General introduction

This book provides an elementary exposition intended for students who have completed four years' study of university level mathematics. A knowledge of the elements of functional analysis, Fourier analysis and distribution theory (including, in particular, Fourier analysis in $\mathcal{S}$ and $\mathcal{S}'$) is assumed. Chapter 0 contains a reminder of the notation, concepts and main results used in the remainder of the book (with references). On the other hand, no knowledge of partial differential equations is needed, although it will be beneficial to have received an initiation to the topic.

The book stems from a course on 'Pseudo-differential operators and the Nash–Moser theorem', presented at the Ecole Normale Supérieure (ENS) from October 1986 onwards, to second-year students studying for the degree of Master of Fundamental and Applied Mathematics and Computer Science.

Although the topics covered largely form the subject of research literature, we have striven to avoid any scholarly discussions, 'veiled references' and sibylline allusions, which might open chasms beneath the reader's footsteps. A particular presentation of the subject is selected and developed in each chapter: the commentary at the end of each chapter indicates the sources, differing approaches, certain current extensions, and the connections between the topics handled.

Finally, we have assembled numerous exercises, divided into two classes. Elementary exercises are intended to help readers assimilate the course and monitor their progress. Other more complex exercises, marked with an asterisk ($*$), present recent developments which have sometimes only been published in journal articles: we crave their authors' forgiveness for this simplification! These exercises, unlike those in certain famous treatises, can be effectively solved by real students, as experience of the teaching at ENS has shown.

It was our wish that this text should also be useful to researchers as a simple and *self-contained* introduction to subjects with which they are unfamiliar.

The dual purpose of these notes led us to keep them short, sometimes at the expense of a certain denseness of the text (which we believe is essentially accessible to motivated students). In particular, we had in mind our many colleagues in 'applied mathematics' who wish to use the Nash–Moser theorem in their research or to keep themselves up to date on microlocal analysis, without delving into the arcana of the specialist literature: they will be able to read the desired chapters independently of each other.

The choice of the material presented is a matter of personal taste and of the fields of research of the authors who, incidentally, believe that certain difficult (nonlinear) problems cannot be solved without a sufficient knowledge of pseudo-differential operators.

The authors are indebted to numerous mathematicians (cited in the commentaries) who have inspired them to present the subjects dealt with, and, in particular, to L. Hörmander, to whom the mathematical contents of Chapter I and Section III.C are largely due. The Bibliography at the end of the book indicates the sources used.

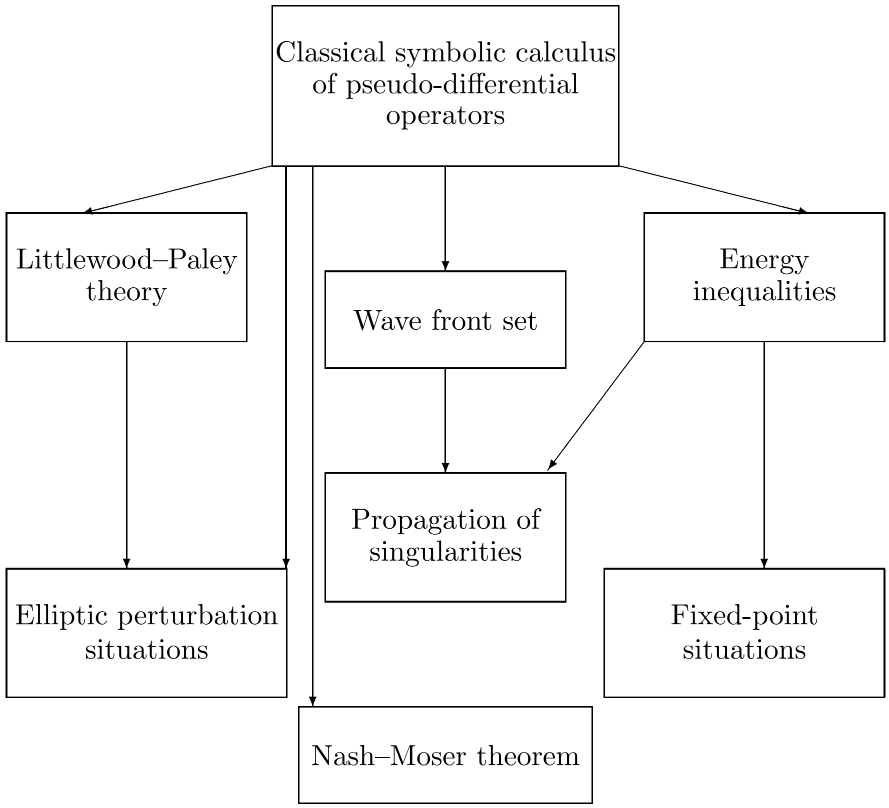
While presenting important concepts which are the true starting points for numerous recent developments, we have sought to end up with real theorems: microlocal elliptic regularity; propagation of singularities; existence of solutions of quasilinear hyperbolic systems; existence of isometric embeddings; the Nash–Moser theorem. The plan of the book is as follows.

In Chapter I we present the 'minimal' theory of pseudo-differential operators, in a global context (on $\mathbb{R}^n$), which turns out to be very nice in practice. The main points here are the notion of the symbol, the symbolic calculus for operators, the action in Sobolev spaces and the invariance under change of coordinates. The text presents only a few concrete applications and the most technical proofs are brought together in the appendix, in order to enable the reader to obtain an overall view of the subject. The exercises in Chapter I, which are particularly numerous, provide an introduction to a number of variants of the theory proposed and present several applications, notably to the analysis on compact manifolds.

Chapter II brings together three themes. Section A presents the Littlewood–Paley theory of 'dyadic decomposition' of distributions: this systematizes the natural division of the space of frequencies ξ according to their size $|\xi|$, associated with the classical symbolic calculus of Chapter I. This very simple theory allows one to rapidly obtain interesting properties of composite functions in Sobolev and Hölder spaces. Section B presents the concept of the wave front set and its links with pseudo-differential operators: this time it is a matter of the conical division of the space of frequencies ξ according to their directions $\xi \in S^{n-1}$, associated with the classical symbol homogeneities. Finally, Section C deals with hyperbolic energy inequalities for which pseudo-differential operators turn out to be an effective tool. Thus, Chapter II serves to present very useful applications of the 'dry theory' of Chapter I, while preparing the material and the concepts which will be needed in Chapter III.

The final chapter discusses certain problems of a nonlinear nature which arise in geometry or in analysis and which may be reduced to perturbation problems. The plan of this chapter reflects the various situations which one may encounter: 'elliptic' situations in which the usual Banach implicit function theorem suffices; 'fixed-point' situations, such as one often finds in nonlinear hyperbolic problems or again in the isometric embedding problem; and, finally, situations where the 'loss of derivatives' is too great and a Nash–Moser technique has to be used. The Nash–Moser theorem relies completely on the acquisition of *a priori* 'tame' inequalities; the reader who is already familiar with *a priori* inequalities (presented in Chapter I and Section III.C) will grasp the concept of 'tame' estimates through its clear link with Littlewood–Paley theory and the paradifferential calculus of J.-M. Bony (Section II.A).

This establishes the underlying cohesion of this book, which can be schematized in the accompanying diagram.



In this spirit, we were recently very happy to learn of the work of L. Hörmander [H9], explaining the links between pseudo-differential and paradifferential operators, fixed-point methods and the Nash–Moser theorem.

Finally, we are grateful to G. Ben Arous and J.B. Bost for their kind and valuable suggestions.