Preface

The theory of quadratic forms has a long and glorious history: launched in ancient Babylonia between 1900 and 1600 BC, taken up again by Brahmagupta in the Seventh Century, and then—another thousand years later—by the great genius Fermat, followed by a succession of extraordinary mathematicians, including Euler, Lagrange, and Gauss, who brought the subject closer to its modern form. The work of Minkowski in the late Nineteenth Century, coupled with the extension of his work by Hasse in the early Twentieth Century, led to a great broadening and deepening of the theory that has served as the foundation for an enormous amount of research that continues today.

Though the roots of the subject are in number theory of the purest sort, the last third of the Twentieth Century brought with it new links of quadratic forms to group theory, topology, and—most recently—to cryptography and coding theory. So there are now many members of the mathematical community who are not fundamentally number theorists but who find themselves needing to learn about quadratic forms, especially over the integers. There is thus a need for an accessible introductory book on quadratic forms that can lead readers into the subject without demanding a heavy background in algebraic number theory or previous exposure to a lot of sophisticated algebraic machinery. My hope is that this is such a book.

One of the special attributes of number theory that distinguishes it from most other areas of mathematics is that soon after a subject is introduced and objects are defined, questions arise that can be understood even by a newcomer to the subject, although the answers may have eluded the experts for centuries. Even though this is an introductory book, it contains a substantial amount of material that has not yet appeared in book form, and the reader will be exposed to topics of current research interest. I will be happy if the readers find themselves wanting to pursue some aspects of the subject in more detail than this book can provide; accordingly, I will offer some references to the literature and recommendations for further study.

Before 1937, quadratic forms were treated primarily as homogeneous polynomials of degree 2 acted on by transformations that could change a given quadratic form into certain other ones. (And a fundamental question was: into *which* other ones?) But a pioneering paper by Witt in 1937 brought a more geometric flavor to the subject, putting it on the border of linear algebra and number theory—roughly speaking, a theory of generalized inner products on modules. Our coefficient ring of interest will most often be the ring $\mathbb{Z}$ of rational integers, though we will also give special attention to the polynomial rings $\mathbb{F}_q[x]$. (Here $\mathbb{F}_q$ denotes a finite field with q elements.) We will see that before we can effectively explore quadratic forms over a given domain R, we may need to extend R, perhaps in many ways, to larger rings. The extended domains (specifically, the p-adic number fields, their rings of integers, and their function-field analogues) may possess complications of their own that require clarification before we can consider quadratic forms over them; but once we have achieved that clarification, we may find that quadratic forms over those extensions are far more tractable than over R. When that happens, the trick is to then bring that information down to R and apply it to the original forms.

This book has evolved from lecture notes for introductory graduate courses on quadratic forms I have taught many times at the University of California, Santa Barbara, and once at Dartmouth College. Typically these courses have been populated by second-year graduate students who have already had a basic course in algebraic structures, and this is the primary audience I have had in mind during the writing process. But in fact the book should be readable by anyone with a strong undergraduate background in linear and abstract algebra who has also seen the construction of the real numbers from the rationals.

Naturally the contents of this book have been shaped by my own interests, experience, and tastes, and I have no doubt that some mathematicians will lament the absence of one or more of their favorite topics in the theory of quadratic forms. But I hope that their concerns will be eased by seeing in these pages some new perspectives—and occasionally something completely new—and that where the material is familiar they will experience the joy of revisiting old friends.

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