Acknowledgments

This book originated as notes by the authors for lectures on Hamilton's Ricci flow given in China during the summer of 2004. These lecture notes were written during the authors' stays at Fudan, East China Normal, Beijing, and Zhejiang Universities. We soon decided to expand these notes into a book which covers many more topics, including some of Perelman's recent results. The work on this book was completed when the authors were at their respective institutions, the University of California at San Diego and the University of Oregon.

The first author gave lectures at the Beijing University Summer School and the Summer School at the Zhejiang University Center of Mathematical Sciences and one lecture each at East China Normal University, Fudan University, and the Nankai Institute. The second author gave lectures at the Summer School at the Zhejiang University Center of Mathematical Sciences. The third author gave lectures at Fudan University and a lecture at East China Normal University.

The first author would like to thank Professors Gang Tian and Mei-Yue Jiang for the invitation to Beijing University and Professors Shiing-Shen Chern and Yiming Long for the invitation to the Nankai Institute. All three authors would like to thank Professors Xinan Ma, Shen-Li Tang, Jiaping Wang, Stephen Yau and Yu Zheng for the invitation to East China Normal University. The first and last authors would like to thank Professors Yuxin Dong, Chaohao Gu, Jiaxing Hong, Hesheng Hu, and Jun Li for the invitation to Fudan University. The first and second authors would like to thank Professors Huai-Dong Cao, Richard Hamilton, Fang-Hua Lin, Kefeng Liu, Yibing Shen, Weimin Sheng and Shing-Tung Yau for the invitation to the Zhejiang University Center of Mathematical Sciences. The second author would like to thank Professors Jiayu Li and Gang Tian for the invitation to the Mathematics Institute at the Chinese Academy and Nanjing University. We would like to thank Ms. Xiang-Hui Yu at Beijing University for her assistance during the 2004 Beijing University Summer School and during a visit to Science Press in August 2005.

We would like to thank the following mathematicians for discussions, interest, and encouragement which were helpful in preparing this book: Scot Adams, Ben Andrews, Salah Baouendi, Huai-Dong Cao, Xiaodong Cao, Albert Chau, Bing-Long Chen, Xiuxiong Chen, Hsiao-Bing Cheng, Li-Tien Cheng, Shiu-Yuen Cheng, Sun-Chin Chu, Jean Cortissoz, Klaus Ecker, José Escobar (1954–2004), David Glickenstein, Christine Guenther, Bob Gulliver, Zhengxu He, Gerhard Huisken, Jim Isenberg, Tom Ivey, Naresh Jain, Nitu Kitchloo, Bruce Kleiner, Dan Knopf, Conan Leung, Peter Li, Qing-Zhong Li, John Lott, Feng Luo, Yuan Luo, Robert McCann, John Morgan, Andre Neves, Wei-Ming Ni, Peter Petersen, Justin Roberts, Xiaochun Rong, Linda Rothschild, Fadil Santosa, Rick Schoen, Natasa Sesum, Weimin Sheng, Miles Simon, Luen-Fai Tam, Gang Tian, Peter Topping, Jeff Viaclovsky, Nolan Wallach, Jiaping Wang, Guo-Fang Wei, Ben Weinkove, Neshan Wickramasekera, Jon Wolfson, Yuan-Long Xin, Deane Yang, Rugang Ye, Yu Yuan, Chongchun Zeng, Fang-Yang Zheng, Yu Zheng, Xi-Ping Zhu, and the first and last authors' Ph.D. students Hongxin Guo and Brett Kotschwar. We would like to thank the participants at the Clay Mathematics Institute Summer School on Ricci Flow, 3-Manifolds and Geometry, June 20–July 16, 2005, at MSRI for the helpful feedback on various parts of the book. We would like to thank David Eisenbud and Hugo Rossi at MSRI and Jim Carlson and David Ellwood at Clay. The first author would like to thank Yu Zheng for the invitation to East China Normal University in August of 2005 during which time some of the finishing work for the book was done. We would especially like to thank Gang Tian for his encouragement to write what originally were lecture notes that have evolved into this book; without his encouragement this particular book would not have been written. We would also like to thank Gang Tian for his help in publishing the book with Science Press as a volume in the Lectures in Contemporary Mathematics series (in conjunction with the Beijing University Summer School).

We would like to thank Esther Cabezas-Rivas, Sun-Chin Chu, Hongxin Guo, Dan Knopf, Jun Ling, and Weimin Sheng for a number of helpful suggestions and corrections of misprints in the preliminary version. We would like to thank Jiaping Wang for very helpful discussions about the style, content, and purpose of the book.

We would like to thank Ms. Hong Lu, senior editor at Science Press, for all of her help in making the publication of our manuscript possible as a co-publication of Science Press and the AMS. We are grateful to Ms. Lu for her work in securing financing for the book, her assistance in many matters related to publishing the book, and her tireless efforts to publish the book in a timely manner while maintaining the highest publishing standards. We would like to thank both Ms. Min-Ling Hu at Science Press and the editorial staff at the AMS for their technical help in editing the .tex file for the book and preparing the final copy. We would like to thank Ed Dunne, editor at the AMS, for his invaluable and continual help in placing our book in the Graduate Studies in Mathematics series at the AMS as a co-publication with Science Press. We would like to thank Ed Dunne, the GSM series editors, and the anonymous reviewers for their help in making numerous improvements to the book. We would also like to thank Arlene O'Sean for all of her efficient and expert work in copy editing the book.

The first author would like to give special thanks to Professor Shiing-Shen Chern (1911–2004) whom he first met in 1979. The author's father, Yutze Chow, invited Professor Chern to the University of Wisconsin at Milwaukee to give a joint mathematics-physics colloquium. Professor Chern was gracious enough to accept the invitation and visited Milwaukee during his return trip from IAS at Princeton where he had given a talk on the occasion of Einstein's centennial. Professor Chern was kind enough to spend time with the author and his father. His visit influenced the author to study differential geometry.

The first author would like to thank his parents, Yutze Chow and Wanlin Wu, for their nurturing, encouragement, and love throughout his life. He would also like to thank his brother, Peter, and sister, Eleanor (1955–1985), for their love and encouragement. Special thanks to daughters Michelle and Isabelle for their inspiring love.

The first author would especially like to thank Dan Knopf for collaboration over the years on the book "The Ricci Flow: An Introduction" [**163**], published recently by the American Mathematical Society. It has been a fun ride and we have enjoyed and benefited from this collaboration quite a bit. We have tried to minimize the overlap between these two books; unfortunately, to retain some degree of completeness, a certain amount of overlap has been unavoidable. Parts of this book are derived from notes written in the 1990's. This author would like to thank editors Ed Dunne, Sergei Gelfand, and Ina Lindemann at the AMS for their help with the AMS Ricci flow series of books and especially to Ed Dunne for all of his help in publishing his book "The Ricci flow: An Introduction" [**163**] with Dan Knopf and the future coauthored book "The Ricci flow: Techniques and Applications" [**153**]. This author would especially like to thank his teachers Professors Shing-Tung Yau and Richard Hamilton for their help, guidance and encouragement over many years. The benefit he has received from them encompasses much of his mathematical career.

The first author was partially supported by NSF grants DMS-0505507 and DMS-0203926. The second author was partially supported by NSF grant DMS-0405255. The last author was partially supported by NSF grants DMS-0504792 and DMS-0203023. The first and last authors were partially supported by NSF FRG grant DMS-0354540.

The first author dedicates this book to Classic Dimension; he would especially like to thank Classic Dimension for constant encouragement, faith, guidance, support, and providing the inspiration to write these books on Ricci flow.

The second author dedicates this book to his teacher Professor Kung-Ching Chang on the occasion of his seventieth birthday.

The last author dedicates this book to both his wife, Jing-Jing Wu, and his teacher Professor Peter Li.

Bennett Chow
UC San Diego and East China Normal University

Peng Lu
University of Oregon

Lei Ni
UC San Diego

rcflow@math.ucsd.edu
December 6, 2005

A Detailed Guide for the Reader

Chapter 1. We present some basic results and facts from Riemannian geometry. The results in this chapter, for the most part, either are used or are analogous to results in the latter chapters on Ricci flow and other geometric flows. Although we have included in this chapter what we feel ideally the reader should know, he or she should be reassured that mastering all of the contents of this chapter is not a prerequisite for studying Ricci flow! Indeed, this chapter may be used as a reference to which the reader may refer when necessary.

In Section 2 we give a quick review of metrics, connections, curvature, and covariant differentiation. Of particular note are the Bianchi identities, the Lie derivative, and the covariant derivative commutator formulas in Section 3, which have applications to the formulas we shall derive for solutions of the Ricci flow. In Section 4 we recall the theory of differential forms and discuss the Laplace operator for tensors and Bochner formulas for differential forms. Since integration by parts is a useful technique in geometric analysis and in particular Ricci flow, in Section 5 we recall the divergence theorem and its consequences. We also give a quick review of the de Rham theorem and the Hodge decomposition theorem. In Section 6 we introduce the Weyl tensor and the decomposition of the Riemann curvature tensor into its irreducible components. We consider some basic aspects of locally conformally flat manifolds. In Section 7 we discuss Cartan's method of moving frames since it is a useful technique for computing curvatures, especially in the presence of symmetry. As an application, we give a proof of the Gauss-Bonnet formula for surfaces using moving frames. We also discuss hypersurfaces from the point of view of moving frames. Metric geometry has important implications in Ricci flow, so we discuss (Section 8) the first and second variation of arc length and energy of paths. Applications are Synge's theorem and the Hessian comparison theorem. We include the application of the second variation formula to long geodesics and a variational proof of the fact that Jacobi fields minimize the index form. We also discuss the first and second variation formulas for the areas of hypersurfaces. Such formulas have applications to minimal surface theory. In Section 9 we recall basic facts about the exponential map such as the Gauss lemma, the Hopf-Rinow theorem, Jacobi fields, conjugate and cut points, and injectivity radius estimates for positively curved manifolds. Next (Section 10) we present geodesic spherical coordinates using the exponential map. This is a convenient way of studying (Hessian) comparison theorems for Jacobi fields, and by taking the determinant, volume (Laplacian) comparison theorems. One may think of these calculations as associated to hypersurfaces (the distance spheres from a point) evolving in their normal directions with unit speed. (More generally, one may consider arbitrary speeds, including the mean curvature flow.) Observe that the Laplacian of the distance function is the radial derivative of the logarithm of the Jacobian, which is the mean curvature of the distance spheres. Similarly, the Hessian of the distance function is the radial derivative of the logarithm of the inner product of Jacobi fields, which is the second fundamental form of the distance spheres. We then begin to discuss in more detail the Laplacian and Hessian comparison theorems in Section 11. These results are essentially equivalent to the Bishop-Gromov volume and Rauch comparison theorems and have important analogues in Ricci flow. As an application, we prove the mean value inequality. In Section 12 we give detailed proofs of the Laplacian, volume and Hessian comparison theorems. In the case of the Laplacian comparison theorem, we prove the inequality holds in the sense of distributions. In Section 13 we discuss the Cheeger-Gromoll splitting theorem, which relies on the Busemann functions associated to a line being subharmonic (and hence harmonic), and the mean value inequality. We then discuss the Toponogov comparison theorem. Left-invariant metrics on Lie groups provide nice examples of solutions on Ricci flow which can often be analyzed, so we introduce some background material in Section 14. In the notes and commentary (Section 15) we review some basic facts about the first and second fundamental forms of hypersurfaces in Euclidean space, since we shall later discuss curvature flows of hypersurfaces to compare with Ricci flow.

Chapter 2. Here we begin the study of Ricci flow proper. Before we describe the contents of this chapter, we suggest that the reader may occasionally refer to Chapter 4 for some explicit examples of solutions to the Ricci flow; these examples may guide the reader's intuition when studying the abstract derivations throughout the book. We start in Section 1 with some historical remarks about geometric evolution equations. We also give a brief layman's description of how Ricci flow approaches the Thurston geometrization conjecture. The Ricci flow is like a heat equation for metrics. One quick way of seeing this (Section 2) is to compute the evolution equation for the scalar curvature using a variation formula we derive later. We get a heat equation with a nonnegative term. Because of this, we can apply the maximum principle (Section 3) to show that the minimum of the scalar curvature increases. The variation formula for the scalar curvature yields a short derivation of Einstein's equations as the Euler-Lagrange equation for the total scalar curvature (Section 4). By modifying the total scalar curvature, we are led to Perelman's energy functional. Next (Section 5) we carry out the actual computations of the variation of the connection and curvatures. When the variation is minus twice the Ricci tensor, we obtain the evolution equations for the connection, scalar and Ricci curvatures under the Ricci flow. This is the first place we encounter the Lichnerowicz Laplacian which arises in the variation formula for the Ricci tensor. Since the Ricci flow is a weakly parabolic equation, to prove short time existence, we use DeTurck's trick, which shows that it is equivalent to a strictly parabolic equation (Section 6). Having derived the evolution equations for the Ricci and scalar curvatures, we derive the evolution equation for the full Riemann curvature tensor (Section 7). This takes the form of a heat equation with a quadratic term on the right-hand side. In dimension 3, the form of the quadratic term is especially simple. In the notes and commentary (Section 8) we discuss the symbol of the linearization of the Ricci tensor.

Chapter 3. We give a proof of Hamilton's classification of closed 3-manifolds with positive Ricci curvature using Ricci flow. Hamilton's theorem (Section 1) says that under the normalized (volume-preserving) Ricci flow on a closed 3-manifold with positive Ricci curvature, the metric converges exponentially fast in every C^k-norm to a constant positive sectional curvature metric. The maximum principle for tensors (Section 2) enables us to estimate the Ricci and sectional curvatures. We first show that positive Ricci curvature and Ricci pinching are preserved. To control the curvatures, it is convenient to generalize the maximum principle for symmetric 2-tensors to a maximum principle for the curvature operator, and more generally, to systems of parabolic equations on a vector bundle (Section 3). Using this formalism, we show that the pinching of the curvatures improves and tends to constant curvature at points and times where the curvature tends to infinity. This is the central estimate in the study of 3-manifolds with positive Ricci curvature. Once we have a pointwise estimate for the curvatures, we need a gradient estimate for the curvature in order to compare curvatures at different points at the same time (Section 4). Based on the fact that the pinching estimate breaks the scale-invariance, we obtain a gradient of the scalar curvature estimate which shows that a scale-invariant measure tends to zero at points where the curvature tends to infinity. Combining the estimates of the previous sections, we show that indeed the curvatures tend to constant (Section 5). With control of the curvatures and assuming derivative estimates derived in a later chapter, we can show that the normalized Ricci flow converges exponentially fast in each C^k-norm to a constant positive sectional curvature metric (Section 6). In the notes and commentary (Section 7) we state some of the basic evolution equations under the mean curvature flow. These formulas are somewhat analogous to the equations for the Ricci flow.

Chapter 4. We discuss Ricci solitons, homogeneous solutions and other special solutions. From the study of singularity formation we are interested in solutions which exist for all negative time. Some of these solutions exist also for all positive time, which makes them even more special. Of fundamental importance are gradient Ricci solitons (Section 1). We carefully formulate the basic equations of a gradient Ricci soliton and show that gradient solitons can be put in a canonical form. It is interesting that Euclidean space is not only a steady soliton but also a shrinking and an expanding soliton (Section 2). We then briefly discuss the cylinder shrinking soliton. For the Ricci flow on low-dimensional manifolds, it is particularly important to consider complete Ricci solitons on surfaces with positive curvature. The cigar soliton (Section 3) is such a soliton on the Euclidean plane, rotationally symmetric, asymptotic to a cylinder at infinity and with curvature decaying exponentially fast. We exhibit the cigar soliton in various coordinate systems. An interesting rotationally symmetric ancient solution on the 2-sphere is the Rosenau solution (Section 4). As time tends to negative infinity, the Rosenau solution looks like a pair of cigars, one at each end. Indeed, backward (in time) limits at the endpoints yield the cigar soliton. Next we describe an explicit rotationally symmetric expanding soliton on the plane with positive curvature (Section 5). The curvature also decays exponentially fast as the distance to the origin tends to infinity. Moving up one dimension, we obtain the Bryant soliton (Section 6). This is a rotationally symmetric steady gradient Ricci soliton on Euclidean 3-space with positive sectional curvature. Here the curvature decays inverse linearly and the metric is essentially asymptotic to a paraboloid at infinity. This means that at infinity the metric, after rescaling, limits to a cylinder in the same sense that a parabola in the plane, after rescaling, limits to two parallel lines after dilating about points which tend to infinity. Some of the most interesting explicit examples are homogeneous solutions (Section 7). We consider SU(2) and Nil and analyze the ODE which arises from Ricci flow. Here, since the isometry group acts transitively on the manifold, the Ricci flow reduces to a system of ordinary differential equations. In dimension 3, these systems are largely well understood. We also consider the Ricci flow of bi-invariant metrics on a compact Lie group. Under the Ricci flow, isometries persist (Section 8). That is, the isometry group is nondecreasing. It is interesting to ask if the isometry group remains constant under the Ricci flow. Analogous to the Ricci flow on surfaces is the curvature shortening flow (CSF) of plane curves (Section 9). The analogue of the cigar soliton is the grim reaper for the CSF. The Rosenau solution also has an analogue for the CSF.

Chapter 5. In this chapter we discuss monotonicity formulas which yield isoperimetric and volume ratio estimates. In Section 1 we discuss isoperimetric inequalities and their relation with Sobolev inequalities. We also derive the logarithmic Sobolev inequality from the L^2 Sobolev inequality. In Section 2 we study the evolution of the length of paths and geodesics in preparation for studying the evolution of the isoperimetric ratio on surfaces. Another proof, due to Hamilton, of the convergence of the Ricci flow on a 2-sphere uses a monotonicity formula for the isoperimetric ratio (Section 3). In Section 4 we give a proof of Perelman's no local collapsing theorem. In Section 5 we present geometric applications of the no local collapsing theorem. Of particular note are the consequent local injectivity radius estimate and the fact that the cigar can be ruled out as a finite time singularity model. The local injectivity radius estimate enables one to take limits of dilations of finite time singular solutions on closed manifolds. To facilitate the exposition of this, we present some preliminaries on the compactness theorem for solutions of the Ricci flow. In Section 6 we give a shorter proof of the classification of closed 3-manifolds with positive Ricci curvature using Perelman's no local collapsing theorem and the compactness theorem. Here we only prove sequential convergence instead of exponential convergence. In Section 7 we discuss Hamilton's isoperimetric estimate for Type I singular solutions in dimension 3.

Chapter 6. In this chapter we collect the analytic results and techniques which are useful for singularity analysis. Estimates for all of the derivatives of the curvature (Section 1) in terms of bounds on the curvature enable one to show that the solution exists as long as the curvature remains bounded (the long time existence theorem). Thus if a solution forms a singularity in finite time (i.e., cannot be continued past a finite time), then the supremum of the curvature is infinity. In Section 2 we present the local derivative of curvature estimates of W.-X. Shi. These estimates are fundamental in the study of singularities of the Ricci flow. Another basic tool to study singular solutions is a Cheeger-Gromov-type compactness theorem for a sequence of solutions of Ricci flow (Section 3). We present both the global and local versions of this result. What is needed for this sequence is a curvature bound and an injectivity radius estimate (to prevent collapsing). The sequences we usually consider arise from dilating a singular solution about a sequence of points and times with the times approaching the singularity time. An interesting application of compactness is Sesum's result (Section 4) that a solution exists as long as its Ricci curvature is bounded. When studying the formation of singularities on a 3-manifold, the Hamilton-Ivey estimate is particularly useful (Section 5). Roughly speaking, it says that at large curvature points, the largest sectional curvature is positive and much larger than any negative sectional curvature in magnitude. It implies that limits of dilations (which we call singularity models) have nonnegative sectional curvature. Since nonnegative sectional curvature metrics are rather limited geometrically and (especially) topologically, this is a crucial first step in the surgery theory for singular 3-dimensional solutions. The Hamilton-Ivey estimate also tells us that ancient 2- and 3-dimensional solutions with bounded curvature have nonnegative sectional curvature. Nonnegative sectional curvature in dimension 3 is a special case of nonnegative curvature operator in all dimensions, a curvature condition which is preserved under the Ricci flow. Such solutions satisfy the strong maximum principle (Section 6), which says that either the solution has (strictly) positive curvature operator or the holonomy reduces and the image and kernel of the curvature operator are constant in time and invariant under parallel translation. This rigidity result is especially powerful in dimension 3 (Section 7), where it implies that a simply connected nonnegative sectional curvature solution of the Ricci flow either has positive sectional curvature, splits as the product of a surface solution with positive curvature (which is topologically a 2-sphere or the plane) and a line, or is the flat Euclidean space. In the notes and commentary (Section 8) we note that the derivative estimates may be improved if we assume bounds on some derivatives of the curvature.

Chapter 7. In Section 1 we present the spherical space form theorem of Huisken, Margerin, and Nishikawa which says that for an initial Riemannian manifold with sectional curvatures pointwise sufficiently close to that of a constant positive curvature space, the normalized Ricci flow exists for all time and converges to a constant positive curvature metric. In Section 2 we outline the proof of Hamilton's classification of 4-manifolds with positive curvature operator. In higher dimensions (Section 3), a solution with nonnegative curvature operator either has positive curvature operator, a Kähler manifold with positive curvature operator on $(1,1)$-forms, or is a locally symmetric space. Except in dimension 4, where it was solved by Hamilton, it is an open problem whether a closed Riemannian manifold with positive curvature operator converges under the Ricci flow to a metric with constant positive sectional curvature (spherical space form). A potentially useful result in this regard, due to Tachibana, says that an Einstein metric with positive curvature operator has constant sectional curvature and an

Einstein metric with nonnegative curvature operator is locally symmetric. The maximum principle discussed in Chapter 2 extends to complete noncompact manifolds. In Section 4 we present some general results about the maximum principle on noncompact manifolds. In Section 5 we survey some results on complete solutions of the Ricci flow on noncompact manifolds.

Chapter 8. Here we begin our study of singularities. To study singularities (Section 1), one takes dilations about sequences of points and times where the time tends to the singularity time. The limit solutions of such sequences, if they exist, are ancient solutions. It is useful to distinguish singular and ancient solutions according to the rate of blowup of the curvature. Type I singularities blow up in finite time at the rate of the standard shrinking sphere. Type II singularities form more slowly in the sense that in terms of the curvature scale, the time to blow up is longer than that of Type I. On the other hand, as a function of time to blow up, the curvature of a Type II singularity is larger than that of a Type I singularity. We also give lower bounds (gap estimates) for the supremum of the curvature as a function of time for singular and ancient solutions. Given a singularity type, we describe ways of picking sequences of points and times about which to dilate (Section 2). Suitable choices of such sequences lead to ancient solution limits, called singularity models. Given an ancient solution, we can also dilate again about a sequence of points and times. We also discuss the prototype for a Type II singularity: the degenerate neckpinch. An interesting open problem is to show that Type II singularities indeed exist, a result which is known for the mean curvature flow. Since complete noncompact ancient solutions are important in the study of singularities, we begin the study of the geometry at infinity of such solutions under various hypotheses (Section 3). Some useful geometric invariants of the geometry at infinity are the asymptotic scalar curvature ratio and the asymptotic volume ratio. To study the geometry at infinity, a useful technique is dimension reduction (Section 4). Here we assume the asymptotic scalar curvature ratio (ASCR) is infinite, that is, the $\limsup$ of the scalar curvature times the square distance to an origin is equal to infinity. Roughly speaking, this says that the scalar curvature has slower than quadratic decay. Using a point-picking-type argument, we find good sequences of points tending to spatial infinity. When ASCR is infinite and $\mathrm{Rm} \geq 0$, there exists a limit which splits as the product of a one lower-dimensional solution with a line. In the notes and commentary we note that numerical studies of a degenerate neckpinch have been carried out by Garfinkle and Isenberg (Section 5).

Chapter 9. Considering ancient solutions on 2-dimensional surfaces (Section 1), we show using the Harnack inequality that ancient solutions with bounded curvature whose maximum is attained in space in time must be isometric to the cigar soliton. On the other hand, using Hamilton's entropy monotonicity, one can show that Type I ancient solutions are compact and in fact isometric to the round shrinking sphere. Complementarily, Type II ancient solutions (such as the Rosenau solution) must have a backward limit which is the cigar soliton. Our discussion includes as a corollary the classification of ancient κ-solutions on surfaces. An interesting result is that noncompact ancient solutions on surfaces with time-dependent bounds on the curvature can be extended to eternal solutions. We conjecture that eternal solutions (without assuming the supremum of the curvature is attained) are cigar solitons. Optimistically, one can hope that the only complete ancient surface solutions with bounded curvature are rotationally symmetric and, in fact, belong to one of three types: either the cigar, or a surface with constant curvature, or the Rosenau solution. In Section 2 we discuss aspects of ancient solutions relating to their type. A Type I ancient solution with positive curvature operator has infinite asymptotic scalar curvature ratio. For Type II ancient solutions with positive curvature operators there exist backward limits which are steady gradient Ricci solitons. We also state and prove Perelman's theorem that positively curved ancient solutions have vanishing asymptotic volume ratio and infinite asymptotic scalar curvature ratio. In Section 3 we give proofs of Hamilton's results that *steady* gradient Ricci solitons have infinite asymptotic scalar curvature ratio and complete noncompact *expanding* gradient Ricci solitons with positive Ricci curvature have positive asymptotic volume ratio. In Section 4 we discuss a local injectivity radius estimate for steady gradient Ricci solitons not assuming they are κ-noncollapsed. In Section 5 we discuss what is known about 3-dimensional singularities from the classical theory as well as Perelman's no local collapsing theorem. In dimension 3, by dimension reduction, there exists a limit which splits as the product of a surface solution with a line. By the no local collapsing theorem, the surface solution cannot be the cigar soliton. By a previous result of Hamilton, the surface must then be a round shrinking 2-sphere. In the case of a Type I ancient solution, we either have a shrinking spherical space form or there exists a backward limit which is a cylinder (2-sphere product with a line). For ancient κ-solutions the latter cannot exist. We conjecture that the κ-noncollapsed condition can be removed. In dimension 3 we also have another basic result of Perelman, the nonexistence of noncompact shrinking gradient solitons with positive sectional curvature (Section 6). In Section 7 we summarize the results in the chapter and pose some conjectures about long-existing solutions, especially in dimensions 2 and 3.

Chapter 10. We discuss various differential Harnack estimates. These are sharp pointwise derivative estimates which usually enable one to compare a solution at different points in space and time. In Ricci flow they have applications toward the classification of singularity models. We begin with the heat equation (Section 1) and present the seminal Li-Yau estimate for positive solutions. For manifolds with nonnegative Ricci curvature, the estimate is sharp in the sense that equality is obtained for the fundamental solution on Euclidean space. Since the Li-Yau estimate has a precursor in the work of Yau on harmonic functions, we discuss the Liouville theorem for complete manifolds with nonnegative Ricci curvature, which relies on a gradient estimate. The Li-Yau estimate has a substantial extension to solutions of the Ricci flow, due to Hamilton. To describe this, we begin with surfaces, since the form of the inequality and its proof are much simpler in this case. In Section 2 we prove a differential Harnack estimate for surfaces with positive curvature. Ricci solitons again motivate the specific quantities we consider. A perturbation of these arguments enables one to prove a Harnack estimate for surfaces with variable signed curvature. Interestingly, the above (trace) inequality may be generalized to a matrix inequality which, roughly speaking, gives a lower bound for the Hessian of the logarithm of the curvature. In this sense, the Harnack inequalities are somewhat analogous to the Laplacian and Hessian comparison theorems of Riemannian geometry discussed in Chapter 1. A fact related to Hamilton's entropy is that in the space of metrics on a surface with positive curvature, its gradient is the matrix Harnack quantity, which in dimension 2 is a symmetric 2-tensor. Next, in dimension 2, we consider a 1-parameter family of Harnack inequalities for the Ricci flow coupled to a linear-type heat equation (Section 3). In one instance we have the Li-Yau inequality, and in another instance we have the linear trace Harnack estimate, which generalizes Hamilton's trace estimate. An open problem is to generalize this to higher dimensions. In the Kähler case, this has been accomplished by one of the authors. With these preliminaries we move on to Hamilton's celebrated matrix Harnack estimate for solutions of the Ricci flow with nonnegative curvature operator (Section 4). The specific Harnack quadratic under consideration is motivated directly from the consideration of expanding gradient Ricci solitons. The trace inequality (obtained from the matrix inequality by summing over an orthonormal basis) is particularly simple to state and in all dimensions is surprisingly similar to the 2-dimensional inequality. We show that the matrix inequality in dimension 2 is the same as the previous estimate which was in the form of a symmetric 2-tensor being nonnegative. The proof of the matrix estimate depends on a calculation, which at first glance looks quite complicated (Section 5). However, using the formalism of considering tensors as vector-valued functions on the principal frame bundle (either $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{O}(n)$), we can simplify the computations. The matrix Harnack quadratic is a bilinear form on the Whitney sum of the bundle of 1-forms with the bundle of 2-forms. It satisfies a heat-type equation with a quadratic nonlinear term analogous to the heat-type equation satisfied by the Riemann curvature tensor. Taking bases of 1-forms and 2-forms, one can exhibit the quadratic nonlinearity as a sum of squares when the curvature operator is nonnegative. This is the main reason for why the matrix Harnack inequality may be proved by a maximum principle argument. The trace Harnack inequality has a generalization to a Harnack inequality for nonnegative definite symmetric 2-tensors satisfying the Lichnerowicz Laplacian heat equation coupled to the Ricci flow (Section 6). This Harnack estimate generalizes the trace Harnack estimate. Using this, we give a simplified proof of Hamilton's result that ancient solutions with nonnegative bounded curvature operator which attain the supremum of their scalar curvatures are steady gradient Ricci solitons. Further investigating the linearized Ricci flow, we give a pinching estimate for solutions to the linearized Ricci flow on closed 3-manifolds. An open problem is to find applications of this general estimate, perhaps in conjunction with the linear trace Harnack estimate or other new estimates. In the notes and commentary (Section 7) we briefly discuss the matrix Harnack estimate for the heat equation, the Harnack estimate for the mean curvature flow, and some tools for calculating evolution equations associated to Ricci flow.

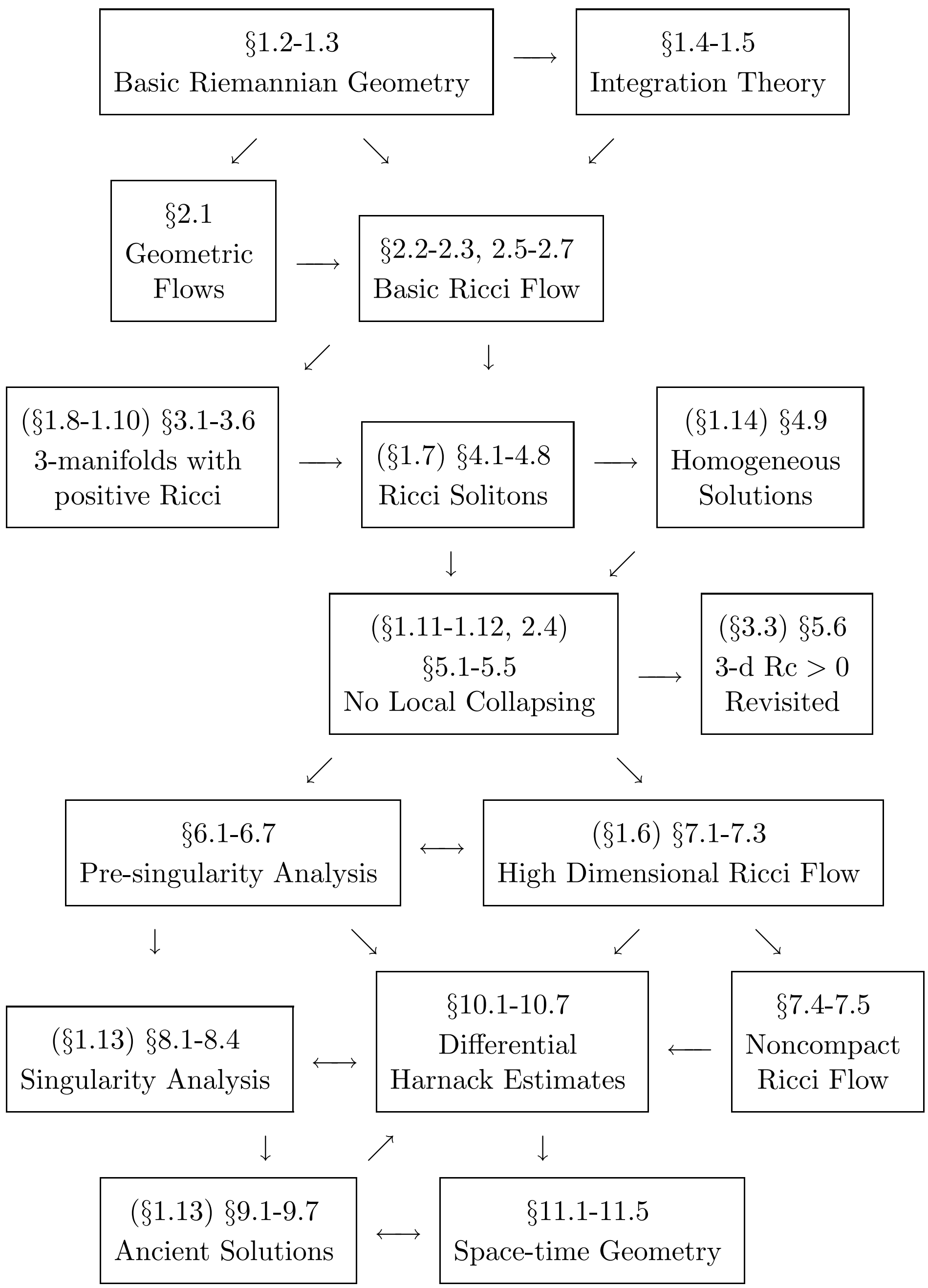
Chapter 11. We discuss various space-time geometries which are rather similar, culminating in Perelman's metric on the product of space-time with large-dimensional and large radius spheres. We begin with Hamilton's notion of the Ricci flow for degenerate metrics and the space-time connection of Chu and one of the authors (Section 1). This connection is compatible with the degenerate space-time metric and as a pair they satisfy the Ricci flow for degenerate metrics. We state the formulas for the Riemann and Ricci tensors of the space-time connection and observe curvature identities which suggest that the metric-connection pair is a Ricci soliton. We observe (Section 2) that the space-time Riemann curvature tensor is the matrix Harnack quadratic and the space-time Ricci tensor is the trace Harnack quadratic. Next we take the product of space-time with Einstein metric solutions of the Ricci flow (Section 3). We also introduce a scalar parameter into the definition of the potentially infinite space-time metric. We calculate the Levi-Civita connections of these metrics and observe that they essentially tend to the space-time connection defined in Section 1. It is particularly interesting that the space-time Laplacians tend to the heat (forward or backward) operator (depending on the sign of the scalar parameter). This fact depends on the dimension tending to infinity. Next we compute the Riemann and Ricci curvature tensors of the potentially infinite metrics. We observe that when the scalar parameter is -1, the metric tends to Ricci flat (this is due to Perelman). This is related to the developments on the ℓ-function discussed in [**153**]. We also recall the observation, again due to Perelman, that the metrics are essentially potentially Ricci solitons (at least he observed this when the parameter is either -1 or 1). Renormalizing the space-time length functional, we obtain the ℓ-function (Section 4). Not only is the space-time metric and connection related to the matrix Harnack estimate, it is also related to the linear trace Harnack estimate (Section 5). Here we need to make some modifications to describe the relation. An auxiliary function f is introduced and its definition is related to Perelman's idea of fixing the measure which he introduced when defining his energy and entropy. In the notes and commentary we discuss a space-time formulation of flows of hypersurfaces (Section 6) and its relation with Andrews' Harnack inequalities.

Appendix A. In this appendix we discuss various aspects of geometric analysis which are related to Ricci flow. In Section 1 we start with a short compendium of some inequalities used in the book. In Section 2 we recall some comparison theorems and explicit formulas for the heat kernel on Riemannian manifolds and in particular on constant curvature spaces. Next (Section 3) we discuss the Green's function, which is related to the heat equation and the geometry of the manifold. In Section 4 we give another proof of the Liouville Theorem 10.8. We also (Section 5) recall some basic facts about eigenvalues and eigenfunctions of the Laplacian. In the Ricci flow it is useful to study the lowest eigenvalue of certain elliptic (Laplace-like) operators. Li and Yau give a lower bound for the first eigenvalue on a closed manifold with nonnegative Ricci curvature. Lichnerowicz's result gives a lower bound assuming a positive lower bound of the Ricci curvature. We recall Reilly's formula and its application to estimating the first eigenvalue on manifolds with boundary. In Section 6 we discuss the definition of the determinant of the Laplacian via the zeta function regularization and compute the difference of determinants on a Riemann surface. The determinant of the Laplacian is an energy functional for the Ricci flow on surfaces. Since the Ricci flow evolves metrics and is related to the heat equation, we discuss (Section 7) the asymptotics of the heat kernel associated to the heat operator with respect to an evolving metric. The method is a slight modification of the fixed metric case. For comparison with Ricci flow monotonicity formulas, in Section 8 we recall some monotonicity formulas for harmonic functions and maps. In Section 9 we recall the Bieberbach theorem on the classification of flat manifolds; we approach the proof along the lines of Gromov's almost flat manifolds theorem.

Appendix B. In this appendix we discuss identities, inequalities, and estimates for various flows including the Ricci flow, Yamabe flow, and the cross curvature flow. It is interesting to compare these techniques. We begin in Section 1 by recalling the statement of the convergence result for the Ricci flow on closed surfaces. Next we consider the Kazdan-Warner and Bourguignon-Ezin identities in Section 2, from which it follows that a Ricci soliton on the 2-sphere has constant curvature. Then we turn our attention to Andrews' Poincaré-type inequality (Section 3), which holds in arbitrary dimensions. In dimension 2, it implies that Hamilton's entropy is monotone. This is useful in certain applications of the maximum principle, especially for systems. One of the proofs of the convergence of the Ricci flow on the 2-sphere relies on Ye's gradient estimate for the Yamabe flow of locally conformally flat manifolds with positive Yamabe invariant (Section 4). The Aleksandrov reflection method is used to obtain the gradient estimate. We also state Leon Simon's asymptotic uniqueness theorem. An interesting fact, which follows from the contracted second Bianchi identity, is that the identity map from a Riemannian manifold to itself, where the image manifold has the Ricci tensor as the metric (assuming it is positive), is harmonic (Section 5). In dimension 3 there is a symmetric curvature tensor, called the cross curvature tensor, which is dual to the Ricci tensor in the following sense. The identity map from a Riemannian 3-manifold to itself, where the domain manifold has the cross curvature tensor as the metric (assuming the sectional curvature is either everywhere negative or everywhere positive), is harmonic. We show two monotonicity formulas for the cross curvature flow which suggest that a negative sectional curvature metric on a closed 3-manifold should converge to a constant negative sectional curvature (hyperbolic) metric. In Section 6 we consider the time-derivative of the supremum function. This has applications to maximum principles. Finally, in the notes and commentary we note that in higher dimensions there are arbitrarily pinched negative sectional curvature metrics which do not support hyperbolic metrics.

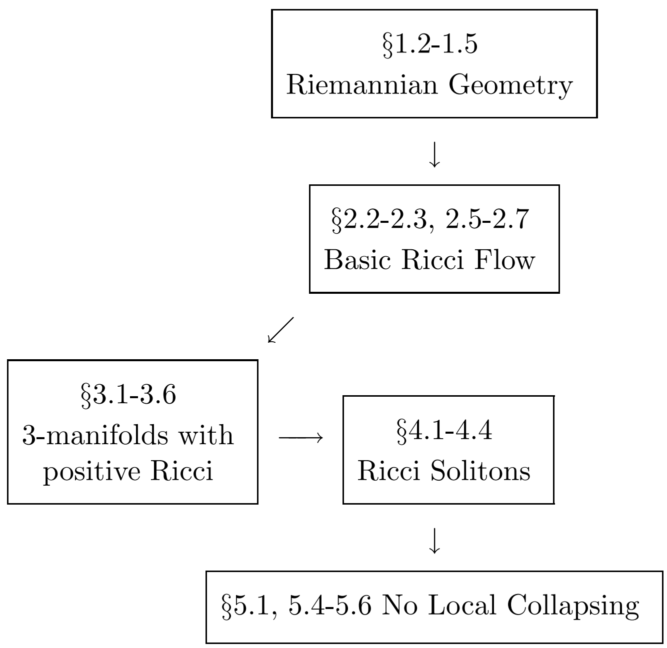
Bibliography. We have included a number of references in the areas of Ricci flow, geometric evolution equations, geometric analysis, and related areas. Not all of the references are cited in the book. The references we do not cite are included so that the interested reader may be aware of various works in these fields which may be related to the topics discussed in this book.

Overall Structure of the Book

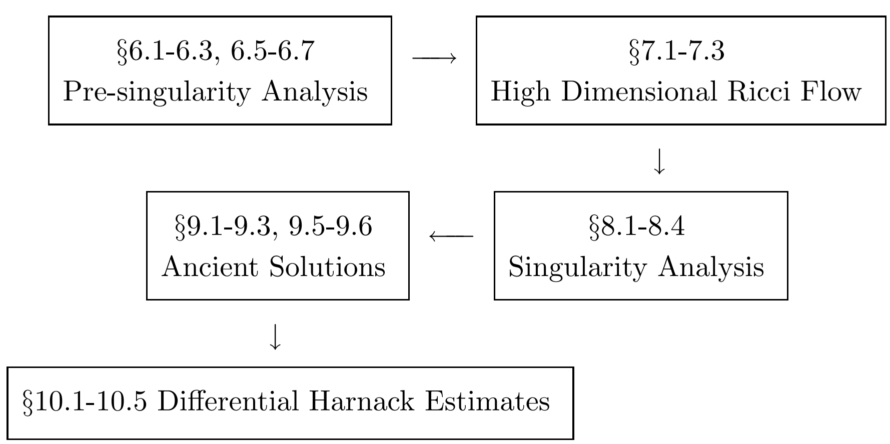


Suggested Course Outline

Semester 1



Semester 2



Chapter 2

Fundamentals of the Ricci Flow Equation

Wake up, Neo.... The Matrix has you.... Follow the white rabbit.[1]

We begin by giving an elementary introduction to how Ricci flow may be used to approach the geometrization conjecture. The intent of this section is only to give the reader some preliminary intuition about what type of results one would like to prove about Ricci flow in dimension 3. Next we describe some of the evolution equations for the curvatures of solutions to Ricci flow. Since the scalar curvature satisfies a particularly simple equation, we start with its evolution. This leads to the heat equation, for which a principal tool is the maximum principle, which we discuss both when the manifold is compact and noncompact. Although the Ricci flow is not exactly variational in the usual sense, we consider the Einstein-Hilbert functional and see that Ricci flow is similar to a gradient flow. In Section 4 we shall see a precise way in which Ricci flow is a gradient flow, due to Perelman. We then discuss some local coordinate calculations which lead to variation formulas for the curvatures. Applying this to the Ricci tensor and its modification, we obtain DeTurck's trick, which is used to prove the short time existence of solutions of Ricci flow on closed manifolds. In the last section we derive the formula for the evolution of the Riemann curvature tensor under Ricci flow. In dimension 3, the associated ODE is particularly simple.

Some conventions that we follow throughout the rest of the book are that RF is short for Ricci flow, $g(s)$ denotes a general 1-parameter family of

[1]The quotations at the beginning of Chapters 2, 8, and 10 are from the movie 'The Matrix'; see Wachowski and Wachowski [**544**].

metrics (with variation field $\frac{\partial}{\partial s} g_{ij} = v_{ij}$,) and $g(t)$ denotes a solution of the Ricci flow. When we compute variation formulas, $\frac{\partial}{\partial s}$ denotes the variation under a general variation $\frac{\partial}{\partial s} g = v$ of the metric, whereas $\frac{\partial}{\partial t}$ denotes the time derivative under the Ricci flow.

1. Geometric flows and geometrization

1.1. Some geometric flows predating Ricci flow. Geometric flows have been around at least since Mullins' paper [**427**] in 1956 proposing the curve shortening flow to model the motion of idealized grain boundaries. In 1964 the seminal paper of Eells and Sampson [**208**] introduced the harmonic map heat flow and used it to prove the existence of harmonic maps into targets with nonpositive sectional curvature. In 1974 Firey [**225**] proposed the Gauss curvature flow to model the shapes of worn stones and considered the case where the surface is invariant under the antipodal map. In 1975 Hamilton [**274**] continued the study of the harmonic map heat flow by considering manifolds with boundary. In 1978 Brakke [**58**] studied the mean curvature flow and proved regularity properties for it.

1.2. Ricci flow and geometrization: a short preview. The first success, in fact the main theorem of the seminal paper [**275**] by Richard Hamilton in 1982, has the following topological consequence.

Theorem 2.1 (Hamilton, 3-manifolds with positive Ricci curvature)**.** *If* (M^3, g) *is a closed* 3*-manifold with positive Ricci curvature, then it is diffeomorphic to a* ***spherical space form****. That is,* M^3 *admits a metric with constant positive sectional curvature.*

Here and throughout this book, closed means compact without boundary. More ambitiously, one would like to use Ricci flow to prove the following **geometrization conjecture**.

Conjecture 2.2 (Thurston geometrization)**.** *Every closed* 3*-manifold admits a geometric decomposition.*

A corollary of the geometrization conjecture is the **Poincaré conjecture**, which says that every simply connected closed topological 3-manifold is homeomorphic to the 3-sphere and this is one of the Millennium prize problems. We refer to the survey papers of Thurston [**528**] and Scott [**488**] for background on the geometrization conjecture. Some recent surveys of Ricci flow and the Poincaré/geometrization conjecture are Milnor [**413**] and Morgan [**418**].

In this section we give a very rough and intuitive description of Hamilton's idea for using Ricci flow to approach Thurston's geometrization conjecture. In the following discussion all 3-manifolds will be closed and orientable.

Start with a closed Riemannian manifold $\left(M_1^3, g_0\right)$. We want to infer the existence of a geometric decomposition for M_1^3 by studying the properties of a solution $\left(M^3(t), g(t)\right)$ of the **Ricci flow with surgeries** with $M^3(0) = M_1^3$ and $g(0) = g_0$. By Ricci flow with surgeries we mean a sequence of solutions $\left(M_i^3, g_i(t)\right)$, $t \in [\tau_{i-1}, \tau_i]$, with $\tau_{i-1} < \tau_i \leq \infty$ and $\tau_0 = 0$, of Ricci flow where $\left(M_{i+1}^3, g_{i+1}(\tau_i)\right)$ is obtained from $\left(M_i^3, g_i(\tau_i)\right)$ by a geometric-topological surgery. $M^3(t) = M_i^3$ and $g(t) = g_i(t)$ for $t \in [\tau_{i-1}, \tau_i]$ so that the manifold and metric are doubly defined at the surgery times τ_i.

The short and long time existence theorems imply that there exists a unique maximal solution to the Ricci flow $\left(M_1^3, g_1(t)\right)$, $t \in [0, T_1)$, where either

$$\sup_{M_1^3 \times [0,T_1)} |\mathrm{Rm}(g_1)| = \infty$$

or $T_1 = \infty$. We say that the Ricci flow develops a **singularity** at time T_1 and we call T_1 the **singular time**. If $T_1 < \infty$, then we want to prove that in high curvature regions (near the singularity time) the metric looks like part of an $S^2 \times \mathbb{R}$ cylinder (this is not quite true, but for the sake of simplicity let's assume this for now). At some appropriate time $\tau_1 \lessapprox T_1$ right before the singularity forms we perform a geometric-topological surgery by cutting out an $S^2 \times B^1$ from the cylindrical region, which has two disjoint 2-spheres as its boundary: $\partial\left(S^2 \times B^1\right) \cong S^2 \times S^0 \cong \partial\left(B^3 \times S^0\right)$, and we replace it by two balls: $B^3 \times S^0$.[2] We call the new manifold $\left(M_2^3, g_1\right)$, which may not be connected. Let $\left(M_2^3, g_2(t)\right)$, $t \in [\tau_1, T_2)$, be the unique maximal solution to the Ricci flow with $g_2(\tau_1) = g_1$. Topologically, M_2^3 and M_1^3 are related as follows. If $M_2^3 \cong M_{2,1}^3 \sqcup M_{2,2}^3$ is the disjoint union of two connected manifolds, then $M_1^3 \cong M_{2,1}^3 \# M_{2,2}^3$. On the other hand, if M_2^3 is connected, then $M_1^3 \cong M_2^3 \# \left(S^2 \times S^1\right)$. The reason for this is that reversing the surgery process, we see that M_1^3 is obtained from M_2^3 by attaching a handlebody. Isotopying this handlebody so that it is attached to two 2-spheres inside a 3-ball in M_2^3, we see that M_1^3 is the connected sum of M_2^3 with an orientable S^2 bundle over S^1 which must be $S^2 \times S^1$.

Now repeat the process so that if $T_2 < \infty$, then we perform a geometric-topological surgery at some time $\tau_2 \lessapprox T_2$ to get $\left(M_3^3, g_2\right)$, etc. As long as $T_i < \infty$ for all i, we have surgery times $\tau_i \lessapprox T_i$. Hence, we either obtain a finite sequence of solutions $\left(M_i^3, g_i(t)\right)$, $t \in [\tau_{i-1}, \tau_i]$, where $1 \leq i \leq I < \infty$ where the last solution $\left(M_I^3, g_I(t)\right)$ is defined for $t \in [\tau_{I-1}, \infty)$, or we obtain an infinite sequence of solutions. In the latter case it is conceivable that the sequence of surgery times $\{\tau_i\}$ accumulates to a finite time. This is a major case which needs to be ruled out. Barring this, we obtain a solution

[2] In Perelman's approach, the surgery is performed *at* the singularity time on the singular manifold limit.

of the Ricci flow with surgeries, consisting of a countable (infinite or finite) sequence of solutions of the Ricci flow, which is defined for all time $t \in [0, \infty)$. Next we want to infer the existence of a geometric structure on M_i^3 for i large enough via a sufficient understanding of the geometric properties of a metric $g_i(t_i)$ where $t_i \in [\tau_{i-1}, \tau_i]$. In particular, we want to show that the manifold may be decomposed into pieces with incompressible tori boundary whose interiors either admit complete, finite volume hyperbolic metrics or are collapsed with bounded curvature in the sense of Cheeger and Gromov.

2. Ricci flow and the evolution of scalar curvature

Given a 1-parameter family of metrics $g(t)$ on a Riemannian manifold M^n, defined on a time interval $\mathcal{I} \subset \mathbb{R}$, Hamilton's **Ricci flow equation** is

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

For any C^∞ metric g_0 on a closed manifold M^n, there exists a unique solution $g(t)$, $t \in [0, \epsilon)$, to the Ricci flow equation for some $\varepsilon > 0$, with $g(0) = g_0$. This was proved in Hamilton [**275**] and shortly thereafter a much simpler proof was given by DeTurck [**191**] (see [**192**] for an improved version). We shall discuss short time existence in Section 6.

First we look at some simple examples. Let $M^n = S^n$ and let g_{S^n} denote the standard metric on the unit n-sphere in Euclidean space. If $g_0 = r_0^2 g_{S^n}$ for some $r_0 > 0$ (r_0 is the radius), then

$$g(t) \doteqdot \left(r_0^2 - 2(n-1)t\right) g_{S^n} \tag{2.1}$$

is a solution to the Ricci flow with $g(0) = g_0$ defined on the maximal time interval $(-\infty, T)$, where $T \doteqdot r_0^2/2(n-1)$. That is, under the Ricci flow, the sphere stays round and shrinks at a steady rate.

Exercise 2.3 (Standard shrinking sphere)**.** *Show that the metric defined by (2.1) is a solution to the Ricci flow.* HINT: *Use the scale-invariance of the Ricci tensor (Exercise 1.11) and* $\mathrm{Rc}(g_{S^n}) = (n-1) g_{S^n}$. *Show that the scalar curvature of the solution is given by*

$$R(g(t)) = \frac{n(n-1)}{r_0^2 - 2(n-1)t}.$$

In particular, the solution with $r_0^2 = n(n-1)$, *which is defined on* $(-\infty, n/2)$, *has scalar curvature* $R(g(t)) = \frac{1}{1-2t/n}$.

Exercise 2.4 (Homothetic Einstein solutions)**.** *Suppose that* g_0 *is an Einstein metric, i.e.,* $\mathrm{Rc}(g_0) \equiv c g_0$ *for some* $c \in \mathbb{R}$. *Derive the explicit formula for the solution* $g(t)$ *of the Ricci flow with* $g(0) = g_0$. *Observe that* $g(t)$ *is homothetic to the initial metric* g_0 *and that it shrinks, is stationary, or expands depending on whether* c *is positive, zero, or negative, respectively.*

Exercise 2.5 (Product solutions). *Let $(M_1, g_1(t))$ and $(M_2, g_2(t))$ be solutions of the Ricci flow on a common time interval $\mathcal{I}$. Show that*

$$(M_1 \times M_2, g_1(t) + g_2(t))$$

is a solution of the Ricci flow. Hint: *See Exercise 1.68. In particular, if $(M^n, g(t))$ is a solution of the Ricci flow, then so is $(M^n \times \mathbb{R}, g(t) + dr^2)$ (we can replace $(\mathbb{R}, dr^2)$ by any static flat manifold).*

Problem 2.6. *What explicit solutions of the Ricci flow are there? Of course we have the constant sectional curvature solutions, products and quotients thereof. Some other solutions are the cigar and Rosenau solutions on $\mathbb{R}^2$ and S^2, respectively (see Chapter 4). In addition, some homogeneous solutions are explicit. It would be interesting to find more explicit solutions in dimensions ≥ 3.*

Given that we have short time existence, we are interested in the long time behavior of the solution. Toward this end we want to derive the evolution equations for the Riemann, Ricci and scalar curvatures

$$\frac{\partial}{\partial t} R_{ijk\ell}, \ \frac{\partial}{\partial t} R_{ij}, \ \frac{\partial}{\partial t} R$$

as well as other geometric quantities. We begin with the **evolution equation for the scalar curvature** since this is the easiest:

$$\boxed{\frac{\partial}{\partial t} R = \Delta R + 2\,|\mathrm{Rc}|^2}\,. \tag{2.2}$$

When $n = 2$, since then $\mathrm{Rc} = \frac{1}{2} R g$, we have

$$\frac{\partial}{\partial t} R = \Delta R + R^2. \tag{2.3}$$

Note the similarity to the heat equation $\frac{\partial u}{\partial t} = \Delta u$. To derive formula (2.2), we recall the **variation of scalar curvature**.

Lemma 2.7 (Variation of scalar curvature). *If $\frac{\partial}{\partial s} g_{ij} = v_{ij}$, then*

$$\boxed{\frac{\partial}{\partial s} R = -\Delta V + \operatorname{div}(\operatorname{div} v) - \langle v, \mathrm{Rc} \rangle}\,, \tag{2.4}$$

where $V = g^{ij} v_{ij} = \operatorname{trace}(v)$ is the trace of v.

For the derivation of this formula see (2.30) below. Note that $\operatorname{div}(\operatorname{div} v) = \nabla_p \nabla_q v_{pq}$. Plugging in $v = -2\,\mathrm{Rc}$ and using the contracted second Bianchi identity $2\nabla_q R_{pq} = \nabla_p R$, we obtain (2.2).

Exercise 2.8. *Let (M^2, h) be a Riemannian surface. Recall that if $g = u \cdot h$ for some function u on M, then*

$$R_g = u^{-1} (R_h - \Delta_h \log u)\,.$$

Using this equation and the fact that $R_{ij} = \frac{1}{2} R g_{ij}$ *when* $n = 2$*, show that* $g(t) = u(t) \cdot h$ *is a solution of the Ricci flow if and only if*

$$\frac{\partial u}{\partial t} = \Delta_h \log u - R_h. \tag{2.5}$$

3. The maximum principle for heat-type equations

When we have heat-type equations, we can apply a powerful tool, the **maximum principle**. The maximum principle is basically the first and second derivative tests in calculus. For elliptic equations on a manifold, the facts we use are that if a function $f : M \to \mathbb{R}$ attains its minimum at a point $x_0 \in M$, then

$$\nabla f(x_0) = 0 \text{ and } \Delta f(x_0) \geq 0.$$

For equations of parabolic type (see Section 6 for the definition of parabolic), such as the heat equation, a simple version gives the following.

Proposition 2.9 (Maximum principle for supersolutions of the heat equation)**.** *Let* $g(t)$ *be a family of metrics on a closed manifold* M^n *and let* $u : M^n \times [0, T) \to \mathbb{R}$ *satisfy*

$$\frac{\partial}{\partial t} u \geq \Delta_{g(t)} u.$$

Then if $u \geq c$ *at* $t = 0$ *for some* $c \in \mathbb{R}$*, then* $u \geq c$ *for all* $t \geq 0$*.*

Remark 2.10. *To be more precise, we should call this the 'minimum principle' since we obtain an estimate for the minimum of the supersolution. But for convenience, we just call all such minimum or maximum principles simply the maximum principle.*

Proof. The idea is simply that given a time $t_0 \geq 0$, if the spatial minimum of u is attained at a point $x_0 \in M$, then

$$\frac{\partial u}{\partial t}(x_0, t_0) \geq \left(\Delta_{g(t)} u\right)(x_0, t_0) \geq 0,$$

so that the minimum should be nondecreasing. Note that at (x_0, t_0) we actually have $(\nabla_i \nabla_j u) \geq 0$ (this is simply the second derivative test from calculus). Recall that when we write $(A_{ij}) \geq 0$ for some matrix A, we mean that it is nonnegative definite.

More rigorously, we proceed as follows. Given any $\varepsilon > 0$, define $u_\varepsilon : M \times [0, T) \to \mathbb{R}$ as

$$u_\varepsilon = u + \varepsilon(1 + t).$$

Since $u \geq c$ at $t = 0$, we have $u_\varepsilon > c$ at $t = 0$. Now suppose for some $\varepsilon > 0$ we have $u_\varepsilon \leq c$ somewhere in $M \times (0, T)$. Then since M is closed, there exists $(x_1, t_1) \in M \times (0, T)$ such that $u_\varepsilon (x_1, t_1) = c$ and $u_\varepsilon (x, t) > c$ for all $x \in M$ and $t \in [0, t_1)$. We then have at (x_1, t_1)

$$0 \geq \frac{\partial u_\varepsilon}{\partial t} \geq \Delta_{g(t)} u_\varepsilon + \varepsilon > 0,$$

which is a contradiction. Hence $u_\varepsilon > c$ on $M \times [0, T)$ for all $\varepsilon > 0$ and by taking the limit as $\varepsilon \to 0$, we get $u \geq c$ on $M \times [0, T)$. □

Applying the maximum principle to the equation for the scalar curvature (2.2) yields

Corollary 2.11 (Lower bound of scalar curvature is preserved under RF)**.** *If $g(t)$, $t \in [0, T)$, is a solution to the Ricci flow on a closed manifold with $R \geq c$ at $t = 0$ for some $c \in \mathbb{R}$, then*

$$R \geq c$$

for all $t \in [0, T)$. In particular, nonnegative (positive) scalar curvature is preserved under the Ricci flow.

A simple extension of Proposition 2.9 that we shall find convenient to use is the following (for a proof, see [**163**], Theorem 4.4 on p. 96).

Lemma 2.12 (Maximum principle—comparing with the ODE)**.** *Suppose $g(t)$ is a family of metrics on a closed manifold M^n and $u : M^n \times [0, T) \to \mathbb{R}$ satisfies*

$$\frac{\partial}{\partial t} u \leq \Delta_{g(t)} u + \langle X(t), \nabla u \rangle + F(u),$$

where $X(t)$ is a time-dependent vector field and F is a Lipschitz function. If $u \leq c$ at $t = 0$ for some $c \in \mathbb{R}$, then $u(x, t) \leq U(t)$ for all $x \in M^n$ and $t \geq 0$, where $U(t)$ is the solution to the ODE

$$\frac{dU}{dt} = F(U)$$

with $U(0) = c$.

Exercise 2.13. *Prove Lemma 2.12.* Hint: *Use*

$$\frac{\partial}{\partial t}(u - U) \leq \Delta_{g(t)}(u - U) + \langle X(t), \nabla(u - U) \rangle + F(u) - F(U) \tag{2.6}$$

and the Lipschitz property of F.

Remark 2.14. *In the statement of Lemma 2.12 we may reverse the signs on all of the inequalities (except $t \geq 0$) to obtain that a supersolution to a semi-linear heat equation is bounded below by any solution to the corresponding* ODE *with initial value less than or equal to the infimum of the initial value of the supersolution of the* PDE.

Exercise 2.15. *Formulate the minimum principle for supersolutions of the equation*

$$\frac{\partial}{\partial t}u \geq \Delta_{g(t)}u + \langle X(t), \nabla u\rangle + F(u). \tag{2.7}$$

Let $(M^n, g(t))$, $t \in [0, T)$, be a solution to the Ricci flow on a closed manifold (or any solution where we can apply the maximum principle to the evolution equation for the scalar curvature). Since $|\mathrm{Rc}|^2 \geq \frac{1}{n}R^2$ (see Exercise 1.50), equation (2.2) implies

$$\frac{\partial}{\partial t}R \geq \Delta R + \frac{2}{n}R^2. \tag{2.8}$$

Since the solutions to the ODE $\frac{d\rho}{dt} = \frac{2}{n}\rho^2$ are $\rho(t) = \frac{n}{n\rho(0)^{-1} - 2t}$, by the maximum principle one has

$$\boxed{R(x,t) \geq \frac{n}{n\left(\inf_{t=0} R\right)^{-1} - 2t}} \tag{2.9}$$

for all $x \in M^n$ and $t \geq 0$. It $\rho(0) > 0$, then $\rho(t)$ tends to infinity in finite time. When M is closed, let $R_{\min}(0) \doteqdot \inf_{t=0} R$. Hence

Corollary 2.16 (Finite singularity time for positive scalar curvature)**.** *If (M^n, g_0) is a closed Riemannian manifold with positive scalar curvature, then for any solution $g(t)$, $t \in [0, T)$, to the Ricci flow with $g(0) = g_0$ we have*

$$T \leq \frac{n}{2R_{\min}(0)} < \infty.$$

Besides studying the Ricci flow on closed manifolds, we shall consider the Ricci flow on noncompact manifolds. This will be especially important in singularity analysis.

Definition 2.17 (Complete solution)**.** *A solution $g(t)$, $t \in \mathcal{I}$, of the Ricci flow is said to be* ***complete*** *if for each $t \in \mathcal{I}$, the Riemannian metric $g(t)$ is complete.*

In addition, we say that a solution of the Ricci flow is **ancient** if it exists on the time interval $(-\infty, 0]$.

Lemma 2.18 (Ancient solutions have nonnegative scalar curvature)**.** *If $(M^n, g(t))$, $t \in (-\infty, 0]$, is a complete solution to the Ricci flow with bounded curvature on compact time intervals, then either $R(g(t)) > 0$ for all $t \in (-\infty, 0]$ or $\mathrm{Rc}(g(t)) \equiv 0$ for all $t \in (-\infty, 0]$.*

Proof. If M^n is closed, then we can apply the maximum principle by Lemma 2.12 and Exercise 2.15. On the other hand, if M^n is noncompact, then since the solution has bounded curvature on compact time intervals, we may still apply the maximum principle to the evolution equation for R by Corollary 7.43 and Exercise 7.47. For any solution of the Ricci flow for which we can apply the maximum principle on a time interval $[0, T]$, by (2.9) we have

$$R(x,t) \geq -\frac{n}{2t}$$

for $t \in (0, T]$. Let α be any negative number. Since the solution is ancient, it exists on the time interval $[\alpha, 0]$. Thus, by translating time, we have

$$R(x,t) \geq -\frac{n}{2(t-\alpha)}$$

for all $x \in M$ and $t \in (\alpha, 0]$. Taking the limit as $\alpha \to -\infty$, we conclude that $R(x,t) \geq 0$ for all $t \in (-\infty, 0]$.

The strong maximum principle (see Corollary 6.55) implies either $R > 0$ always or $R \equiv 0$ always. In the latter case, by the evolution equation for R, we deduce $\mathrm{Rc} \equiv 0$. □

Exercise 2.19. *Let u be a solution to the heat equation with respect to a metric $g(t)$ evolving by the Ricci flow:*

$$\frac{\partial u}{\partial t} = \Delta_{g(t)} u.$$

Recall from Exercise 1.40 that

$$\frac{\partial}{\partial t} |\nabla u|^2 = \Delta |\nabla u|^2 - 2 |\nabla \nabla u|^2 .$$

(1) *From this deduce*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(t |\nabla u|^2 + \frac{1}{2} u^2\right) \leq 0.$$

(2) *Apply the maximum principle to conclude that if M^n is closed, then*

$$|\nabla u| \leq \frac{U}{\sqrt{2} t^{1/2}}, \tag{2.10}$$

where $U \doteqdot \max_{t=0} |u|$.

Exercise 2.20. *Let u be a solution to the heat equation on a Riemannian manifold (M^n, g). Show that*

$$\frac{\partial}{\partial t} |\nabla u|^2 = \Delta |\nabla u|^2 - 2 |\nabla \nabla u|^2 - 2 R_{ij} \nabla_i u \nabla_j u.$$

What estimate, analogous to the previous exercise, for $|\nabla u|$ do you get assuming $\mathrm{Rc} \geq 0$? How about the case where $\mathrm{Rc} \geq -(n-1)K$ for some constant K? Which curvature condition do you need to get decay of $|\nabla u|$ as $t \to \infty$?

4. The Einstein-Hilbert functional

The game of Ricci flow, so to speak, is to **control geometric quantities associated to the metric as it evolves**. The above Corollary 2.11 and (2.9) are nice examples of this. Let's now move onto the volume form $d\mu$ (we assume that M is **oriented**). In general, if $\frac{\partial}{\partial s} g_{ij} = v_{ij}$, then

$$\frac{\partial}{\partial s} d\mu = \frac{1}{2} V d\mu, \tag{2.11}$$

where $V \doteqdot g^{ij} v_{ij}$. This is easily seen from the local coordinate formula

$$d\mu = \sqrt{\det (g_{ij})} dx^1 \wedge \cdots \wedge dx^n \tag{2.12}$$

in a positively oriented local coordinate system $\{x^i\}$ and the formula for the variation of a determinant of a matrix $A(s)$

$$\frac{d}{ds} \log \det A = \left(A^{-1}\right)^{ij} \frac{d}{ds} A_{ij}, \tag{2.13}$$

where log denotes the natural logarithm. We can see this from the standard definition

$$\det A = \sum_{\sigma} \operatorname{sign}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)},$$

where the summation is over all permutations σ of $\{1, 2, \ldots, n\}$. Differentiating this formula, we get

$$\frac{d}{ds} \det A = \sum_{i,j=1}^{n} \frac{d}{ds} A_{ij} \sum_{\sigma:\sigma(i)=j} \operatorname{sign}(\sigma) A_{1\sigma(1)} \cdots \widehat{A_{i\sigma(i)}} \cdots A_{n\sigma(n)},$$

where $\widehat{A_{i\sigma(i)}}$ means to omit this factor and the second summation is over all permutations σ such that $\sigma(i) = j$. Equation (2.13) now follows from Cramer's rule:

$$\left(A^{-1}\right)_{ij} = \frac{1}{\det A} \sum_{\sigma:\sigma(i)=j} \operatorname{sign}(\sigma) A_{1\sigma(1)} \cdots \widehat{A_{i\sigma(i)}} \cdots A_{n\sigma(n)}.$$

Exercise 2.21 (Variation of the inverse of g)**.** *By differentiating the formula $g^{ij} g_{jk} = \delta^i_k$, show that*

$$\frac{\partial}{\partial s} g^{ij} = -g^{ik} g^{j\ell} \frac{\partial}{\partial s} g_{k\ell}. \tag{2.14}$$

Of course, the global formula (2.11) is independent of the coordinates we choose to derive the formula. The above formulas give a quick derivation of the first variation formula for the Einstein-Hilbert (total scalar curvature) functional

$$E(g) \doteqdot \int_M R d\mu.$$

Namely, if $\frac{\partial}{\partial s} g_{ij} = v_{ij}$, then by (2.4) and (2.11)

$$\frac{d}{ds} E = \int_M \left(-\Delta V + \nabla_p \nabla_q v_{pq} - \langle v, \mathrm{Rc} \rangle + \frac{1}{2} R V \right) d\mu$$

$$= \int_M \left\langle v, \frac{1}{2} R g - \mathrm{Rc} \right\rangle d\mu. \tag{2.15}$$

Note that (twice) the **gradient flow** of E is

$$\frac{\partial}{\partial t} g_{ij} = 2 \left(\nabla E \left(g \right) \right)_{ij} = R g_{ij} - 2 R_{ij}. \tag{2.16}$$

This looks sort of like Ricci flow, but this equation in fact is not parabolic, and as such, short time existence is not expected to hold. Dropping the Rg term on the RHS of (2.16) yields the Ricci flow.

Exercise 2.22. *Given a metric g_0, let*

$$\mathcal{C} \doteqdot \{ u g_0 : u > 0 \text{ and } \operatorname{Vol}(u g_0) = 1 \}$$

be the space of unit volume metrics conformal to g_0. Show that subject to the constraint of lying in $\mathcal{C}$, the critical points E have constant scalar curvature.

Exercise 2.23. *Show that when $n \geq 3$, the negative gradient flow of E in a fixed conformal class (i.e., in $[g_0] \doteqdot \{ g : g = e^u g_0 \text{ for some } u \in C^\infty(M) \}$) is the Yamabe flow $\frac{\partial}{\partial t} g = -Rg$.*

Now let's look at the total scalar curvature functional (2.15) from the point of view of the Ricci flow. Instead of (2.16), we would like to get the equation $\frac{\partial}{\partial t} g_{ij} = -2 R_{ij}$, which is the Ricci flow. The undesirable term $R g_{ij}$ in (2.16) is due to the variation of the volume form $d\mu$ in $E(g)$. How can we get rid of this term? First, we should consider metrics up to geometric equivalence, that is, up to the pull back by diffeomorphisms. In other words, we consider the more general class of flows

$$\frac{\partial}{\partial t} g_{ij} = -2 \left(R_{ij} + \nabla_i \nabla_j f \right),$$

where f is a time-dependent function. Flows of this form are equivalent to the Ricci flow since $\nabla_i \nabla_j f = \left(\mathcal{L}_{\nabla f} g \right)_{ij}$ (see (1.24)). The following exercise verifies this.

Exercise 2.24. *Define a 1-parameter family of diffeomorphisms $\Psi(t) : M \to M$ by*

$$\frac{d}{dt} \Psi(t) = \nabla_{g(t)} f(t),$$
$$\Psi(0) = \mathrm{id}_M.$$

Show that $\bar{g}(t) \doteqdot \Psi(t)^* g(t)$ *and* $\bar{f}(t) \doteqdot f \circ \Psi(t)$ *satisfy*

$$\frac{\partial \bar{g}_{ij}}{\partial t} = -2\bar{R}_{ij}, \tag{2.17}$$

$$\frac{\partial \bar{f}}{\partial t} = -\bar{\Delta}\bar{f} + \left|\bar{\nabla}\bar{f}\right|^2 - \bar{R}. \tag{2.18}$$

Now one way of making the variation of $d\mu$ zero is to introduce such a function f and impose the condition

$$\frac{\partial}{\partial s}\left(e^{-f} d\mu\right) = 0 \tag{2.19}$$

and to consider the functional

$$\mathcal{E}(g) \doteqdot \int_M R e^{-f} d\mu.$$

This does not quite work since the e^{-f} factor introduces new terms in the variation (the divergence-type terms no longer integrate out to zero). In particular, $\frac{d}{ds}\mathcal{E}$ now has terms of the form

$$\begin{aligned}&\int_M \left(-\Delta V + \nabla_i \nabla_j v_{ij}\right) e^{-f} d\mu \\ &= \int_M v_{ij}\left(-\Delta\left(e^{-f}\right) g_{ij} + \nabla_i \nabla_j \left(e^{-f}\right)\right) d\mu,\end{aligned}$$

which were not there when $f = 0$ (the original case of E). More specifically,

$$\begin{aligned}\frac{d}{ds}\mathcal{E} &= \int_M \frac{\partial R}{\partial s} e^{-f} d\mu \\ &= -\int_M \langle v, \mathrm{Rc}\rangle e^{-f} d\mu + \int_M \left(-\Delta V + \nabla_i \nabla_j v_{ij}\right) e^{-f} d\mu.\end{aligned}$$

We can cancel the undesirable terms by introducing the Dirichlet energy-type term

$$\int_M |\nabla f|^2 e^{-f} d\mu = 4\int_M \left|\nabla e^{-f/2}\right|^2 d\mu.$$

Assuming $\frac{\partial}{\partial s}\left(e^{-f} d\mu\right) = 0$, we compute

$$\begin{aligned}&\int_M \left(\frac{\partial}{\partial s}|\nabla f|^2\right) e^{-f} d\mu \\ &= \int_M \left(-v_{ij}\nabla_i f \nabla_j f + 2\nabla f \cdot \nabla\left(\frac{\partial f}{\partial s}\right)\right) e^{-f} d\mu\end{aligned}$$

since $|\nabla f|^2 = g^{ij}\nabla_i f \nabla_j f$. Integrating by parts and using $\frac{\partial}{\partial s} f = \frac{1}{2}V$ (by (2.19)), we find that the above is equal to

$$\int_M \left(-v_{ij}\left(\nabla_i f \nabla_j f\right) e^{-f} + V\Delta\left(e^{-f}\right)\right) d\mu.$$

Adding the above equations together, we obtain

$$
\begin{aligned}
&\frac{d}{ds}\int_M \left(R+|\nabla f|^2\right) e^{-f} d\mu \\
&= \int_M -\langle v, \mathrm{Rc}\rangle e^{-f} d\mu \\
&+ \int_M \left(-V\Delta\left(e^{-f}\right) + \left(\nabla_i f \nabla_j f - \nabla_i \nabla_j f\right) e^{-f} v_{ij}\right) d\mu \\
&+ \int_M \left(-v_{ij}\left(\nabla_i f \nabla_j f\right) e^{-f} + V\Delta\left(e^{-f}\right)\right) d\mu \\
&= -\int_M \langle v, \mathrm{Rc} + \nabla\nabla f\rangle e^{-f} d\mu. \qquad (2.20)
\end{aligned}
$$

So if we define

$$
\mathcal{F}(g,f) \doteqdot \int_M \left(R+|\nabla f|^2\right) e^{-f} d\mu, \qquad (2.21)
$$

then the gradient flow for $\mathcal{F}$, under the constraint that $e^{-f}d\mu$ is fixed, is

$$
\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -2\left(R_{ij} + \nabla_i \nabla_j f\right), \\
\frac{\partial}{\partial t} f &= -R - \Delta f.
\end{aligned}
$$

These are the equations Perelman imposed (see §1 of [**452**]). By (2.20), we have the monotonicity formula

$$
\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_M |\mathrm{Rc} + \nabla\nabla f|^2 e^{-f} d\mu \geq 0. \qquad (2.22)
$$

Remark 2.25. *The functional $\mathcal{F}(g,f)$ has been considered in string theory. Perelman introduced it to the study of Ricci flow in* [**452**].

Exercise 2.26. *Show that*

$$
\int_M |\nabla f|^2 e^{-f} d\mu = \int_M (\Delta f) e^{-f} d\mu.
$$

This shows that we may rewrite $\mathcal{F}$ as

$$
\mathcal{F}(g,f) = \int_M \left(R + 2\Delta f - |\nabla f|^2\right) e^{-f} d\mu.
$$

Motivations for doing this are given by (1.84) and (5.17).

In Section 4 of Chapter 5 we shall discuss the extension of the monotonicity formula (2.22) for the energy functional to the entropy functional, which has geometric applications in Ricci flow.

5. Evolution of geometric quantities

We now proceed to discuss the variation of the Ricci tensor. Before we can do this, we need to recall the **variation of the Christoffel symbols**.

Lemma 2.27 (Variation of Christoffel symbols)**.** *If $g(s)$ is a 1-parameter family of metrics with $\frac{\partial}{\partial s} g_{ij} = v_{ij}$, then*

$$\frac{\partial}{\partial s}\Gamma_{ij}^{k} = \frac{1}{2} g^{k\ell}\left(\nabla_i v_{j\ell} + \nabla_j v_{i\ell} - \nabla_\ell v_{ij}\right). \tag{2.23}$$

Proof. The derivation of this formula illustrates a nice trick in computing evolutions of various geometric quantities such as the connection and the curvatures. We compute at an arbitrarily chosen point $p \in M$ in normal coordinates centered at p so that $\Gamma_{ij}^{k}(p) = 0$. Note that $\frac{\partial}{\partial x^i} g_{jk}(p) = 0$. In such coordinates, $\nabla_k a_{i_1\cdots i_r}^{j_1\cdots j_q}(p) = \frac{\partial}{\partial x^k} a_{i_1\cdots i_r}^{j_1\cdots j_q}(p)$ for any (r,q)-tensor a. Thus, at p we have

$$\frac{\partial}{\partial s}\Gamma_{ij}^{k} = \frac{1}{2} g^{k\ell}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial s} g_{j\ell} + \frac{\partial}{\partial x^j}\frac{\partial}{\partial s} g_{i\ell} - \frac{\partial}{\partial x^\ell}\frac{\partial}{\partial s} g_{ij}\right)$$

and (2.23) follows since $\nabla_i v_{j\ell}(p) = \frac{\partial}{\partial x^i} v_{j\ell}(p)$. Finally we note that since both sides of (2.23) are the components of tensors, equation (2.23) in fact holds as a tensor equation, that is, it is true for any coordinate system, not just normal coordinates. □

Remark 2.28. *In coordinate-free notation, (2.23) is*

$$\left\langle \left(\frac{\partial}{\partial s}\nabla\right)(X,Y), Z\right\rangle = \frac{1}{2}\left(\left(\nabla_X v\right)(Y,Z) + \left(\nabla_Y v\right)(X,Z) - \left(\nabla_Z v\right)(X,Y)\right). \tag{2.24}$$

This formula may be derived directly from differentiating (1.4).

Corollary 2.29 (Evolution of Christoffel symbols under RF)**.** *Under the Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, we have*

$$\boxed{\frac{\partial}{\partial t}\Gamma_{ij}^{k} = -g^{k\ell}\left(\nabla_i R_{j\ell} + \nabla_j R_{i\ell} - \nabla_\ell R_{ij}\right)}. \tag{2.25}$$

A nice consequence of this is the evolution of the Laplacian operator acting on functions.

Lemma 2.30 (Evolution of Laplacian under RF)**.** *If $(M^n, g(t))$ is a solution to the Ricci flow, then*

$$\frac{\partial}{\partial t}\left(\Delta_{g(t)}\right) = 2R_{ij}\cdot\nabla_i\nabla_j,$$

where $\Delta_{g(t)}$ is the Laplacian acting on functions. In particular, when $n = 2$, $\frac{\partial}{\partial t}(\Delta) = R\Delta$.

Proof. We compute

$$\frac{\partial}{\partial t}\left(\Delta_{g(t)}\right) = \frac{\partial}{\partial t}\left(g^{ij}\nabla_i\nabla_j\right) = -\frac{\partial}{\partial t}g_{ij}\cdot\nabla_i\nabla_j - g^{ij}\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right)\nabla_k$$

and the result follows from

$$g^{ij}\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right) = -g^{k\ell}\left(2g^{ij}\nabla_i R_{j\ell} - \nabla_\ell R\right) = 0, \tag{2.26}$$

where we used the contracted second Bianchi identity. □

Exercise 2.31. *Given* $\frac{\partial}{\partial s}g_{ij} = v_{ij}$, *compute* $\frac{\partial}{\partial s}\left(\Delta_{g(s)}\right)$.

Now we recall how to get the components of the curvature tensors from the Christoffel symbols. Recall that the components of the Riemann curvature $(3,1)$-tensor defined by $R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R_{ijk}^\ell\frac{\partial}{\partial x^\ell}$ are given by (1.12):

$$R_{ijk}^\ell = \partial_i\Gamma_{jk}^\ell - \partial_j\Gamma_{ik}^\ell + \Gamma_{jk}^p\Gamma_{ip}^\ell - \Gamma_{ik}^p\Gamma_{jp}^\ell \tag{2.27}$$

and the Ricci tensor is $R_{ij} = R_{pij}^p$. From this we calculate the **variation of the Ricci tensor in terms of the variation of the connection**

$$\frac{\partial}{\partial s}R_{ij} = \nabla_p\left(\frac{\partial}{\partial s}\Gamma_{ij}^p\right) - \nabla_i\left(\frac{\partial}{\partial s}\Gamma_{pj}^p\right). \tag{2.28}$$

Just as in the proof of Lemma 2.27, this follows from computing at the center in normal coordinates. This is a nice formula and we shall use this later again. For now we just substitute (2.23) into this to obtain that if $\frac{\partial}{\partial s}g_{ij} = v_{ij}$, then

$$\frac{\partial}{\partial s}R_{ij} = \frac{1}{2}\nabla_\ell\left(\nabla_i v_{j\ell} + \nabla_j v_{i\ell} - \nabla_\ell v_{ij}\right) - \frac{1}{2}\nabla_i\nabla_j V. \tag{2.29}$$

Recall that $(\operatorname{div} v)_k \doteqdot g^{ij}\nabla_i v_{jk}$. Taking the trace, we obtain the variation formula (2.4) for R

$$\begin{aligned}\frac{\partial}{\partial s}R &= g^{ij}\left(\frac{\partial}{\partial s}R_{ij}\right) - \frac{\partial}{\partial s}g_{ij}\cdot R_{ij}\\ &= \nabla_\ell\nabla_i v_{i\ell} - \Delta V - v_{ij}\cdot R_{ij}.\end{aligned} \tag{2.30}$$

Commuting derivatives in (2.29) yields the **variation of Ricci formula**:

$$\boxed{\frac{\partial}{\partial s}R_{ij} = -\tfrac{1}{2}\left(\Delta_L v_{ij} + \nabla_i\nabla_j V - \nabla_i\left(\operatorname{div} v\right)_j - \nabla_j\left(\operatorname{div} v\right)_i\right)}. \tag{2.31}$$

Here Δ_L denotes the **Lichnerowicz Laplacian**, which is defined by

$$\boxed{\Delta_L v_{ij} \doteqdot \Delta v_{ij} + 2R_{kij\ell}v_{k\ell} - R_{ik}v_{jk} - R_{jk}v_{ik}} \tag{2.32}$$

acting on symmetric 2-tensors. Recall that $R_{kij\ell} \doteqdot g_{\ell p} R_{kij}^p$. We obtain (2.31) from (2.29) by the computation

$$\begin{aligned}\nabla_\ell \left(\nabla_i v_{j\ell} + \nabla_j v_{i\ell}\right) &= \nabla_i \nabla_\ell v_{j\ell} - R_{\ell ijm} v_{m\ell} - R_{\ell i\ell m} v_{jm} \\ &\quad + \nabla_j \nabla_\ell v_{i\ell} - R_{\ell jim} v_{m\ell} - R_{\ell j\ell m} v_{im} \\ &= \nabla_i \left(\operatorname{div} v\right)_j + \nabla_j \left(\operatorname{div} v\right)_i - 2R_{\ell ijm} v_{\ell m} \\ &\quad + R_{im} v_{jm} + R_{jm} v_{im},\end{aligned}$$

where we used the commutator formula (1.30). Note that (2.31) may be rewritten as

$$\frac{\partial}{\partial s}\left(-2R_{ij}\right) = \Delta_L v_{ij} + \nabla_i X_j + \nabla_j X_i,$$

where $X = \frac{1}{2}\nabla V - \operatorname{div} v$. This is related to DeTurck's trick in proving short time existence (see (2.47)).

Remark 2.32. *By (1.50) we see that the Hodge Laplacian acts on 2-forms formally in the same way that the Lichnerowicz Laplacian acts on symmetric 2-tensors.*

The Lichnerowicz Laplacian is a fundamental operator; when acting on symmetric 2-tensors in the context of Ricci flow, it is perhaps more natural than the rough Laplacian $\Delta = g^{ij}\nabla_i\nabla_j$. Examples of this naturality are the appearance of Δ_L in the linearized Ricci flow equation (formula (2.43) below is an example of this) and the following identity.

Lemma 2.33 (Commutator formula for the Hessian and the Lichnerowicz heat operator)**.** *Under the Ricci flow, the Hessian and the Lichnerowicz Laplacian heat operator commute. That is, for any function f of space and time we have*

$$\boxed{\nabla_i\nabla_j\left(\frac{\partial f}{\partial t} - \Delta f\right) = \left(\frac{\partial}{\partial t} - \Delta_L\right)\nabla_i\nabla_j f}\,. \tag{2.33}$$

Proof. Using (1.30), we compute

$$\begin{aligned}\nabla_i\nabla_j\Delta f &= \nabla_i\nabla_j\nabla_k\nabla_k f \\ &= \nabla_i\nabla_k\nabla_k\nabla_j f - \nabla_i\left(R_{j\ell}\nabla_\ell f\right) \\ &= \nabla_k\nabla_i\nabla_k\nabla_j f - R_{ikj\ell}\nabla_\ell\nabla_k f - R_{i\ell}\nabla_j\nabla_\ell f \\ &\quad - \nabla_i R_{j\ell}\nabla_\ell f - R_{j\ell}\nabla_i\nabla_\ell f \\ &= \nabla_k\nabla_k\nabla_i\nabla_j f - \nabla_k\left(R_{ikj\ell}\nabla_\ell f\right) - R_{ikj\ell}\nabla_\ell\nabla_k f - R_{i\ell}\nabla_j\nabla_\ell f \\ &\quad - \nabla_i R_{j\ell}\nabla_\ell f - R_{j\ell}\nabla_i\nabla_\ell f \\ &= \Delta\nabla_i\nabla_j f + \left(-\nabla_j R_{i\ell} + \nabla_\ell R_{ij} - \nabla_i R_{j\ell}\right)\nabla_\ell f - 2R_{ikj\ell}\nabla_\ell\nabla_k f \\ &\quad - R_{i\ell}\nabla_j\nabla_\ell f - R_{j\ell}\nabla_i\nabla_\ell f \\ &= \Delta_L\nabla_i\nabla_j f - \left(\nabla_i R_{j\ell} + \nabla_j R_{i\ell} - \nabla_\ell R_{ij}\right)\nabla_\ell f \qquad (2.34)\end{aligned}$$

where we used $\nabla_k R_{ikj\ell} = \nabla_j R_{i\ell} - \nabla_\ell R_{ij}$ (from the second Bianchi identity) to get the last equality. Second, using (2.25), we compute

$$\frac{\partial}{\partial t}(\nabla_i \nabla_j f) = \nabla_i \nabla_j \left(\frac{\partial f}{\partial t}\right) + (\nabla_i R_{j\ell} + \nabla_j R_{i\ell} - \nabla_\ell R_{ij}) \nabla_\ell f. \tag{2.35}$$

Formula (2.33) now follows from combining the above two calculations. □

In other words,

$$\left[\nabla_i \nabla_j, \frac{\partial}{\partial t} - \Delta_L\right] = 0,$$

where $\Delta_L \doteqdot \Delta$ acting on functions. An immediate consequence of the above lemma is the following.

Corollary 2.34. *If $g(t)$ satisfies the Ricci flow and $f(t)$ satisfies the heat equation $\frac{\partial f}{\partial t} = \Delta f$, then the Hessian satisfies the Lichnerowicz Laplacian heat equation:*

$$\frac{\partial}{\partial t}(\nabla \nabla f) = \Delta_L (\nabla \nabla f). \tag{2.36}$$

For further discussion of (2.36), see Section 6 in Chapter 10. Instead of considering the heat operator, we may consider the backward heat operator.

Exercise 2.35 (Commutator of $\frac{\partial}{\partial t} + \Delta_L$ and $\nabla\nabla$)**.** *Using the formulas derived in the proof of Lemma 2.33, establish under the Ricci flow that we have the identity*

$$\nabla_i \nabla_j \left(\frac{\partial f}{\partial t} + \Delta f\right) = \left(\frac{\partial}{\partial t} + \Delta_L\right) \nabla_i \nabla_j f - 2(\nabla_i R_{j\ell} + \nabla_j R_{i\ell} - \nabla_\ell R_{ij}) \nabla_\ell f. \tag{2.37}$$

Generalizing the above lemma, we have

Exercise 2.36. *Show that under the Ricci flow, for any 1-form X*

$$\left(\frac{\partial}{\partial t} - \Delta_L\right)(\mathcal{L}_{X^\natural} g) = \mathcal{L}_{\left[\left(\frac{\partial}{\partial t} - \Delta_d\right) X\right]^\natural} g\ ;$$

that is,

$$\left(\frac{\partial}{\partial t} - \Delta_L\right)(\nabla_i X_j + \nabla_j X_i) = \nabla_i Y_j + \nabla_j Y_i, \tag{2.38}$$

where $Y \doteqdot \left(\frac{\partial}{\partial t} - \Delta_d\right) X$.

A useful consequence of the above exercise is the following.

Lemma 2.37. *If $(M^n, g(t))$ is a solution to the Ricci flow and if X is a vector field evolving by*

$$\frac{\partial}{\partial t} X^i = \Delta X^i + R^i_k X^k, \tag{2.39}$$

then $h_{ij} \doteqdot \nabla_i X_j + \nabla_j X_i = (\mathcal{L}_X g)_{ij}$ *evolves by*

$$\frac{\partial}{\partial t} h_{ij} = \Delta_L h_{ij} \doteqdot \Delta h_{ij} + 2R_{kij\ell} h_{k\ell} - R_{ik} h_{kj} - R_{jk} h_{ik}. \tag{2.40}$$

Proof. The dual 1-form evolves by

$$\frac{\partial}{\partial t} X_i = \Delta X_i - R_{ij} X_j = \Delta_d X_i,$$

where $\Delta_d \doteqdot -(d\delta + \delta d)$. The result now follows from Exercise 2.36. □

Remark 2.38. *A special case of (2.38) is formula (2.33). We may see this as follows. Since* $\left(\frac{\partial}{\partial t} - \Delta_d\right) df = d\left(\frac{\partial}{\partial t} - \Delta\right) f$*, by taking* $X = df$*, we have from (2.38)*

$$\left(\frac{\partial}{\partial t} - \Delta_L\right)(2\nabla_i \nabla_j f) = \nabla_i \left(\left(\frac{\partial}{\partial t} - \Delta_d\right)(df)\right)_j + \nabla_j \left(\left(\frac{\partial}{\partial t} - \Delta_d\right)(df)\right)_i$$
$$= 2\nabla_i \nabla_j \left(\left(\frac{\partial}{\partial t} - \Delta\right) f\right),$$

which is (2.33).

Exercise 2.39. *Show that if* X *is a Killing vector field, then*

$$\nabla_k \nabla_j X_i + R_{\ell k j i} X_\ell = 0. \tag{2.41}$$

Tracing (2.41), we have

$$\Delta X_i + R_{\ell i} X_\ell = 0. \tag{2.42}$$

Hence, if M^n is closed and the Ricci curvature is negative, then there are no nontrivial Killing vector fields. For a generalization of this to conformal Killing vector fields, see Proposition B.13.

Since, by the contracted second Bianchi identity,

$$\nabla_i \nabla_j R - \nabla_i (\operatorname{div} \mathrm{Rc})_j - \nabla_j (\operatorname{div} \mathrm{Rc})_i = 0,$$

equation (2.31) implies the following.

Lemma 2.40 (Evolution of the Ricci tensor under RF)**.** *Under the Ricci flow,*

$$\boxed{\frac{\partial}{\partial t} R_{ij} = \Delta_L R_{ij} = \Delta R_{ij} + 2R_{kij\ell} R_{k\ell} - 2R_{ik} R_{jk}}. \tag{2.43}$$

Now, just as with the evolution equation for the scalar curvature, we have a heat-type equation. However, there is an important difference: R is a scalar function whereas R_{ij} is a tensor. It is nice to know that we can still apply the maximum principle. This (the maximum principle for tensors) is the subject of the second section of the next chapter.

Exercise 2.41. *Calculate the evolution equation for* $R_{ij} - \alpha R g_{ij}$*, where* $\alpha \in \mathbb{R}$.

Exercise 2.42. *Using (2.27), show that*

$$R_{ijk\ell} = \frac{1}{2}\left(\partial_i \partial_k g_{j\ell} + \partial_j \partial_\ell g_{ik} - \partial_i \partial_\ell g_{jk} - \partial_j \partial_k g_{i\ell}\right) + Q\left(g, \partial g\right),$$

where Q *is quadratic in* ∂g*. (For a related formula, see (2.67).)*

6. DeTurck's trick and short time existence

The fundamental short time existence theorem for the Ricci flow on closed manifolds is the following.

Theorem 2.43 (Hamilton, DeTurck—short time existence)**.** *If* M^n *is a closed Riemannian manifold and if* g_0 *is a* C^∞ *Riemannian metric, then there exists a unique smooth solution* $\bar{g}(t)$ *to the Ricci flow defined on some time interval* $[0, \delta)$, $\delta > 0$, *with* $\bar{g}(0) = g_0$.

Using (2.31), we shall now present what is essentially DeTurck's proof (or **DeTurck's trick**) of short time existence. First note that the principal symbol of the nonlinear partial differential operator $-2\operatorname{Rc}(g)$ of the metric g is nonnegative definite and has a nontrivial kernel which is due exactly to the diffeomorphism invariance of the Ricci tensor (see Section 2.3 of [**163**] for details). For this reason the Ricci flow equation is only **weakly parabolic**. We search for an equivalent flow which is **strictly parabolic** (i.e., where the principal symbol of the second-order operator on the RHS is positive definite). Motivated by formula (2.31), given a fixed background connection $\tilde{\Gamma}$, which for convenience we assume to be the Levi-Civita connection of a metric $\tilde{g}$, we define the **Ricci-DeTurck flow** by

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= -2R_{ij} + \nabla_i W_j + \nabla_j W_i\,, \\ g(0) &= g_0, \end{aligned} \tag{2.44}$$

where the time-dependent 1-form $W = W(g)$ is defined by

$$W_j \doteqdot g_{jk} g^{pq}\left(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}\right). \tag{2.45}$$

Note that if $g(s)$ is a 1-parameter family of metrics with $g(0) = g$ and

$$\left.\frac{\partial}{\partial s}\right|_{s=0} g_{ij} = v_{ij},$$

then

$$\left.\frac{\partial}{\partial s}\right|_{s=0} W\left(g\left(s\right)\right)_j = -X_j + \text{zeroth-order terms in } v, \tag{2.46}$$

where $X = \frac{1}{2}\nabla V - \operatorname{div} v$ as above. From (2.31) and (2.23) we compute

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \left(-2R_{ij} + \nabla_i W_j + \nabla_j W_i\right) = \Delta_L v_{ij} + \text{first-order terms in } v. \tag{2.47}$$

Exercise 2.44. *Verify formula (2.47).*

From (2.47) it follows that the Ricci-DeTurck flow is strictly parabolic and that given any smooth initial metric g_0 on a closed manifold, there exists a unique solution $g(t)$ to the Ricci-DeTurck flow with $g(0) = g_0$.

Another way of showing that the Ricci-DeTurck flow is a strictly parabolic system, besides computing the linearization of the modified Ricci tensor as above, is to compute an expression for the modified Ricci tensor of a metric g as an elliptic operator of g using a background metric $\tilde{g}$. We proceed to do this in the following; see (2.51) below for the end result. Recall that the difference of two connections is a tensor:

$$A_{ij}^k(t) \doteqdot \Gamma_{ij}^k(t) - \tilde{\Gamma}_{ij}^k = \frac{1}{2} g^{k\ell}\left(\tilde{\nabla}_i g_{j\ell} + \tilde{\nabla}_j g_{i\ell} - \tilde{\nabla}_\ell g_{ij}\right). \tag{2.48}$$

From (2.27) and the trick of computing the following tensor expressions at any point (p,t) in a coordinate system where $\tilde{\Gamma}_{ij}^k(p) = 0$, we have

$$R_{ijk}^\ell(t) - \tilde{R}_{ijk}^\ell = \tilde{\nabla}_i A_{jk}^\ell - \tilde{\nabla}_j A_{ik}^\ell + A_{jk}^p A_{ip}^\ell - A_{ik}^p A_{jp}^\ell. \tag{2.49}$$

Substituting (2.48) into the above expression, we have

$$R_{ijk}^\ell(t) - \tilde{R}_{ijk}^\ell = \tilde{\nabla}\left(g^{-1}\tilde{\nabla} g\right) + \left(g^{-1}\tilde{\nabla} g\right)^2 = g^{-1}\tilde{\nabla}\tilde{\nabla} g + g^{-2}\left(\tilde{\nabla} g\right)^2. \tag{2.50}$$

Replacing (2.50) by a more careful computation while still substituting (2.48) in (2.49) yields

$$\begin{aligned} 2R_{ijk}^\ell - 2\tilde{R}_{ijk}^\ell = {} & g^{\ell m}\tilde{\nabla}_i\left(\tilde{\nabla}_j g_{km} + \tilde{\nabla}_k g_{jm} - \tilde{\nabla}_m g_{jk}\right) \\ & - g^{\ell m}\tilde{\nabla}_j\left(\tilde{\nabla}_i g_{km} + \tilde{\nabla}_k g_{im} - \tilde{\nabla}_m g_{ik}\right) + g^{-2}\left(\tilde{\nabla} g\right)^2. \end{aligned}$$

Tracing over i and ℓ, we have

$$\begin{aligned} -2R_{jk} = {} & 2\tilde{R}_{jk} - g^{\ell m}\tilde{\nabla}_\ell\left(\tilde{\nabla}_j g_{km} + \tilde{\nabla}_k g_{jm} - \tilde{\nabla}_m g_{jk}\right) \\ & + g^{\ell m}\tilde{\nabla}_j\left(\tilde{\nabla}_\ell g_{km} + \tilde{\nabla}_k g_{\ell m} - \tilde{\nabla}_m g_{\ell k}\right) + g^{-2}\left(\tilde{\nabla} g\right)^2. \end{aligned}$$

Since by definition (2.45)

$$W_j = g_{jk} g^{\ell m} A_{\ell m}^k,$$

we have

$$\begin{aligned} \nabla_i W_j &= g_{jk} g^{\ell m} \nabla_i A_{\ell m}^k = g_{jk} g^{\ell m} \tilde{\nabla}_i A_{\ell m}^k + g * g^{-3} * \left(\tilde{\nabla} g\right)^2 \\ &= \frac{1}{2} g^{\ell m}\tilde{\nabla}_i\left(\tilde{\nabla}_\ell g_{mj} + \tilde{\nabla}_m g_{\ell j} - \tilde{\nabla}_j g_{\ell m}\right) + g * g^{-3} * \left(\tilde{\nabla} g\right)^2 \end{aligned}$$

and similarly for $\nabla_j W_i$. Hence

$$
\begin{aligned}
&-2R_{ij} + \nabla_i W_j + \nabla_j W_i \\
&= 2\tilde{R}_{ij} + g^{\ell m}\tilde{\nabla}_\ell \tilde{\nabla}_m g_{ij} + g^{\ell m}\left(\tilde{\nabla}_i \tilde{\nabla}_\ell - \tilde{\nabla}_\ell \tilde{\nabla}_i\right) g_{jm} \\
&+ g^{\ell m}\left(\tilde{\nabla}_j \tilde{\nabla}_\ell - \tilde{\nabla}_\ell \tilde{\nabla}_j\right) g_{mi} + \frac{1}{2} g^{\ell m}\left(\tilde{\nabla}_i \tilde{\nabla}_j - \tilde{\nabla}_j \tilde{\nabla}_i\right) g_{\ell m} + g * g^{-3} * \left(\tilde{\nabla} g\right)^2 \\
&= g^{\ell m}\tilde{\nabla}_\ell \tilde{\nabla}_m g_{ij} + g^{-1} * g * \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} + g * g^{-3} * \left(\tilde{\nabla} g\right)^2 .
\end{aligned}
$$

Hence the Ricci-DeTurck (2.44) flow may be expressed as

$$
\frac{\partial}{\partial t} g_{ij} = g^{\ell m}\tilde{\nabla}_\ell \tilde{\nabla}_m g_{ij} + g^{-1} * g * \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} + g * g^{-3} * \left(\tilde{\nabla} g\right)^2 . \tag{2.51}
$$

This also exhibits the strict parabolicity of the flow (recall that given tensors A and B, $A * B$ denotes a linear combination of contractions of $A \otimes B$).

Now given a solution of the Ricci-DeTurck flow, we can solve the following ODE at each point in M :

$$
\begin{aligned}
\frac{\partial}{\partial t}\varphi_t &= -W^*, \\
\varphi_0 &= \mathrm{id},
\end{aligned} \tag{2.52}
$$

where $W^*(t)$ is the vector field dual to $W(t)$ with respect to $g(t)$. Pulling back $g(t)$ by the diffeomorphisms φ_t, we obtain a solution

$$
\bar{g}(t) \doteqdot \varphi_t^* g(t) \tag{2.53}
$$

to the Ricci flow with $\bar{g}(0) = g_0$. Indeed,

$$
\begin{aligned}
\frac{\partial}{\partial t}\bar{g}(t) &= \varphi_t^*\left(\frac{\partial}{\partial t} g(t)\right) + \left.\frac{\partial}{\partial s}\right|_{s=0}\left(\varphi_{t+s}^* g(t)\right) \\
&= -2\,\mathrm{Rc}\left(\varphi_t^* g(t)\right) + \varphi_t^*\left(\mathcal{L}_{W(t)} g(t)\right) - \mathcal{L}_{\left(\varphi_t^{-1}\right)_* W(t)}\left(\varphi_t^* g(t)\right) \\
&= -2\,\mathrm{Rc}\left(\bar{g}(t)\right).
\end{aligned}
$$

Next we shall show that this solution is unique by considering the harmonic map heat flow. What is important to the study of Ricci flow is that the following discussion also holds for the most part in the noncompact case. This reduces the uniqueness problem on noncompact manifolds to an existence problem about harmonic maps. Recall that given a map $f : (M^n, g) \to (N^m, h)$, the **map Laplacian** of f is defined by

$$
\begin{aligned}
\left(\Delta_{g,h} f\right)^\gamma &= \Delta_g\left(f^\gamma\right) + g^{ij}\left(\Gamma(h)_{\alpha\beta}^\gamma \circ f\right)\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j} \\
&= g^{ij}\left(\frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f^\gamma}{\partial x^k} + \left(\Gamma(h)_{\alpha\beta}^\gamma \circ f\right)\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\right),
\end{aligned} \tag{2.54}
$$

where $f^\gamma \doteqdot y^\gamma \circ f$, and $\left\{x^i\right\}$ and $\{y^\alpha\}$ are coordinates on M and N, respectively. Note that $\Delta_{g,h} f \in C^\infty\left(f^* TN\right)$, where $f^*(TN) \to M$ is the pullback vector bundle of TN by f. In (2.54), $\Delta_g\left(f^\gamma\right)$ denotes the Laplacian with respect to g of the function f^γ. As a special case, if $M = N$ and f is the identity map and we choose the x and y coordinates to be the same, then

$$(\Delta_{g,h} \operatorname{id})^k = g^{ij}\left(-\Gamma(g)_{ij}^k + \Gamma(h)_{ij}^k\right).$$

Remark 2.45. *The derivative df of a map $f : M^n \to N^m$ is a section of the vector bundle $E \doteqdot T^*M \otimes f^*TN$. On E there is a natural metric and compatible connection $\nabla^{g,h}$ defined by the (dual of the) Riemannian metric g, its associated Levi-Civita connection on T^*M, the pull back by f of the metric h, and its associated Levi-Civita connection on TN. So $\nabla^{g,h} df$ is a section of the bundle $T^*M \otimes_S T^*M \otimes f^*TN$. The map Laplacian is the trace with respect to g of $\nabla^{g,h} df$*

$$\Delta_{g,h} f = \operatorname{tr}_g\left(\nabla^{g,h} df\right).$$

A map $f : (M^n, g) \to (N^m, h)$ is called a **harmonic map** if $\Delta_{g,h} f = 0$. Harmonic maps are critical points of the **harmonic map energy**

$$E(u) \doteqdot \int_M |du|^2 \, d\mu_g,$$

where

$$|du|^2 \doteqdot g^{ij} h_{\alpha\beta} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j}.$$

In the case where $N = \mathbb{R}$, a harmonic map is the same as a harmonic function and the harmonic map energy is the same as the Dirichlet energy (A.27). If M is 1-dimensional, then a harmonic map is the same as a constant speed geodesic.

Remark 2.46. *Given a diffeomorphism $f : (M^n, g) \to (N^n, h)$, the map Laplacian satisfies the following identity:*

$$(\Delta_{g,h} f)(x) = \left(\Delta_{(f^{-1})^* g, h} \operatorname{id}_N\right)(f(x)) \in C^\infty\left(f^* TN\right). \tag{2.55}$$

This corresponds to considering f as

$$(M^n, g) \xrightarrow{f} \left(N^n, \left(f^{-1}\right)^* g\right) \xrightarrow{\operatorname{id}} (N^n, h)$$

where the map on the left f is an isometry. More generally, if (P^n, k) and (N^m, h) are Riemannian manifolds, $F : P^n \to N^m$ is a map and $\varphi : M^n \to P^n$ is a diffeomorphism, then

$$(\Delta_{k,h} F)(\varphi(y)) = (\Delta_{\varphi^* k, h}(F \circ \varphi))(y), \tag{2.56}$$

which corresponds to

$$(M^n, \varphi^* k) \xrightarrow{\varphi} (P^n, k) \xrightarrow{F} (N^m, h).$$

Formula (2.55) is the special case of (2.56) where $n = m$, $P^n = N^m$, $F = \mathrm{id}_N$, $\varphi = f$ *and* $k = \left(f^{-1}\right)^* g$. *To prove (2.56), we compute using local coordinates* $\{x^i\}$ *on* M, $\{z^a\}$ *on* P, *and* $\{y^\alpha\}$ *on* N:

$$\begin{aligned}
&(\Delta_{\varphi^* k,h} (F \circ \varphi))^\ell \\
&= \Delta_{\varphi^* k} \left(x^\ell \circ F \circ \varphi\right) + (\varphi^* k)^{\alpha\beta} \left(\Gamma(h)_{ij}^\ell \circ F \circ \varphi\right) \frac{\partial (F \circ \varphi)^i}{\partial y^\alpha} \frac{\partial (F \circ \varphi)^j}{\partial y^\beta} \\
&= \left(\Delta_k \left(x^\ell \circ F\right)\right) \circ \varphi + \left(k^{ab} \circ \varphi\right) \left(\Gamma(h)_{ij}^\ell \circ F \circ \varphi\right) \left(\frac{\partial F^i}{\partial z^a} \circ \varphi\right) \left(\frac{\partial F^j}{\partial z^b} \circ \varphi\right) \\
&= (\Delta_{k,h} F)^\ell \circ \varphi,
\end{aligned}$$

where $\varphi^i = \varphi \circ x^i$, $\left(k^{ab}\right)$ *is the inverse of* $(k_{ab}) \doteqdot \left(k\left(\frac{\partial}{\partial z^a} \frac{\partial}{\partial z^b}\right)\right)$ *and since*

$$\Delta_{\varphi^* g} (u \circ \varphi) = \Delta_g u$$

for a function u.

If $\tilde{\Gamma}$ is the Levi-Civita connection of a metric $\tilde{g}$, then equation (2.52) is equivalent to

$$\frac{\partial}{\partial t} \varphi_t = g^{pq} \left(-\Gamma_{pq}^k + \tilde{\Gamma}_{pq}^k\right) \frac{\partial}{\partial x^k} = \Delta_{g,\tilde{g}} \,\mathrm{id} = \Delta_{\bar{g}(t),\tilde{g}} \varphi_t.$$

In other words, if $\bar{g}(t)$ is a solution to the Ricci flow and $\varphi_t : M^n \to M^n$ is a solution to the **harmonic map heat flow**

$$\frac{\partial}{\partial t} \varphi_t = \Delta_{\bar{g}(t),\tilde{g}} \varphi_t,$$

then the metrics $g(t) \doteqdot \left(\varphi_t^{-1}\right)^* \bar{g}(t)$ satisfy the Ricci-DeTurck flow.

Now we can finally return to the uniqueness of solutions of the Ricci flow on closed manifolds. We shall also indicate the issue which arises in the complete noncompact case. Let $g_1(t)$ and $g_2(t)$ be two complete solutions of the Ricci flow with $g_1(0) = g_2(0) = g_0$ on M^n. Suppose that we can show that there exist solutions $\varphi_1(t), \varphi_2(t) : M^n \to M^n$ to the harmonic map heat flow

$$\begin{aligned}
\frac{\partial}{\partial t} \varphi_i(t) &= \Delta_{g_i(t),g_0} \varphi_i(t), \\
\varphi_i(0) &= \mathrm{id}_M
\end{aligned}$$

on some short time interval $[0, \varepsilon)$. This is true when M is closed. Then the metrics

$$\hat{g}_i(t) = \varphi_i(t)_* g_i(t) = \left(\varphi_i(t)^{-1}\right)^* g_i(t)$$

are solutions of the Ricci-DeTurck flow with $\hat{g}_1(0) = \hat{g}_2(0) = g_0$. By the uniqueness theorem for the Ricci-DeTurck flow we have

$$\hat{g}_1(t) = \hat{g}_2(t) \doteqdot \hat{g}(t)$$

for $t \in [0, \varepsilon)$. We also have that $\varphi_1(t)$ and $\varphi_2(t)$ are solutions of the ODE (2.52)

$$\frac{d}{dt}(\varphi_i(t)) = -W(t) \circ \varphi_i(t),$$
$$\varphi_i(0) = \mathrm{id}_M,$$

where

$$W(t)^k \doteqdot \hat{g}(t)^{pq}\left(\Gamma(\hat{g}(t))^k_{pq} - \Gamma(g_0)^k_{pq}\right).$$

Hence $\varphi_1(t) = \varphi_2(t)$ for $t \in [0, \varepsilon)$. We conclude that $g_1(t) = g_2(t)$ for $t \in [0, \varepsilon)$. Thus the uniqueness problem reduces to the short time existence problem for the harmonic map heat flow with respect to a time-dependent domain metric. When M is compact, this follows from standard parabolic theory and we are done with the proof of short time existence and uniqueness.

We make some remarks about the complete noncompact case. For any C^∞ complete metric g_0 with bounded sectional curvature on a noncompact manifold M^n, a short time existence result for solutions to the Ricci flow was proved by W.-X. Shi [**501**].

Definition 2.47 (Bounded curvature solution)**.** *We say that a solution $g(t)$, $t \in \mathcal{I}$, of the Ricci flow has **bounded curvature** (or **bounded curvature on compact time intervals**) if on every compact subinterval $[a, b] \subset \mathcal{I}$ the Riemann curvature tensor is bounded. In particular, we do not assume the curvature bound is uniform in time on noncompact time intervals.*

Theorem 2.48 (W.-X. Shi)**.** *Given a complete metric g_0 with bounded sectional curvature on a noncompact manifold M^n, there exists a complete solution $g(t)$, $t \in [0, T)$, of the Ricci flow on M^n with $g(0) = g_0$ and bounded curvature such that either $\sup_{M \times [0,T)} |\mathrm{Rm}| = \infty$ or $T = \infty$.*

Problem 2.49 (Uniqueness of Ricci flow on noncompact manifolds)**.** *Under what conditions does uniqueness hold for complete solutions to the Ricci flow on noncompact manifolds?*

Remark 2.50. *On noncompact manifolds some technical issues arise in establishing uniform estimates. The paper of Chen and Zhu* [**127**] *considers the case of bounded curvature, whereas the paper of Tian and one of the authors addresses the uniqueness of the standard solution* [**388**]*. For the Kähler-Ricci flow, see* [**218**].

7. Reaction-diffusion equation for the curvature tensor

In this section we discuss the evolution equation satisfied by the Riemann curvature tensor.

7.1. Evolution equation for $R_{ijk\ell}$.

Lemma 2.51 (Evolution of Rm). *The* ***evolution of the Riemann curvature tensor*** *is given by*

$$\frac{\partial}{\partial t}R_{ijk\ell} = \Delta R_{ijk\ell} + 2\left(B_{ijk\ell} - B_{ij\ell k} + B_{ikj\ell} - B_{i\ell jk}\right) - \left(R_{ip}R_{pjk\ell} + R_{jp}R_{ipk\ell} + R_{kp}R_{ijp\ell} + R_{\ell p}R_{ijkp}\right), \tag{2.57}$$

where

$$B_{ijk\ell} \doteqdot -g^{pr}g^{qs}R_{ipjq}R_{kr\ell s} = -R_{pijq}R_{q\ell kp}. \tag{2.58}$$

This is a **reaction-diffusion equation**. In particular, the Laplacian term is the diffusion term whereas the rest forms the reaction term. Note that the reaction term on the RHS is essentially a quadratic in the Riemann curvature tensor (we are also using the metric to perform contractions). The reason for why there are four like terms for both the B and $\mathrm{Rc} * \mathrm{Rm}$ terms is that Rm satisfies the basic symmetry properties $R_{ijk\ell} = -R_{jik\ell} = R_{k\ell ij}$, etc. Although the formula looks complicated, as we shall see below, the derivation is actually straightforward.

Proof. From (2.27) we see that

$$\boxed{\frac{\partial}{\partial t}R_{ijk}^{\ell} = \nabla_i\left(\frac{\partial}{\partial t}\Gamma_{jk}^{\ell}\right) - \nabla_j\left(\frac{\partial}{\partial t}\Gamma_{ik}^{\ell}\right)}. \tag{2.59}$$

Then substituting (2.23) into this yields

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijk\ell} &= \frac{\partial}{\partial t}R_{ijk}^{\ell} + \left(\frac{\partial}{\partial t}g_{\ell p}\right)R_{ijk}^{p} \\ &= -\nabla_i\nabla_j R_{k\ell} - \nabla_i\nabla_k R_{j\ell} + \nabla_i\nabla_\ell R_{jk} + \nabla_j\nabla_i R_{k\ell} \\ &\quad + \nabla_j\nabla_k R_{i\ell} - \nabla_j\nabla_\ell R_{ik} - 2R_{\ell p}R_{ijkp}.\end{aligned} \tag{2.60}$$

This does not look anything like a heat-type equation. Fortunately the second Bianchi identity enables us to convert this into the desired heat-type equation (2.57). In particular, we start out with

$$\Delta R_{ijk\ell} = \nabla_p\nabla_p R_{ijk\ell} = -\nabla_p\nabla_i R_{jpk\ell} - \nabla_p\nabla_j R_{pik\ell}. \tag{2.61}$$

Next we commute the p index closer to Rm, i.e.,

$$\Delta R_{ijk\ell} = -\nabla_i\nabla_p R_{jpk\ell} - \nabla_j\nabla_p R_{pik\ell} + \mathrm{Rm} * \mathrm{Rm}, \tag{2.62}$$

where $\mathrm{Rm} * \mathrm{Rm}$ denotes a 4-tensor quadratic in Rm. Next we apply the second Bianchi identity again to get

$$\begin{aligned} \Delta R_{ijk\ell} &= \nabla_i \nabla_k R_{jp\ell p} + \nabla_i \nabla_\ell R_{jppk} + \nabla_j \nabla_k R_{pi\ell p} \\ &\quad + \nabla_j \nabla_\ell R_{pipk} + \mathrm{Rm} * \mathrm{Rm} \qquad (2.63) \\ &= -\nabla_i \nabla_k R_{j\ell} + \nabla_i \nabla_\ell R_{jk} + \nabla_j \nabla_k R_{i\ell} \\ &\quad - \nabla_j \nabla_\ell R_{ik} + \mathrm{Rm} * \mathrm{Rm}. \qquad (2.64) \end{aligned}$$

Comparing this with (2.60) yields a formula of the form

$$\frac{\partial}{\partial t} R_{ijk\ell} = \Delta R_{ijk\ell} + \mathrm{Rm} * \mathrm{Rm} + \mathrm{Rc} * \mathrm{Rm}.$$

A detailed version of this calculation yields (2.57); see Exercise 2.54 below. We also compute a more general version of equation (2.57) in detail in Chapter 11 when we consider the evolution of the space-time Riemann curvature tensor. □

Exercise 2.52. *Show that $B_{ijk\ell}$ satisfies the symmetries*

$$B_{ijk\ell} = B_{ji\ell k} = B_{k\ell ij}. \qquad (2.65)$$

Exercise 2.53. *Show that if (M^n, g) has constant sectional curvature, then*

$$B_{ijk\ell} = -\frac{R^2}{n^2 (n-1)^2} \left((n-2) g_{ij} g_{\ell k} + g_{ik} g_{\ell j} \right).$$

Exercise 2.54. *Give a complete proof of (2.57). In particular, show that* $\mathrm{Rm} * \mathrm{Rm}$ *in (2.62), (2.63) and (2.64) are given by*

$$\begin{aligned} \mathrm{Rm} * \mathrm{Rm} &= R_{pijq} R_{qpk\ell} - R_{iq} R_{jqk\ell} + R_{pikq} R_{jpq\ell} + R_{pi\ell q} R_{jpkq} \\ &\quad - R_{pjiq} R_{qpk\ell} + R_{jq} R_{iqk\ell} - R_{pjkq} R_{ipq\ell} - R_{pj\ell q} R_{ipkq}. \end{aligned}$$

Then use the definition of $B_{ijk\ell}$ and the first Bianchi identities in a suitable way together with the fact that two terms in (2.60), i.e.,

$$-\nabla_i \nabla_j R_{k\ell} + \nabla_j \nabla_i R_{k\ell},$$

yield a quadratic curvature term.

Exercise 2.55 (Variation of Rm)**.** *Show that if $\frac{\partial}{\partial s} g_{ij} = v_{ij}$, then*

$$\frac{\partial}{\partial s} R_{ijk}^{\ell} = \frac{1}{2} g^{\ell p} \left\{ \begin{array}{c} \nabla_i \nabla_j v_{kp} + \nabla_i \nabla_k v_{jp} - \nabla_i \nabla_p v_{jk} \\ -\nabla_j \nabla_i v_{kp} - \nabla_j \nabla_k v_{ip} + \nabla_j \nabla_p v_{ik} \end{array} \right\} \qquad (2.66)$$

$$\begin{aligned} &= \frac{1}{2} g^{\ell p} \left(\nabla_i \nabla_k v_{jp} - \nabla_i \nabla_p v_{jk} - \nabla_j \nabla_k v_{ip} + \nabla_j \nabla_p v_{ik} \right) \qquad (2.67) \\ &\quad - \frac{1}{2} g^{\ell p} \left(R_{ijkq} v_{qp} + R_{ijpq} v_{kq} \right). \end{aligned}$$

7.2. Uhlenbeck's trick. The Riemann curvature tensor may be considered as an operator

$$\mathrm{Rm} : \Lambda^2 M^n \to \Lambda^2 M^n$$

defined by

$$\mathrm{Rm}\,(\alpha)_{ij} \doteqdot R_{ijk\ell}\alpha_{\ell k}. \tag{2.68}$$

Definition 2.56. *We call* Rm *the* ***Riemann curvature operator*** *(or simply* ***curvature operator****). We say that* (M^n, g) *has* ***positive (nonnegative) curvature operator*** *if the eigenvalues of* Rm *are positive (nonnegative), and we denote this by* $\mathrm{Rm} > 0$ *(*$\mathrm{Rm} \geq 0$*).*

We can define the square of Rm by $\mathrm{Rm}^2 = \mathrm{Rm} \circ \mathrm{Rm} : \Lambda^2 M^n \to \Lambda^2 M^n$. It is interesting that there is another quadratic, like a square, which is relevant to the evolution of Rm. To describe this, we recall the Lie algebra structure on $\Lambda^2 M^n$ defined by $[U,V]_{ij} \doteqdot g^{k\ell}(U_{ik}V_{\ell j} - V_{ik}U_{\ell j})$ for $U, V \in \Lambda^2 M^n$. Then $\Lambda^2 M^n \cong \mathfrak{so}(n)$. Choose a basis $\{\varphi^\alpha\}$ of $\Lambda^2 M^n$ and let $C_\gamma^{\alpha\beta}$ denote the structure constants defined by $\left[\varphi^\alpha, \varphi^\beta\right] \doteqdot \sum_\gamma C_\gamma^{\alpha\beta}\varphi^\gamma$. We define the **Lie algebra square** $\mathrm{Rm}^\# : \Lambda^2 M^n \to \Lambda^2 M^n$ by

$$(\mathrm{Rm}^\#)_{\alpha\beta} \doteqdot C_\alpha^{\gamma\delta} C_\beta^{\varepsilon\zeta}\,\mathrm{Rm}_{\gamma\varepsilon}\,\mathrm{Rm}_{\delta\zeta}. \tag{2.69}$$

Note that if we choose $\{\varphi^\alpha\}$ so that Rm is diagonal, then for any 2-form η, we have

$$(\mathrm{Rm}^\#)_{\alpha\beta}\eta^\alpha\eta^\beta = \left(C_\alpha^{\gamma\delta}\eta^\alpha\right)^2 \mathrm{Rm}_{\gamma\gamma}\,\mathrm{Rm}_{\delta\delta}.$$

Hence, we see

Lemma 2.57. *If* $\mathrm{Rm} \geq 0$, *then* $\mathrm{Rm}^\# \geq 0$.

We have the following nice form for the evolution equation for Rm.

Lemma 2.58 (Evolution of the curvature operator)**.**

$$\boxed{\frac{\partial}{\partial t}\mathrm{Rm} = \Delta\,\mathrm{Rm} + \mathrm{Rm}^2 + \mathrm{Rm}^\#}\,. \tag{2.70}$$

Actually we have cheated a little bit; the actual equations include additional terms of the form $\mathrm{Rc} * \mathrm{Rm}$. By using what is known as **Uhlenbeck's trick**, one obtains (2.70). The idea is to choose a vector bundle $E \to M$ isomorphic to the tangent bundle $TM \to M$ and a bundle isomorphism $\iota_0 : E \to TM$. Pulling back the initial metric, we get a bundle metric $h \doteqdot \iota_0^*(g_0)$ on E. By using the metric $g(t)$ to identify TM and T^*M, we may consider the Ricci tensor as a bundle map $\mathrm{Rc}(t) : TM \to TM$. We define a 1-parameter family of bundle isomorphisms $\iota(t) : E \to TM$ by the ODE

$$\frac{d}{dt}\iota(t) = \mathrm{Rc}(t) \circ \iota(t), \quad \iota(0) = \iota_0. \tag{2.71}$$

Let $\{e_a\}$ be a local basis of sections of E and let $h_{ab} \doteqdot h(e_a, e_b)$. We compute that

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\iota(t)^* g(t)\right]_{ab} \\
&= \frac{\partial}{\partial t}\left[\iota(t)_a^i \, \iota(t)_b^j \, g_{ij}(t)\right] \\
&= \left[\frac{\partial}{\partial t}\iota(t)\right]_a^i \iota(t)_b^j \, g_{ij}(t) + \iota(t)_a^i \left[\frac{\partial}{\partial t}\iota(t)\right]_b^j g_{ij}(t) + \iota(t)_a^i \, \iota(t)_b^j \frac{\partial}{\partial t} g_{ij}(t) \\
&= \mathrm{Rc}(t)_k^i \, \iota(t)_a^k \, \iota(t)_b^j \, g_{ij}(t) + \mathrm{Rc}(t)_k^j \, \iota(t)_a^i \, \iota(t)_b^k \, g_{ij}(t) - 2\iota(t)_a^i \, \iota(t)_b^j \, R_{ij}(t) \\
&= 0.
\end{aligned}$$

Hence $h = \iota(t)^* g(t)$ is independent of t. Using the bundle isomorphisms $\iota(t)$, we can pull back tensors on M. In particular, we consider $\iota(t)^*\mathrm{Rm}[g(t)]$, which is a section of $\Lambda^2 E^* \otimes_S \Lambda^2 E^*$. It is this tensor which satisfies (2.70), which is equivalent to

$$\left(\frac{\partial}{\partial t} - \Delta\right) R_{abcd} = 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}) \tag{2.72}$$

where R_{abcd} are the components of $\iota(t)^* \mathrm{Rm}[g(t)]$, and

$$B_{abcd} \doteqdot -R_{aebf} R_{cedf}. \tag{2.73}$$

To see this, first recall from (2.57) that

$$\begin{aligned}
\frac{\partial}{\partial t} R_{ijk\ell} = \Delta R_{ijk\ell} &+ 2\left(B_{ijk\ell} - B_{ij\ell k} + B_{ikj\ell} - B_{i\ell jk}\right) \\
&- \left(R_{ip}R_{pjk\ell} + R_{jp}R_{ipk\ell} + R_{kp}R_{ijp\ell} + R_{\ell p}R_{ijkp}\right).
\end{aligned}$$

Now

$$\begin{aligned}
\frac{\partial}{\partial t} R_{abcd} &= \frac{\partial}{\partial t}\left(\iota_a^i \iota_b^j \iota_c^k \iota_d^\ell R_{ijk\ell}\right) \\
&= \iota_a^i \iota_b^j \iota_c^k \iota_d^\ell \frac{\partial}{\partial t} R_{ijk\ell} + R_a^p R_{pbcd} + R_b^p R_{apcd} + R_c^p R_{abpd} + R_d^p R_{abcp}
\end{aligned}$$

and (2.72) follows from combining the above two equations (see also [**277**] or [**163**], p. 180ff).

7.3. The curvature operator in dimension 3. Now we get to what is nice about dimension 3. Here, the Lie algebra structure of $\mathfrak{so}(3) \cong \mathbb{R}^3$ is simple, namely $[U, V] = U \times V$ is the cross product. This implies $\mathrm{Rm}^{\#}$ is the adjoint of Rm. If we diagonalize, i.e.,

$$\mathrm{Rm} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix},$$

then $\mathrm{Rm}^2 + \mathrm{Rm}^{\#}$ is also diagonal and

$$\mathrm{Rm}^2 + \mathrm{Rm}^{\#} = \begin{pmatrix} \lambda^2 + \mu\nu & 0 & 0 \\ 0 & \mu^2 + \lambda\nu & 0 \\ 0 & 0 & \nu^2 + \lambda\mu \end{pmatrix}. \tag{2.74}$$

So we understand the evolution equation for Rm in dimension 3 pretty well.

Exercise 2.59. *Derive (2.74) from (2.69).*

HINT: Use (7.14).

Exercise 2.60 ($n = 3$, principal sectional curvatures)**.** *When* $\dim M = 3$, *show that at each point there exists an orthonormal frame* $\{e_1, e_2, e_3\}$ *such that the 2-forms* $\varphi_1 \doteqdot e_2^* \wedge e_3^*$, $\varphi_2 \doteqdot e_3^* \wedge e_1^*$, $\varphi_3 \doteqdot e_1^* \wedge e_2^*$ *are eigenvectors of* Rm. *In this case* $\lambda = 2\langle \mathrm{Rm}(e_2, e_3) e_3, e_2 \rangle$, $\mu = 2\langle \mathrm{Rm}(e_1, e_3) e_3, e_1 \rangle$, $\nu = 2\langle \mathrm{Rm}(e_1, e_2) e_2, e_1 \rangle$ *are twice the sectional curvatures.*

Exercise 2.61. *Show that if* g *has constant sectional curvature, then* $\mathrm{Rm} \equiv \frac{2R}{n(n-1)} \mathrm{Id}_{\Lambda^2}$.

Exercise 2.62. *By examining the evolution equation for the Einstein tensor* $\frac{1}{2}Rg_{ij} - R_{ij}$ *(whose eigenvalues are the principal sectional curvatures), verify (2.70) where* $\mathrm{Rm}^2 + \mathrm{Rm}^{\#}$ *is given by (2.74) when* $n = 3$. *(Compare with Exercise 3.8.) Note that one needs to apply Uhlenbeck's trick to the evolution equation for* $\frac{1}{2}Rg - \mathrm{Rc}$ *to get the exact correspondence.*

8. Notes and commentary

Section 1. For more details about collapsing manifolds and their role in Ricci flow, see Cheeger and Gromov [**115**], [**116**], Hamilton [**291**], Perelman [**452**], [**453**], and Shioya and Yamaguchi [**504**].

Section 6. For a discussion of the relation between DeTurck's trick and the harmonic map heat flow, see Schoen [**479**] and §6 of Hamilton [**287**] (there is also an exposition in Chapter 3, §4 of [**163**]). Let $S^2T^*M^n$ denote the bundle of symmetric covariant 2-tensors and let

$$\sigma \doteqdot \sigma D(-2\,\mathrm{Rc})(\zeta) : S^2T^*M^n \to S^2T^*M^n,$$

where $\zeta \in T^*M$, denote the **symbol** of the linearization of the Ricci tensor as a function of the metric. Recall from (2.31) that the linearization of $-2\,\mathrm{Rc}$ is given by

$$D\left(-2\,\mathrm{Rc}\right)(v)_{ij} = \Delta_L v_{ij} + \nabla_i \nabla_j V - \nabla_i \left(\operatorname{div} v\right)_j - \nabla_j \left(\operatorname{div} v\right)_i ,$$

where $V \doteqdot g^{ij} v_{ij}$. The symbol is obtained by replacing ∇ by $\zeta \in T^*M$ in the highest- (i.e., second-) order terms. Thus

$$\sigma(v)_{ij} = \left[\sigma D\left(-2\,\mathrm{Rc}\right)(\zeta)\right](v)_{ij} = |\zeta|^2 v_{ij} + \zeta_i \zeta_j V - \zeta_i \zeta_k v_{kj} - \zeta_j \zeta_k v_{ki}.$$

Assuming that $\zeta_1 = 1$ and $\zeta_i = 0$ for $i \neq 1$, one sees that for a symmetric 2-tensor v

$$\begin{aligned} \sigma(v)_{ij} &= v_{ij} && \text{if } i,j \neq 1, \\ \sigma(v)_{1j} &= 0 && \text{if } j \neq 1, \\ \sigma(v)_{11} &= \sum_{k=2}^{n} v_{kk} \end{aligned} \tag{2.75}$$

(see also Hamilton [**275**]). One checks that σ is given by a nonnegative $N \times N$ matrix, where $N = n(n+1)/2$, and its kernel is the n-dimensional subspace given by

$$\ker \sigma D\left(-2\,\mathrm{Rc}\right)(\zeta) = \left\{ v : v_{ij} = 0 \text{ for } i,j \neq 1 \text{ and } \sum_{k=2}^{n} v_{kk} = 0 \right\}.$$

This kernel is due exactly to the diffeomorphism invariance of the operator $g \mapsto -2\,\mathrm{Rc}$ which we see as follows. Define the linear **Bianchi operator**

$$B_g : C^\infty\left(S^2 T^* M^n\right) \to C^\infty\left(T^* M^n\right)$$

by

$$B_g(h)_k \doteqdot g^{ij}\left(\nabla_i h_{jk} - \frac{1}{2}\nabla_k h_{ij}\right)$$

so that $B_g(-2\,\mathrm{Rc}) = 0$. One finds that

$$K \doteqdot \ker \sigma B_g(\zeta) = \operatorname{image} \sigma D\left(-2\,\mathrm{Rc}\right)(\zeta) \subset S^2 T^* M^n$$

is equal to

$$K = \left\{ v : v_{1j} = 0 \text{ for } j \neq 1 \text{ and } v_{11} = \sum_{k=2}^{n} v_{kk} \right\}.$$

From (2.75) we see that

$$\sigma|_K(v) = |\zeta|^2 v$$

for any $\zeta \in T^* M^n$. We note that when $n = 3$, $\sigma = \sigma D(-2\,\mathrm{Rc})(\zeta)$ is given by

$$\sigma \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{22} \\ v_{33} \\ v_{23} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{22} \\ v_{33} \\ v_{23} \end{pmatrix}.$$

See Buckland [**62**] for comparison with the cross curvature flow (see also Section 5 of Appendix B for a short discussion of the cross curvature flow).

Section 7. A Riemannian manifold has **2-positive curvature operator** if

$$\lambda_1(\mathrm{Rm}) + \lambda_2(\mathrm{Rm}) > 0.$$

That is, the sum of the lowest two eigenvalues of Rm is positive at every point. H. Chen [**128**] has shown that if $(M^n, g(0))$ is a closed Riemannian manifold with 2-positive curvature operator, then under the Ricci flow $g(t)$ has 2-positive curvature operator for all $t > 0$.