classical solution (subsolution, supersolution).

2.2. Asymptotic developments

The definitions of C^2-viscosity subsolutions, supersolutions and solutions can be restated in terms of asymptotic linear behavior near a regular point of the free boundary.

Indeed, let u be a C^2-viscosity solution and suppose that a classical strict subsolution v touches u from below at 0, say. Then the origin is a regular point from the right, i.e., there is a ball B touching $F(u)$ at zero, inside $\Omega^+(w)$. By Lemma 11.17, if ν denotes the unit normal to ∂B inward to $\Omega^+(u)$, we have, near zero, in B

$$u^+(x) = \alpha\langle x, \nu\rangle^+ + o(|x|)$$

with $\alpha > 0$ and in $\mathcal{C}B$

$$u^-(x) = \beta\langle x, \nu\rangle^- + o(|x|)$$

with $\beta \geq 0$, finite. The free boundary condition translates into a relation between α and β. Which is the correct one? Since v is a classical strict subsolution, we have

$$G(v_\nu^+(0), v_\nu^-(0)) > 0 \ .$$

On the other hand, the Hopf maximum principle gives

$$v_\nu^+(0) < \alpha \quad \text{and} \quad v_\nu^-(0) > \beta$$

so that from the ellipticity of G

$$0 < G(v_\nu^+(0), v_\nu^-(0)) < G(\alpha, \beta) \ .$$

We reach a contradiction if $G(\alpha, \beta) \le 0$, which is, therefore, the correct inequality. Analogously, if v is a supersolution touching u from above at zero, there exists a ball B touching $F(u)$ at zero, inside $\Omega^-(u) = \{u \le 0\}^o$. Then if ν denotes the unit normal to ∂B inward to $\Omega^+(u)$, we have, near zero, in B

$$u^-(x) = \beta \langle x, \nu \rangle^- + o(|x|)$$

with $\beta > 0$, while in $\mathcal{C}B$

$$u^+(x) = \alpha \langle x, \nu \rangle^+ + o(|x|)$$

with $\alpha \ge 0$.

This time we have

$$0 > G(v_\nu^+(0), v_\nu^-(0)) > G(\alpha, \beta) \ ,$$

a contradiction if $G(\alpha, \beta) \ge 0$.

Substantially, we require that u satisfy a supersolution condition at regular points from the right (touching ball inside $\Omega^+(w)$) and a subsolution condition at regular points from the left (touching ball inside $\Omega^-(u)$).

The strict condition $G(\bar{\alpha}, \bar{\beta}\,) > 0$ compared to $G(\alpha, \beta) = 0$ indicates that $\bar{\alpha} > \alpha$ and/or $\bar{\beta} < \beta$ and therefore a supersolution is "more concave" than a solution at a common point of the free boundary. This clearly prevents a supersolution from touching a solution from above and it is perfectly analogous to the fact that a superharmonic function is more concave than a harmonic function at a common point of their graphs.

We summarize the conclusions by giving an alternative definition.

Definition 2.4. A continuous function u is a viscosity solution in D if the following hold.

(i) It is harmonic in $\Omega^+(u)$ and $\Omega^-(u)$.

(ii) Along $F(u)$, u satisfies the free boundary condition in the following sense:
(a) If at x_0 there is a ball $B \subset \Omega^+(u)$, $B \cap \Omega^+(u) = \{x_0\}$ and near x_0,
in B

$$u^+(x) \ge \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|) \qquad (\alpha > 0), \tag{2.5}$$

in $\mathcal{C}B$

$$u^-(x) \le \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \qquad (\beta \ge 0) \tag{2.6}$$

with equality along every nontangential domain in both cases, then

$$G(\alpha, \beta) \le 0. \tag{2.7}$$

(b) If at x_0 there is a ball $B \subset \Omega^-(u)$, $B \cap \Omega^-(u) = \{x_0\}$ and near x_0, in B

$$u^-(x) \ge \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \qquad (\beta > 0), \tag{2.8}$$

in $\mathcal{C}B$

$$u^+(x) \le \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \qquad (\alpha \ge 0) \tag{2.9}$$

with equality along every nontangential domain in both cases, then

$$G(\alpha, \beta) \ge 0\ . \tag{2.10}$$

We leave it as an exercise to prove that Definitions 2.3 and 2.4 are equivalent. (Hint: if (2.5) and (2.6) hold and $G(\alpha, \beta) > 0$, construct a strict subsolution touching u from below at x_0.)

Definition 2.4 is based on the asymptotic behavior of u. We get an equivalent definition involving the linear behavior of classical test functions touching the free boundary at regular points (see Definition 2.3) replacing condition (ii) by the following.

(ii)* (a) If x_0 has a touching ball B inside $\Omega^+(u)$ and in B, near x_0

$$u^+(x) \ge \bar{\alpha}\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \qquad (\alpha \ge 0), \tag{2.11}$$

then in $\mathcal{C}B$, near x_0

$$u^-(x) \ge \bar{\beta}\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \qquad (\beta \ge 0) \tag{2.12}$$

for any $\bar{\beta}$ such that

$$G(\,\bar{\alpha}, \bar{\beta}\,) > 0\ .$$

(b) If x_0 has a touching ball B inside $\Omega^-(u)$ and in B, near x_0

$$u^-(x) \ge \bar{\beta}\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \qquad (\bar{\beta} \ge 0), \tag{2.13}$$

then in $\mathcal{C}B$, near x_0

$$u^+(x) \ge \bar{\alpha}\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \qquad (\bar{\alpha} \ge 0) \tag{2.14}$$

for any $\bar{\alpha}$ such that

$$G(\,\bar{\alpha}, \bar{\beta}\,) < 0\ . \tag{2.15}$$

Let us check that conditions (ii) and (ii)* are equivalent. Assume (ii)(b) holds. Then if (2.13) holds, we have $\beta \ge \bar{\beta}$. If $\bar{\alpha}$ is such that $G(\bar{\alpha}, \bar{\beta}) < 0$, since $G(\alpha, \bar{\beta}) \geq G(\alpha, \beta) \geq 0$, it must be that $\alpha \geq \bar{\alpha}$. Since equality holds in (2.9) along nontangential domains, (2.14) follows.

Assume now (ii)*(b). Let (2.8) and (2.9) hold; we want to show that $G(\alpha, \beta) \geq 0$. If not, $G(\alpha, \beta) < 0$ and, for a small $\varepsilon > 0$, $G(\alpha + 2\varepsilon, \beta) < 0$. Then (2.9) gives

$$u^+(x) \leq (\alpha + \varepsilon)\langle x - x_0, \nu\rangle^+ + o(|x - x_0|)$$

while (2.13) and (2.14) with $\bar{\alpha} = \alpha + 2\varepsilon$ and $\bar{\beta} = \beta$ give

$$u^+(x) \geq (\alpha + 2\varepsilon)\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \ .$$

Contradiction.

Analogously one can check that (ii)(a) and (ii)*(a) are equivalent.

Remark. As we have seen, viscosity solutions can be characterized in different ways. The definition is clearly closed under uniform limits.

The disadvantage is that it could produce undesirable solutions like

$$u(x) = \alpha_1 x_1^+ + \alpha_2 x_1^-$$

with any α_1, α_2 such that

$$G(\alpha_1, 0) \leq 0, \quad G(\alpha_2, 0) \leq 0 \ .$$

Extra care will be necessary to construct solutions with the desired geometric measure-theoretic properties.

2.3. Comparison principles

Strictly speaking, the definition of viscosity subsolution (supersolution) involves a condition at those points of the free boundary that are regular from the left (right). These conditions turn out to be not enough for comparison purposes.

Suppose we intend to prove that a viscosity subsolution v cannot touch a solution u from below at a point $x_0 \in F(u) \cap F(v)$. Then it is natural to require the existence at x_0 of a touching ball from *the right* (not from the left) and a proper asymptotic behavior for v near x_0 that could force a contradiction. Therefore it is useful to introduce another kind of "subsolution" with these characteristics. It turns out that a natural way to construct such a type of functions is to start from a subsolution or a solution (in the viscosity sense) and to build parallel surfaces. Here is the simplest example.

Let u be a solution in D and take

$$v_t(x) = \sup_{B_t(x)} u, \qquad t > 0. \tag{2.16}$$

Let us examine the properties of v_t. Since v_t is the supremum of a family of translations of u, it is subharmonic both in $\Omega^+(v_t)$ and $\Omega^-(v_t)$. Now let

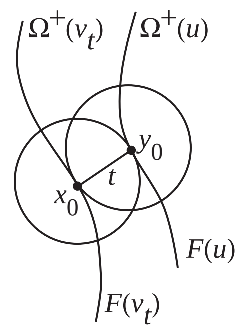

Figure 2.1

$x_0 \in F(v_t)$. This means that $B_t(x_0)$ is touching $F(u)$ from $\Omega^-(u)$ at a point y_0 (see Figure 2.1). Therefore we have the following.

(a) x_0 is regular from the right since $B_t(y_0) \subset \Omega^+(v_t)$ and $B_t(y_0)$ touches $F(v_t)$ at x_0.

(b) y_0 is a regular point from the left for $F(u)$, thus, near y_0,

$$u^-(x) = \beta\langle x - y_0, \nu\rangle^- + o(|x - x_0|) \qquad (\beta > 0)$$

in $B_t(x_0)$, while

$$u^+(x) = \alpha\langle x - y_0, \nu\rangle^+ + o(|x - x_0|) \qquad (\alpha \geq 0)$$

in $\mathcal{C}B_t(x_0)$, with $G(\alpha, \beta) \geq 0$.
Hence, since $v_t(x) \geq u(x + y_0 - x_0)$, near x_0,

$$v_t^+(x) \geq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|)$$

in $B_t(y_0)$, while

$$v_t^-(x) \leq \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|)$$

in $\mathcal{C}B_t(y_0)$.

Let us summarize the properties of v_t:

(i) $\Delta v \geq 0$ both in $\Omega^+(v)$ and $\Omega^-(v)$.

(ii) Whenever $x_0 \in F(v)$ has a touching ball $B \subset \Omega^+(v)$, then near x_0, in B

$$v^+(x) \geq \bar{\alpha}\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \tag{2.17}$$

and in $\mathcal{C}B$

$$v^-(x) \leq \bar{\beta}\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \tag{2.18}$$

with $\bar{\alpha}, \bar{\beta} \geq 0$, and

$$G(\bar{\alpha}, \bar{\beta}) \geq 0\ .$$

We call a function v with properties (i), (ii) an *R-subsolution.* We can now prove the following comparison result:

Lemma 2.1. *Let u, v be a (viscosity) solution and an R-subsolution in D, respectively. Then if $u \geq v$, $u > v$ in $\Omega^+(v)$ and $x_0 \in F(u) \cap F(v)$, x_0 cannot be a regular point from the right.*

Proof. If x_0 is a regular point from the right, we have, near x_0, in a touching ball $B \subset \Omega^+(v)$, nontangentially,

$$u^+(x) = \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \ , \quad v^+(x) \geq \bar{\alpha}\langle x - x_0, \nu\rangle^- + o(|x - x_0|)$$

and in $\mathcal{C}B$, nontangentially,

$$u^-(x) = \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \ , \quad v^-(x) \geq \bar{\beta}\langle x - x_0, \nu\rangle^+ + o(|x - x_0|)$$

with

$$G(\alpha, \beta) \leq 0 \quad \text{and} \quad G(\bar{\alpha}, \bar{\beta}) \geq 0 \ . \tag{2.19}$$

Since $u \geq v$, we have $\alpha \geq \bar{\alpha}$ and $\beta \leq \bar{\beta}$. The strict monotonicity of G and (2.19) imply $\alpha = \bar{\alpha}$, $\beta = \bar{\beta}$. But $u - v$ is a positive superharmonic function in $\Omega^+(v)$. By the Hopf principle, since x_0 is a regular point, we have $(u - v)(x) \geq |x - x_0|$ with $\varepsilon > 0$, radially into $\Omega^+(v)$ along ν from x_0, a contradiction. □

A more refined version, of a "continuous deformation" nature, is the following theorem that we will use later.

Theorem 2.2. *Let v_t, $0 \leq t \leq 1$, be a family of R-subsolutions, continuous in $\bar{\Omega} \times [0, 1]$. Let u be a solution continuous in $\bar{\Omega}$. Assume that*

(i) *$v_0 \leq u$ in Ω,*

(ii) *$v_t \leq u$ on $\partial\Omega$ and $v_t < u$ in $\overline{\Omega^+(v_t)} \cap \partial\Omega$ for $0 \leq t \leq 1$,*

(iii) *every point $x_0 \in F(v_t)$ is a regular point from the right,*

(iv) *the family $\Omega^+(v_t)$ is continuous, that is, for every $\varepsilon > 0$,*

$$\Omega^+(v_{t_1}) \subset \mathcal{N}_\varepsilon(\Omega^+(v_{t_2}))$$

whenever $|t_1 - t_2| \leq \delta(\varepsilon)$.

Then

$$v_t \leq u \quad in \quad \Omega$$

for every $t \in [0, 1]$.

Proof. Let $E = \{t \in [0, 1] : v_t \leq u \text{ in } \bar{\Omega}\}$. E is obviously closed. Let us show that it is open. If $v_{t_0} \leq u$, from (ii) and the strong maximum principle it follows that $v_{t_0} < u$ in $\Omega^+(v_{t_0})$. Since every point of $F(v_{t_0})$ is regular from the right, Lemma 2.1 and (ii) imply that $\overline{\Omega^+(v_{t_0})}$ is compactly contained in $\Omega^+(w)$, up to $\partial\Omega$.

From assumption (iv), the openness of E follows. □

Remark. Comparisons of this nature are necessary when a maximum principle (or uniqueness) is not available. For instance, the classical reflection method of Alexandrov, of Serrin and of Gidas, Ni, and Nirenberg is of this nature.

The family v_t constructed in (2.16) is an admissible family for the comparison in the previous theorem. It can be used for a comparison principle that says: *if u_1, u_2 are solutions such that $u_1 \leq u_2$ and near $\partial\Omega$, $\sup_{B_t(x)} u_1 \leq u_2(x)$, then also in the interior of Ω, $\sup_{B_t(x)} u_1 \leq u_2(x)$*, keeping in particular $F(u_2)$ t-away from $F(u_1)$.

We shall see the usefulness of this kind of principle in proving strong regularity results for the free boundary.

Chapter 7

Parabolic Free Boundary Problems

7.1. Introduction

In this part we will discuss several free boundary problems of evolution type.

The typical problem consists mainly of finding a function u in a space time domain that satisfies some parabolic equation (usually the heat equation) when u is different from zero (one or two phases), and across the surface where u vanishes, there is a balance condition between the speed of the interphase, that is, of the zero level surface, and may be the flux discontinuity, i.e., the jump on the normal derivative of u, and may also be some geometrical quantity along the free boundary, like its curvature.

As in the elliptic case, and different from, for instance, conservation laws, where the conservation law or constitutive relation is a smooth function across a shock, and where it is the solution itself that jumps from an admissible value to another, in the case of a parabolic free boundary problem, it is the law itself that changes discontinuously.

Some examples are Stefan-like problems, that is, problems of melting or solidification, where caloric energy changes discontinuously across the melting temperature (see, e.g., [D], [F1], [F2], [K]), problems of flame propagation, where the existence of a sharp flame front is assumed (sometimes constructed as a limit of singular perturbation problems as in our example of Chapter 1 ([CV], [CLW1], [CLW2])), and more recently problems arising from financial mathematics where the edging strategy changes when the present value of an option goes through a certain threshold.

Mathematically, the general lines of attack of the problems follow naturally those of the elliptic case, and the necessary tools are similar: Harnack type inequalities in the interior and at the boundary of one of the phases, optimal regularity, monotonicity formulas, and the perturbation techniques, through continuous families of supersolutions.

But one soon realizes that the study of these problems entails new, serious difficulties.

The first obvious difficulty is the role of time, and it is already present in the standard Harnack inequality, which says that for a nonnegative solution, in a parabolic cylinder, the past controls the future only from below, since one can obviously "inject" as much heat as one likes from the sides of the cylinder, something that the bottom will never see.

This implies that on one hand stronger hypotheses have to be made in the geometry of the problem, or the starting configuration, and that the very strong local conclusions we obtained in the elliptic case do not necessarily hold.

An example of this phenomenon is the waiting time counterexample to instantaneous regularization of the free boundary for a two-phase Stefan problem, which we discuss below.

A second difficulty comes from the different homogeneities corresponding on one hand to the evolution equation that u satisfies away from the transition surface (being a parabolic equation, it remains invariant under dilations that are linear in space and quadratic in time), and on the other hand to that of the free boundary that, relating speed with flux, is of a Hamilton-Jacobi nature and as such scales homogeneously of degree one, both in time and space.

Hence we are faced with the dilemma that parabolic scaling will keep the heat equation but will, generically, make the free boundary vertical and information will be lost while hyperbolic scaling will preserve the asymptotic geometry of the free boundary but we will lose the time derivative in the parabolic part, disconnecting u in time away from the free boundary.

From these difficulties comes the very delicate balance in the intermediate rescaling that we will use in the Stefan problem, which allows us (an almost miraculous fact) to reconstruct a Dini domain out of our iteration process.

Finally, the third ingredient we are missing is that of the very strong, local geometric measure-theoretical properties of the free boundary, and this can usually be seen in the case in which there is focusing (total melting of a solid region or extinction of a flame) and all hopes of universal regularity or nondegeneracy are broken.

This is one more reason why we can only treat geometries for which one can assert a priori that, at least after finite time, focusing is ruled out (for instance, flatness hypotheses).

7.2. A class of free boundary problems and their viscosity solutions

Perhaps the best known example of parabolic two-phase free boundary problem is the Stefan problem, a simplified model describing the melting (or solidification) of a material with a solid-liquid interphase.

The concept of solution can be stated in several ways: classical solutions, weak solutions on divergence form or viscosity solutions.

Locally, a classical solution of the Stefan problem may be described as follows. In the unit cylinder $\mathcal{C}_1 = B_1' \times (-1,1) \subset R^m \times R$ we have two complementary domains, Ω^+ and $\Omega^- = \mathcal{C}_1 \setminus \bar{\Omega}^+$, separated by a smooth surface $F = \partial\Omega^+ \cap \mathcal{C}_1$. In Ω^+ and Ω^-, respectively, we have two solutions u_1 and u_2 of the heat equations

$$\Delta u_1 - a_1 \partial_t u_1 = 0 = \Delta u_2 - a_2 \partial_t u_2$$

with $u_2 \le 0 \le u_1$.

The functions u_1 and u_2 are C^1 up to F, and along F both $u_1 = u_2 = 0$ and the interphase energy balance condition

$$\frac{\partial_t u_i}{|\nabla u_i|} = |\nabla u_1| - |\nabla u_2| \tag{7.1}$$

are satisfied. Note that the ratios (for $i = 1, 2$) on the left-hand side of (7.1) represent the speed of F in the direction $-\nu$, $\nu = \nabla u^+ / |\nabla u^+|$.

What can be constructed for all times are weak solutions to the equation

$$\Delta u \in \beta(u)_t$$

with $\beta(u) = a_1 u^+ - a_2 u^- + \operatorname{sign} u$, subject to proper initial and boundary conditions ([K], [F1], [F2]).

From [CE], u is continuous in $\mathcal{C}_1$ and heuristically, $u_1 = u^+$ in $\Omega^+ = \{u > 0\}$, $u_2 = -u^-$ in $\Omega^- = \{u \le 0\}^0$ and $F = \partial\{u > 0\} \cap \mathcal{C}_1$ becomes the *free boundary*.

Modeled on the example of the Stefan problem, we formally introduce the following class of free boundary problems (f.b.p. in the sequel): *to find a function u, continuous in $\mathcal{C}_1 = B_1' \times (-1,1)$, such that the following hold.*

(a) $\Delta u - a_1 u_t = 0$ in $\Omega^+(u) = \{u > 0\} \cap \mathcal{C}_1$, and
$\Delta u - a_2 u_t = 0$ in $\Omega^-(u) = \{u \le 0\}^0 \cap \mathcal{C}_1$.

(b) On $F(u) = \partial\Omega^+(u) \cap \mathcal{C}_1$, the (*free boundary*) condition,

$$V_\nu = -G(u_\nu^+, u_\nu^-) \tag{7.2}$$

must be satisfied, where $a_1 > 0$, $a_2 > 0$ and $V_\nu(\cdot, \tau)$ is the speed of the surface $F(u) \cap \{t = \tau\}$ in the direction $\nu = \nabla u^+ / |\nabla u^+|$.

The basic requirements on the function $G : [0,\infty)^2 \mapsto \mathbb{R}$ follow.

(i) G is continuous in $[0,\infty)^2$.

(ii) G is strictly increasing in u_ν^+ and strictly decreasing in u_ν^-.

(iii) $G \to +\infty$ when $u_\nu^+ - u_\nu^- \to \infty$.

Note that there is no nondegeneracy property of G, since this is the case of the Stefan problem. In fact, harmonic functions are stationary solutions of a Stefan problem and we cannot hope for nondegeneracy properties in this case.

Let us now define what we mean by a C^k-classical ($k \geq 1$) subsolution, supersolution and solution.

Definition 7.1. A function $v \in C(\mathcal{C}_1)$ is a C^k-classical subsolution if the following hold.

(a) $v \in C^k(\bar{\Omega}^+(v)) \cap C^k(\bar{\Omega}^-(v))$.

(b) $\Delta v - a_1 v_t \geq 0$ in $\Omega^+(v)$ and $\Delta v - a_2 v_t \geq 0$ in $\Omega^-(v)$.

(c) The free boundary $F(v) = \partial\Omega^+(v) \cap C_1$ is a C^k-surface, $|\nabla v^+| > 0$ on $F(v)$ and

$$-V_\nu = \frac{v_t^+}{v_\nu^+} \leq G(v_\nu^+, v_\nu^-) \tag{7.3}$$

where $\nu = \frac{\nabla v^+}{|\nabla v^+|}$.

The free boundary condition (7.3) indicates that the speed of $F(v)$ towards the "solid phase" $\{u \leq 0\}^0$ is "smaller" for a subsolution than for a solution. If the inequality in (7.3) is *strict*, we call v a *strict subsolution.*

A C^k-classical supersolution is defined by reversing the inequalities in (b) and (7.3). A C^k-classical solution is both a C^k-classical subsolution and supersolution.

As in the elliptic case, we use classical strict subsolutions and supersolutions as test functions to define viscosity solutions.

Again, from now on, it is understood that $k = 2$. Let

$$Q_r(x_0, t_0) = B_r'(x_0) \times (t_0 - r^2, t_0] \ .$$

Definition 7.2. A function $u \in C(\mathcal{C}_1)$ is a viscosity subsolution (supersolution) to f.b.p. if, whenever $(x_0, t_0) \in \mathcal{C}_1$ and v is a classical strict supersolution (subsolution) in $Q_r(x_0, t_0) \subset \mathcal{C}_1$, then v cannot touch u from above (below) at (x_0, t_0); u is a viscosity solution if it is both a subsolution and a supersolution.

In other words, if u is a subsolution and v is a classical strict supersolution in $Q_r(x_0, t_0) = Q_r$, and $(x_0, t_0) \in F(u)$, *it is not possible that* $v > u$ in $Q_r \setminus \{(x_0, t_0)\}$ and $v(x_0, t_0) = u(x_0, t_0)$. Analogously if u is a supersolution and v is a subsolution in $Q_r(x_0, t_0)$, it is not possible that $v < u$ in $Q_r \setminus \{(x_0, t_0)\}$ and $v(x_0, t_0) = u(x_0, t_0)$.

It is easy to check that a C^2-classical (sub, super) solution is a solution in the viscosity sense and that a viscosity solution u which is of class C^2, with its free boundary, in $\bar{\Omega}^+(u)$ and $\bar{\Omega}^-(u)$ is also a C^2-classical solution.

Sometimes it is more convenient to use the definition of viscosity solution in the following equivalent form.

Definition 7.2′. A function $u \in C(\mathcal{C}_1)$ is a *viscosity subsolution (supersolution)* to f.b.p. if, for any cylinder Q, $Q \subset\subset \mathcal{C}_1$, and for every classical strict supersolution (subsolution) v in Q, $u < v$ on $\partial_p Q$ implies $u < v$ in $\bar{Q}$; u is a *viscosity solution* if it is both a subsolution and supersolution.

Indeed Definitions 7.2 and 7.2′ are equivalent.

(a) Definition 7.2′ $\Rightarrow$ Definition 7.2. Let u be a viscosity subsolution, $(x_0, t_0) \in \mathcal{C}_1$, $Q_r = Q_r(x_0, t_0) \subset \mathcal{C}_1$ and let v be a classical strict supersolution in Q_r such that $v > u$ in $Q_r \setminus \{(x_0, t_0)\}$. If $u(x_0, t_0) = v(x_0, t_0)$, we have a contradiction to Definition 7.2′ (in a slightly smaller cylinder $Q_{r'}(x_0, t_0)$, $r' < r$).

(b) Definition 7.2 $\Rightarrow$ Definition 7.2′. Let u be a viscosity subsolution, $Q \subset\subset \mathcal{C}_1$, and let v be a classical strict supersolution in Q such that $v > u$ on $\partial_p Q$. Suppose $v > u$ in $\bar{Q}$ is not true. Then there is a first time τ and a point $(x_0, \tau) \in F(u) \cap F(v)$ such that $u(x_0, \tau) = v(x_0, \tau)$. Assume e_n is the normal direction to $F(v)$ at (x_0, τ). Then near (x_0, τ) we can write

$$
\begin{aligned}
v(x,t) = &\big(\alpha^+(x - x_0)_n + \beta^+(t - \tau)\big)^+ \\
&- \big(\alpha^-(x - x_0)_n + \beta^-(t - \tau)\big)^- + o(|x - x_0| + |t - \tau|)
\end{aligned}
$$

where $\alpha^\pm = v^\pm_{x_n}$, $\beta^\pm = v^\pm_t$, with $\alpha^+ > 0$, $\alpha^- > 0$ and

$$
\frac{\beta^-}{\alpha^-} = \frac{\beta^+}{\alpha^+} > G(\alpha^+, \alpha^-) \ .
$$

On the other hand, for $t \leq \tau$, $u(x,t) < v(x,t)$, and therefore, using Theorem 7.1, we deduce that

$$\frac{\beta^-}{\alpha^-} \leq C(\alpha^+, \alpha^-) \ .$$

Contradiction. □

Remark. Weak solutions of the two-phase Stefan problem are viscosity solutions. This follows from the following comparison theorem in [F1]: if u and v are a subsolution and supersolution, respectively, in $\mathcal{C}_1$ and $u > v$ on $\partial_p \mathcal{C}_1$, then $u > v$ in $\mathcal{C}_1$.

7.3. Asymptotic behavior and free boundary relation

From the results in Section 13.3, and in particular from Lemma 13.19, we can deduce asymptotic inequalities at regular points of the free boundary and give a weak formulation of the free boundary condition. First, let us clarify what we mean by regular point in this case.

Definition 7.3. A point $(0,0)$ on $F(u)$ is a regular point from the right (from the left) if it has a touching ball $B \subset \Omega^+(u)$ $(B \subset \Omega^-(u))$ with tangent plane, say $\alpha e_n + \beta t = 0$, of finite slope β/α.

The following theorem holds ([ACS1]):

Theorem 7.1. *Let u be a viscosity solution in $\mathcal{C}_1$ of the f.b.p., according to either Definition 7.2 or 7.2′. Suppose $(0,0) \in F(u)$ and that near $(0,0)$, for $t \leq 0$, the following asymptotic inequality holds.*

(a) $u(x,t) \geq (\beta^+ t + \alpha^+ \langle x, \nu \rangle)^+ - (\beta^- t + \alpha^- \langle x, \nu \rangle)^- + o(d(x,t))$
with $\alpha^+, \alpha^-, \beta^+, \beta^- \in \mathbb{R}$, $\alpha^+ > 0$, $\alpha^- \geq 0$

or

(b) $u(x,t) \leq (\beta^+ t + \alpha^+ \langle x, \nu \rangle)^+ - (\beta^- t + \alpha^- \langle x, \nu \rangle)^- + o(d(x,t))$
with $\alpha^+, \alpha^-, \beta^+, \beta^- \in \mathbb{R}$, $\alpha^+ \geq 0$, $\alpha^- > 0$

where ν denotes the inward spatial direction to $\Omega^+(u)$ at $(0,0)$ and $d(x,t) = |x| + |t|$.

Then

$$\frac{\beta^+}{\alpha^+} \geq G(\alpha_+, \alpha_-) \qquad \textit{(supersolution condition)} \tag{7.4}$$

in case (a), while

$$\frac{\beta^-}{\alpha^-} \leq G(\alpha_+, \alpha_-) \qquad \textit{(subsolution condition)} \tag{7.5}$$

in case (b).

Proof. We give it only for case (a), the other case being completely analogous.

Let $\mathcal{N}$ denote the neighborhood in which (a) is valid and assume $\nu = e_n$. Define, for $\lambda > 0$,

$$u_\lambda(x,t) = \frac{1}{\lambda} u(\lambda x, \lambda t) \ .$$

Then, in $\mathcal{N}$,

$$u_\lambda(x,t) \geq (\alpha^+ x_n + \beta^+ t)^+ - (\alpha^- x_n + \beta^- t) + o(1)$$

where $o(1) \to$ uniformly in $\mathcal{N}$, when $\lambda \to 0$.

Now if (7.4) were false, there would exist $\eta > 0$ such that

$$\frac{\beta^+}{\alpha^+} \leq G(\alpha^+, \alpha^-) - \eta \ .$$

We show that this leads to a contradiction. Let R be a small parabolic neighborhood of $(0,0)$ contained in $\mathcal{N}$, i.e.,

$$R = \{(x', x_n, t) : |x'| < a, \ |x_n| < b, \ -t_0 < t \leq 0\}$$

with a, b, t_0 small. Define

$$\psi(x', x_n, t) = \bar{\alpha}^+ x_n + \bar{\beta}^+ t - ct^2 + x_n^2 - \frac{|x'|^2}{2(n-1)}$$

where $\bar{\alpha}^+ = \alpha^+ + \varepsilon$, $\bar{\beta}^+ = \beta^+ + \varepsilon$ for some ε, positive and small, to be determined later. Choose $C > 0$, large, so that the level surface $\{\psi = 0\}$ is strictly convex and $\{\psi > 0\} \cap R \subset \Omega^+(u)$.

Observe that

$$\Delta\psi - \lambda a_1 \psi_t = 1 - \lambda a_1(\bar{\beta}^+ - 2Ct) > 0$$

in R if λ is small enough.

Claim. *If ε, a, b, t_0 are small enough, depending on η, G, then the function*

$$\varphi = \psi^+ - \frac{\bar{\alpha}^-}{\bar{\alpha}^+}\psi^- \quad \textit{with} \quad \bar{\alpha}^- = \alpha^- + \varepsilon$$

is a classical strict subsolution to the f.b.p. in R.

To prove the claim, it is enough to show that

$$\frac{\varphi_t^+}{|\nabla\varphi^+|} \leq G(|\nabla\varphi^+|, |\nabla\varphi^-|) \tag{7.6}$$

on $\{\varphi = 0\} \cap R$. By continuity, $\varphi_t^+ / |\nabla\varphi^+|$ is close to

$$\frac{\varphi_t^+(0,0)}{|\nabla\varphi^+(0,0)|} = \frac{\bar{\beta}^+}{\bar{\alpha}^+}$$

uniformly in R. On the other hand

$$\frac{\bar{\beta}^+}{\bar{\alpha}^+} = \frac{\beta^+}{\alpha^+} + O(\varepsilon) \leq G(\alpha^+, \alpha^-) - \eta/2$$

if $\varepsilon = \varepsilon(\eta)$ is small enough.

Since G is continuous in all of its arguments, we obtain (7.6) if the size of R is small enough.

If we now choose λ very small, we see that

$$u_\lambda(x,t) \geq (\alpha^+ x_n + \beta^+ t)^+ - (\alpha^- x_n + \beta^- t)^- + o(1) > \varphi(x,t)$$

on the parabolic boundary of R, hence in $R \setminus \{(0,0)\}$. Since u_λ is a viscosity solution of the f.b.p., we must have $u_\lambda > \varphi$ in $\bar{R}$. But then $0 = u_\lambda(0,0) > \varphi(0,0) = 0$ gives a contradiction. □

Remark. Let u be a viscosity solution in $\mathcal{C}_1$ and let $(0,0) \in F(u)$. If $(0,0)$ is a regular point from the right (resp. left), then (a) (resp. (b)) holds near $(0,0)$. Precisely, let B an $(n+1)$-dimensional ball touching $F(u)$ at $(0,0)$ from the positive side. Let $\nu = e_n$ be the spatial inward unit normal to B at $(0,0)$ and let $\bar{\alpha} x_n + \bar{\beta} t = 0$ be the equation of the tangent plane. Assume that the slope $-\bar{\beta}/\bar{\alpha}$ is finite ($\bar{\alpha} > 0$). Then apply Lemma 13.19 to u^+ and u^-. We obtain

$$u^+(x,t) \geq (\alpha^+ x_n + \beta^+ t)^+ + o(d(x,t)) \tag{7.7}$$

in B, with $\alpha^+ > 0$ (which we may assume finite) and $\beta^+ \in \mathbb{R}$, and

$$u^-(x,t) \leq (\alpha^- x_n + \beta^- t)^- + o(d(x,t)) \tag{7.8}$$

in $\mathcal{C}B$, with $\alpha^- > 0$ (finite) and $\beta^+ \in \mathbb{R}$.

Then the conclusion of Theorem 7.1 follows. In particular, note that

$$\infty > \frac{\bar{\beta}}{\bar{\alpha}} = \frac{\beta^+}{\alpha^+} \geq G(\alpha_+, \alpha_-)$$

and therefore $\sup\{\alpha : (7.7) \text{ holds}\} < \infty$.

Regular points from the left can be handled in the same way. Also, we know, from Lemma 13.19, that equality holds in (7.7) and (7.8), along paraboloids of the form $t = -\gamma x_n^2$, $\gamma > 0$.

7.4. R-subsolutions and a comparison principle

As in Section 2.3, for comparison purposes, it is useful to introduce another class of subsolutions. We say that $v \in C(\mathcal{C}_1)$ is an *R-subsolution* if the following hold.

(i) $\Delta v - a_1 v_t \geq 0$ in $\Omega^+(v)$, $\Delta v - a_2 v_t \geq 0$ in $\Omega^-(v)$.

(ii) Whenever $(x_0, t_0) \in F(v)$ has a touching $(n+1)$-dimensional ball $B \subset \Omega^+(v)$, then near (x_0, t_0), in B,

$$v^+(x,t) \geq (\alpha^+\langle x - x_0, \nu\rangle + \beta^+(t - t_0))^+ + o(|x - x_0| + |t - t_0|) \tag{7.9}$$

and in $\mathcal{C}B$

$$v^-(x,t) \leq (\alpha^-\langle x - x_0, \nu\rangle + \beta^-(t - t_0))^- + o(|x - x_0| + |t - t_0|) \tag{7.10}$$

where $\alpha^{\pm} \geq 0$, $\beta^{\pm} \in \mathbb{R}$, $\nu = \nu(x_0, t_0)$ is the spatial inward unit normal to B at (x_0, t_0),

$$\beta^+ \leq \alpha^+ G(\alpha^+, \alpha^-) \qquad (\text{or } \beta^- \leq \alpha^- G(\alpha^+, \alpha^-)) \ . \tag{7.11}$$

Here

$$\alpha^+\langle x - x_0, \nu\rangle + \beta^+(t - t_0) = 0 = \alpha^-\langle x - x_0, \nu\rangle + \beta^-(t - t_0)$$

are equations for the tangent plane to B at (x_0, t_0).

The following analog of Lemma 2.1 holds.

Theorem 7.2. *Let u, v be a viscosity solution and an R-subsolution in $\mathcal{C}_1$, respectively, with $u \geq v$. Moreover let $u > v$ in $\Omega^+(v)$ and $(0,0) \in F(v) \cap F(u)$. Then $(0,0)$ cannot be a regular point from the right.*

Proof. Suppose $B \subset \Omega^+(v)$ touches $(0,0)$, with tangent plane $\alpha x_n + \beta t = 0$, $-\beta/\alpha$ finite. Then near $(0,0)$ in B, we have

$$v^+(x,t) \geq (\alpha^+ x_n + \beta^+ t)^+ + o(d(x,t)) \qquad (\alpha^+ > 0) \tag{7.12}$$

and in $\mathcal{C}B$,

$$v^-(x,t) \leq (\alpha^- x_n + \beta^- t)^- + o(d(x,t)) \ . \tag{7.13}$$

From Theorem 7.1 and the Remark after it, we know that near $(0,0)$

$$u^+(x,t) \geq (\bar{\alpha}^+ x_n + \bar{\beta}^+ t)^+ + o(d(x,t)) \tag{7.14}$$

in B and

$$u^-(x,t) \geq (\bar{\alpha}^- x_n + \bar{\beta}^- t)^- + o(d(x,t)) \tag{7.15}$$

in $\mathcal{C}B$, with equality along any paraboloid of the form $t = -\gamma x_n^2$, $\gamma > 0$.

We have the following.

(a) Computing along the paraboloid $t = -\gamma x_n^2$, $\gamma > 0$, from (7.12)–(7.15) and $u \geq v$, we deduce that

$$\bar{\alpha}^+ \geq \alpha^+ \ , \quad \bar{\alpha}^- \leq \alpha^-$$

and therefore

$$G(\bar{\alpha}^+, \bar{\alpha}^-) \geq G(\alpha^+, \alpha^-) \ . \tag{7.16}$$

(b) From (7.11) and (7.4) of Theorem 7.1, since $\alpha^+ > 0$,

$$G(\bar{\alpha}^+, \bar{\alpha}^-) \leq \frac{\bar{\beta}^+}{\bar{\alpha}^+} = \frac{\beta^+}{\alpha^+} \leq G(\alpha^+, \alpha^-) \tag{7.17}$$

which implies, by the strict monotonicity of G,

$$\bar{\alpha}^+ = \alpha^+ \ , \quad \bar{\alpha}^- = \alpha^- \ .$$

On the other hand $u - v$ is supercaloric in $\Omega^+(v)$, so that the Hopf principle along $t = -\gamma x_n^2$, gives, for some $\varepsilon > 0$,

$$u - v \geq \varepsilon x_n^+ \ .$$

Contradiction. □