Introduction

Local cohomology was invented by Grothendieck to prove Lefschetz-type theorems in algebraic geometry. This book seeks to provide an introduction to the subject which takes cognizance of the breadth of its interactions with other areas of mathematics. Connections are drawn to topological, geometric, combinatorial, and computational themes. The lectures start with basic notions in commutative algebra, leading up to local cohomology and its applications. They cover topics such as the number of defining equations of algebraic sets, connectedness properties of algebraic sets, connections to sheaf cohomology and to de Rham cohomology, Gröbner bases in the commutative setting as well as for D-modules, the Frobenius morphism and characteristic p methods, finiteness properties of local cohomology modules, semigroup rings and polyhedral geometry, and hypergeometric systems arising from semigroups.

The subject can be introduced from various perspectives. We start from an algebraic one, where the definition is elementary: given an ideal $\mathfrak{a}$ in a Noetherian commutative ring, for each module consider the submodule of elements annihilated by some power of $\mathfrak{a}$. This operation is not exact, in the sense of homological algebra, and local cohomology measures the failure of exactness. This is a simple-minded algebraic construction, yet it results in a theory rich with striking applications and unexpected interactions.

On the surface, the methods and results of local cohomology concern the algebra of ideals and modules. Viewing rings as functions on spaces, however, local cohomology lends itself to geometric and topological interpretations. From this perspective, local cohomology is sheaf cohomology with support on a closed set. The interplay between invariants of closed sets and the topology of their complements is realized as an interplay between local cohomology supported on a closed set and the de Rham cohomology of its complement. Grothendieck's local duality theorem, which is inspired by and extends Serre duality on projective varieties, is an outstanding example of this phenomenon.

Local cohomology is connected to differentials in another way: the only known algorithms for computing local cohomology in characteristic zero employ rings of differential operators. This connects the subject with the study of Weyl algebras and holonomic modules. On the other hand, the combinatorics of local cohomology in the context of semigroups turns out to be the key to understanding certain systems of differential equations.

Prerequisites. The lectures are designed to be accessible to students with a first course in commutative algebra or algebraic geometry, and in point-set topology. We take for granted familiarity with algebraic constructions such as localizations, tensor products, exterior algebras, and topological notions such as homology and fundamental groups. Some material is reviewed in the lectures, such as dimension theory for commutative rings and Čech cohomology from topology. The main body of the text assumes knowledge of the structure theory of injective modules and resolutions; these topics are often omitted from introductory courses, so they are treated in the Appendix.

Local cohomology is best understood with a mix of algebraic and geometric perspectives. However, while prior exposure to algebraic geometry and sheaf theory is helpful, it is not strictly necessary for reading this book. The same is true of homological algebra: although we assume some comfort with categories and functors, the rest can be picked up along the way either from references provided, or from the twenty-four lectures themselves. For example, concepts such as resolutions, limits, and derived functors are covered as part and parcel of local cohomology.

Suggested reading plan. This book could be used as a text for a graduate course; in fact, the exposition is directly based on twenty-four hours of lectures in a summer school at Snowbird (see the Preface). That being said, it is unlikely that a semester-long course would cover all of the topics; indeed, no single one of us would choose to cover all the material, were we to teach a course based on this book. For this reason, we outline possible choices of material to be covered in, say, a semester-long course on local cohomology.

Lectures 1, 2, 3, 6, 7, 8, and 11 are fundamental, covering the geometry, sheaf theory, and homological algebra leading to the definition and alternative characterizations of local cohomology. Many readers will have seen enough of direct and inverse limits to warrant skimming Lecture 4 on their first pass, and referring back to it when necessary.

A course focusing on commutative algebra could include also Lectures 9, 10, 12, and 13. An in-depth treatment in the same direction would follow up with Lectures 14, 15, 18, 21, and 22.

For those interested mainly in the algebraic geometry aspects, Lectures 12, 13, and 18 would be of interest, while Lectures 18 and 19 are intended to describe connections to topology.

For applications to combinatorics, we recommend that the core material be followed up with Lectures 5, 16, 20, and 24, although Lecture 24 also draws on Lectures 17 and 23. Much of the combinatorial material—particularly the polyhedral parts—needs little more than linear algebra and some simplicial topology.

From a computational perspective, Lectures 5, 17, and 23 give a quick treatment of Gröbner bases and related algorithms. These lectures can also serve as an introduction to the theory of Weyl algebras and D-modules.

A feature that should make the book more appealing as a text is that there are exercises peppered throughout. Some are routine verifications of facts used later, some are routine verifications of facts not used later, and others are not routine. None are open problems, as far as we know. To impart a more comprehensive feel for the depth and breadth of the subject, we occasionally include landmark theorems with references but no proof. Results whose proofs are omitted are identified by the end-of-proof symbol $\square$ at the conclusion of the statement.

There are a number of topics that we have not discussed: Grothendieck's parafactoriality theorem, which was at the origins of local cohomology; Castelnuovo-Mumford regularity; the contributions of Lipman and others to the theory of residues; vanishing theorems of Huneke and Lyubeznik, and their recent work on local cohomology of the absolute integral closure. Among the applications, a noteworthy absence is the use of local cohomology by Benson, Carlson, Dwyer, Greenlees, Rickard, and others in representation theory and algebraic topology. Moreover, local cohomology remains a topic of active research, with new applications and new points of view. There have been a number of spectacular developments in the two years that it has taken us to complete this book. In this sense, the book is already dated.

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