Preface

This text is meant to serve as an introduction to the theory and applications of asymptotic approximations. Such approximations are characterized by the property that they are made more accurate by the tuning of an auxiliary parameter. Frequently (but not always) asymptotic approximations arise as the partial sums of a formal power series in a small parameter ϵ, and almost as frequently the formal series is divergent for all $\epsilon \neq 0$. The fact that the series cannot be summed is in no way in contradiction to the utility of the partial sums as asymptotic approximations made more accurate by reducing the magnitude of ϵ. The distinction between such "asymptotic series" and convergent series was established concretely as recently as the end of the nineteenth century, although asymptotic approximations had been in practical use for a very long time before then. This indicates that the distinction between asymptotic and convergent series is a potential point of confusion, especially for students considering series in applications for the first time.

These days, it is not long after students begin studying mathematics, physics, engineering, or another of the quantitative applied sciences when they first encounter in their reading an expression of the form $x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$ that is presented as a solution of some equation for an unknown x involving a small parameter ϵ. On the one hand, students learn quickly how to find the coefficients x_n; one just substitutes the expression into the equation to be solved and gathers together the like powers of ϵ. On the other hand, without further analysis of a different sort it is not at all clear what the meaning of the so-obtained formal series actually is. Is the series convergent, asymptotic, both, or neither? What exactly does the calculation of the first few x_n's buy? More often than not students are left to interpret the series on their own, assuming that they have been trained to think about the precise meaning of an infinite series in the first place. It is my hope that this text helps to clear up some of the potential confusion associated with asymptotic series (and asymptotic approximations more generally). I also hope that this text will generate interest in the fascinating subject of asymptotic analysis and its applications.

This book was written for students with a background in differential equations, advanced calculus (rigorous limits), complex variables, and matrix algebra at the level of undergraduate courses. Some of the applications involve partial differential equations and Fourier and Laplace transform methods, but for the most part this material is introduced where it is needed. In fact, several topics are introduced in this text that are not ordinarily viewed as part of a course on this subject but that are easy to introduce to students with the specified background. These topics include the method of characteristics for partial differential equations (Chapter 3), as well as the Contraction Mapping Principle (Chapter 6) and pseudoinverse operators (Chapter 9) from functional analysis. The applications and problems touch on the theory of weakly viscous shock waves, quantum mechanics and the semiclassical limit, long-time behavior of diffusion processes and dispersive waves, how to count lattice points in the plane using Fourier theory, random matrix theory, orthogonal polynomials, zeros of Taylor polynomials for nonvanishing functions, aspects of special functions (in particular the central role in many problems played by the Airy function $Ai(z)$), nonlinear lattices (coupled pendula and the Fermi-Pasta-Ulam model), and water waves.

This text was developed over a period of five years and was used four times to teach a first-year graduate course in asymptotic methods. I would like to thank the students who gave me valuable feedback on my course notes as they developed into this text. I also want to specifically acknowledge many useful conversations I had with Charlie Doering, Jeffrey Rauch, and Alexander Tovbis.

I have dedicated this book to my teachers. Among many I would like to single out Rich Haberman and Doug Reinelt for getting me started, Marty Greenlee for explaining how it all starts with the beautiful basics of functional analysis, Bill Faris for elucidating how quantum mechanics is the premier application thereof, Al Scott for teaching me how to carry out practical calculations in quantum theory, and for showing me the unexpected complexity of discrete systems (*e.g.* pendulum chains), Dave Levermore for explaining the right way to think about semiclassical limits, Nick Ercolani for showing me the structure and beauty behind integrable systems, Alan Newell for giving me a "big picture" of nonlinear waves, and Hermann Flaschka for

teaching me how important it is to keep it all honest. Anyone who knows these people will see their footprints on every page of this book.

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