Preface

This book is a natural continuation of the author's earlier book *Elements of Combinatorial and Differential Topology* (American Mathematical Society, Providence, RI, 2006), which we refer to as Part I here. (A corrected Russian version of Part I is available at `http://www.mccme.ru/prasolov`.)

In Chapter 1, we define simplicial homology and cohomology and give many examples of their calculations and applications. At this point, the book diverges from standard modern courses in algebraic topology, which usually begin with introducing singular homology. Simplicial homology has a simpler and more natural definition. Moreover, it is simplicial homology that is usually involved in calculations. For this reason, we introduce singular homology near the end of the book and use it only when it is indeed necessary, mainly in studying topological manifolds.

Homology and cohomology groups with arbitrary coefficients are expressed in terms of integral homology by means of the functors Tor and Ext. The properties of these functors are very important for homology theory, so we discuss them in detail.

We first prove the Poincaré duality theorem for simplicial (co)homology. This proof applies only to smooth (to be more precise, triangulable) manifolds. There is no triangularization theorem for topological manifolds, and the proof of the Poincaré duality theorem for them uses, of necessity, singular (co)homology. This proof is given in Chapter 4; it is very cumbersome.

Chapter 2 considers an important algebraic structure on cohomology, the cup product of Kolmogorov and Alexander. It is particularly useful in the case of manifolds. Multiplication in cohomology is related to many topological invariants of manifolds, such as the intersection form and signature.

One possible approach to constructing multiplication in cohomology is based on a theorem of Künneth, which expresses the (co)homology of $X \times Y$ in terms of those of X and Y and is of independent interest.

Chapter 3 is devoted to various applications of (simplicial) homology and cohomology. Many of them are related to obstruction theory. One of such applications is the construction of the characteristic classes of vector bundles. Other approaches to constructing characteristic classes (namely, the universal bundle and axiomatic approaches) are also discussed. Then, we consider the (co)homological properties of spaces with actions of groups; we construct transfers and Smith's exact sequences. We conclude the chapter with constructing Steenrod squares, which generalize multiplication in cohomology.

In Chapter 4, we define singular (co)homology and describe some of its applications; in particular, we obtain certain properties of characteristic classes. (Technically, it is more convenient to prove them by using singular cohomology, although the assertions themselves can be stated for simplicial cohomology.)

Chapter 5 considers yet another approach to constructing cohomology theory, namely, Čech cohomology and de Rham cohomology, which are closely related to each other. For the de Rham cohomology, we prove the Poincaré duality theorem. Then, we carry over the construction of de Rham, which was originally introduced for smooth manifolds, to arbitrary simplicial complexes.

The final Chapter 6 is devoted to various applications of homology theory, largely to the topology of manifolds. We begin with a detailed account of the Alexander polynomials, which we construct by using the homology of cyclic coverings; the Arf invariant is also considered. Then, we prove the strong Whitney embedding theorem. We also give a formula for calculating the Chern classes of complete intersections and discuss some homological properties of Lie groups and H-spaces.

The book contains many problems (with solutions) and exercises. The problems are based on the materials of topology seminars for second-year students held by the author at the Independent University of Moscow in 2003.

The basic notation, as well as theorems and other assertions, of Part I are mostly used without explanations; in some cases, we give references to the corresponding places in Part I.