Introduction

This book and its companion, [**Row2**] (henceforth, referred to as "Volume 2"), grew out of a year-long graduate course in algebra, whose goal was to provide algebraic tools to students from all walks of mathematics. These particular students had already seen modules and Galois theory, so in the first semester we focused on affine algebras and their role in algebraic geometry (Chapters 5 through 10 of this book), and the second semester treated those aspects of noncommutative algebra pertaining to the representation theory of finite groups (Chapters 19 and 20 of Volume 2). The underlying philosophy motivating the lectures was to keep the material as much to the point as possible.

As the course material developed over many years, I started putting it into book form, but also rounding it out with some major developments in algebra in the twentieth century. There already are several excellent works in the literature, including the books by N. Jacobson, S. Lang, and P.M. Cohn, but developments of the last thirty years warrant another look at the subject. For example, Zelmanov's solution of the restricted Burnside problem, affine Lie algebras, and the blossoming of quantum group theory all took place after these books were written, not to mention Wiles' proof of Fermat's Last Theorem. During this time, research in algebra has been so active that there is no hope of encompassing all of the advances in one or two volumes, as evidenced by Hazewinkel's ongoing project, *Handbook of Algebra*. Nevertheless, several major themes recur, including the interplay of algebra and geometry, and the study of algebraic structures through their representations into matrices, thereby bringing techniques of linear algebra into play. The goal in these two volumes is to bring some harmony to these themes.

Since a thorough account of the interesting advances in algebra over the last 50 years would require a work of epic proportions, selections were made at every step. The task was further complicated when some respected colleagues and friends pointed out that any text suitable for graduate students in the United States must contain a treatment of modules over a principal domain as well as the basics of Galois theory, which, together with their exercises, account for some 100 pages. So the project grew to two volumes. Nevertheless, the two volumes were planned and written together, and many results were included in the first volume for use in the second. Accordingly, a unified introduction seems justified.

The first volume features (commutative) affine algebras, and aims quickly for the fundamental theorems from commutative algebra needed for affine algebraic geometry. This means a thorough treatment of transcendence degree and its relation to the lengths of chains of prime ideals in an affine algebra, and some theory of Noetherian rings. In the process, we take the time to develop enough module theory to present basic tools from linear algebra (e.g., rational form, Jordan form, Jordan decomposition) and also enough field theory for the applications later on in the two volumes. For example, Chapter 4 contains a thorough account of the norm and trace in a field extension; its appendices include transcendental field extensions, Luroth's theorem, and the discriminant and resultant. Other appendices along the way include Makar-Limanov's application of derivations to basic questions of affine geometry, and Gröbner bases (related to computational algebra). The later chapters involve applications used in arithmetic and algebraic curves and surfaces, culminating in the algebraic aspects of elliptic curves and the underlying number theory.

The second volume focuses on structures arising from matrices. The twentieth century saw the sprouting of algebraic structures — groups and Lie algebras are used throughout mathematics, and are unified in Hopf algebras — and these are treated, mostly in the context of representation theory. On the other hand, some of the most beautiful results in algebra in the last 50 years have involved interrelations among different structures, such as Gabriel's classification of indecomposable modules of finite representation type in terms of Dynkin diagrams from graph theory and Lie algebra, and Zelmanov's solution of the restricted Burnside problem from group theory by means of Lie algebras and Jordan algebras. The reader is offered a sample of these various theories as well.

To keep within the allotted space and to permit the book to serve its original purpose as a general text, the following decisions were made:

1. The organization is at four levels — main text, supplements, appendices, and exercises. An effort is made to keep this hierarchy consistent,

in the sense that theorems from the main text do not rely on results from the supplements or appendices; the appendices do not rely on results in the exercises, and so forth. As a result, almost every chapter branches out with at least one appendix related to its material. (The exception to this rule is Chapter 25 in [**Row2**], which requires the categorical framework supplied in Appendix 1A.) The exercises often are extensions of the text, containing material which I did not have the heart to exclude altogether, often with very extensive hints amounting to almost complete proofs.

2. Most of the theory is presented over algebraically closed fields of characteristic 0, although there is material on central simple algebras and Azumaya algebras (Chapters 24 and 25 in [**Row2**]), to indicate how algebra looks over more general commutative rings.

3. The material in the main text is handled in as elementary a fashion as possible. For example, categorical methods appear only in the appendices and exercises.

4. Far-reaching generalizations to noncommutative rings, although close to my heart, often are relegated to the exercises, if not cut altogether.

5. Homological methods, used throughout algebra in the last 50 years, are only touched on, and the student should supplement this book with a text on homological algebra.

Prerequisites are discussed in length in Chapter 0. The reader should be familiar with the basic structures of undergraduate algebra: groups and rings. Most of the requisite theory can be found in any standard undergraduate text, such as Herstein's *Topics in Algebra* or Gallian's *Contemporary Abstract Algebra*, referred to as [**Gal**]. When needed, results are cited from my undergraduate text (*Algebra: Groups, Rings, and Fields)*, which is referred to as [**Row1**]. More details can be often be found in Jacobson's text *Basic Algebra I*, referred to as [**Ja1**].

Some prerequisites, such as the Chinese Remainder Theorem, are needed throughout; others, such as Lagrange interpolation, are included as a convenient reference when needed. The hierarchy of the book also is followed here; quadratic forms, covered in Appendix A, are needed only in the appendices; likewise for ordered groups (Appendix B).

As mentioned earlier, this volume developed from a one-semester course, that coverered the main parts (without appendices) of Chapters 1, 2 (through Theorem 2.31), 3, Chapter 5, Chapter 6 (through Lemma 6.39), Chapters 7 through 9, and Chapter 10 (as far as time permitted). One could easily add another semester with Chapter 4 and perhaps the remainder of Chapter 2. Parts of Chapters 2, 4, or 6 have been used at times in graduate student seminars. I hope this book will prove useful to students who are looking for a solid basis in algebra.

Many people have generously contributed of their time and energy to help bring this work to fruition. First thanks are due to Lance Small, who years ago suggested that this book be published with the AMS, and who provided useful advice about topics. Most of an earlier draft of this manuscript was written during my Sabbatical visiting Darrell Haile at Indiana University at Bloomington (in 2000(!)), and the final draft of Volume 1 was made while visiting David Saltman at the University of Texas at Austin, in 2005. I thank both Indiana University and the University of Texas for their gracious hospitality during these periods. Lenny Makar-Limanov graciously permitted free use of his unpublished notes on the use of derivations in studying affine geometry, and went over the material with me. Sue Montgomery explained the newer results about the classification of finite dimensional Hopf algebras. Improvements and corrections were suggested by Jonathan Beck, Alexei Belov, Boris Kunyavski, Malka Schaps, Roy Ben-Ari, Eli Matzri, Tal Perri, and Shai Sarussi. But I would like to express special gratitude to David Saltman, editor of this AMS series, and Uzi Vishne, both of whom spent hours going through the text at various stages and suggested many crucial corrections and improvements. Saltman also explained some key points in the geometry involved. From the publishing side, Sergei Gelfand has been patient for years, while this work was maturing. Finally, as always, Miriam Beller has helped enormously with the technical preparation of the manuscript.