Introduction

As indicated in the title, this volume is concerned primarily with noncommutative algebraic structures, having grown from a course introducing complex representations of finite groups via the structure of group algebras and their modules. Our emphasis is on algebras, although we also treat some major classes of finite and infinite groups. Since this volume was conceived as a continuation of Volume 1 (*Graduate Algebra: Commutative View*, Graduate Studies in Mathematics, volume 73), the numeration of chapters starts with Chapter 13, Part IV, and we use the basics of rings and modules developed in Part I of Volume 1 (Chapters 1–3). Nevertheless, Chapters 13–15 and 18 can largely be read independently of Volume 1.

In the last one hundred years there has been a vast literature in noncommutative theory, and our goal here has been to find as much of a common framework as possible. Much of the theory can be cast in terms of representations into matrix algebras, which is our major theme, dominating our treatment of algebras, groups, Lie algebras, and Hopf algebras. A secondary theme is the description of algebraic structures in terms of generators and relations, pursued in the appendices of Chapter 17, and leading to a discussion of free structures, growth, word problems, and Zelmanov's solution of the Restricted Burnside Problem.

One main divergence of noncommutative theory from commutative theory is that left ideals need not be ideals. Thus, the important notion of "principal ideal" from commutative theory becomes cumbersome; whereas the principal left ideal Ra is described concisely, the smallest ideal of a noncommutative ring QR containing an element a includes all elements of the form

$$r_{1,1}ar_{1,2} + \cdots + r_{m,1}ar_{m,2}, \quad \forall r_{i,1}, r_{i,2}, \in R,$$

where m can be arbitrarily large. This forces us to be careful in distinguishing "left" (or "right") properties from two-sided properties, and leads us to rely heavily on modules.

There are many approaches to structure theory. We have tried to keep our proofs as basic as possible, while at the same time attempting to appeal to a wider audience. Thus, projective modules (Chapter 25) are introduced relatively late in this volume.

The exposition is largely self-contained. Part IV requires basic module theory, especially composition series (Chapter 3 of Volume 1). Chapter 16 draws on material about localization and Noetherian rings from Chapters 8 and 9 of Volume 1. Chapter 17, which goes off in a different direction, requires some material (mostly group theory) given in the prerequisites of this volume. Appendix 17B generalizes the theory of Gröbner bases from Appendix 7B of Volume 1. Chapter 18 has applications to field theory (Chapter 4 of Volume 1).

Parts V and VI occasionally refer to results from Chapters 4, 8, and 10 of Volume 1. At times, we utilize quadratic forms (Appendix 0A) and, occasionally, derivations (Appendix 6B). The end of Chapter 24 draws on material on local fields from Chapter 12. Chapters 25 and 26 require basic concepts from category theory, treated in Appendix 1A.

There is considerable overlap between parts of this volume and my earlier book, *Ring Theory* (student edition), but the philosophy and organization is usually quite different. In *Ring Theory* the emphasis is on the general structure theory of rings, via Jacobson's Density Theorem, in order to lay the foundations for applications to various kinds of rings.

The course on which this book is based was more goal-oriented — to develop enough of the theory of rings for basic representation theory, i.e., to prove and utilize the Wedderburn-Artin Theorem and Maschke's Theorem. Accordingly, the emphasis here is on semisimple and Artinian rings, with a short, direct proof. Similarly, the treatment of Noetherian rings here is limited mainly to Goldie's Theorem, which provides most of the non-technical applications needed later on.

Likewise, whereas in *Ring Theory* we approached representation theory of groups and Lie algebras via ring-theoretic properties of group algebras and enveloping algebras, we focus in Part V of this volume on the actual groups and Lie algebras.

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Part IV

The Structure of Rings

Introduction to the Structure of Rings

Whereas much of commutative theory stems from the polynomial algebra $F[\lambda_1, \dots, \lambda_n]$ over a field F, the matrix algebra $M_n(F)$ is arguably the most important example of a noncommutative algebra, since the tools of matrix theory are available, including the trace, determinant, transpose, and so forth. Much of representation theory involves comparing a given algebraic structure to such a matrix algebra, via suitable homomorphisms.

Our goal in this part is to introduce and develop enough structure theory of algebras and modules to enable us to carry out the applications in the next two parts, especially to representation theory. Since our main objective is representations of finite degree, we focus on finite-dimensional (f.d.) algebras over a field F.

We lay out the territory in Chapter 13, studying matrices (and related notions) from a structural point of view; any f.d. algebra can be embedded into a matrix algebra via the **regular representation**. We also study matrices as rings of endomorphisms. In Chapter 14, we introduce semisimple rings, leading to the Wedderburn-Artin theorem describing semisimple rings as direct products of matrix rings over division rings. Since any semisimple ring is Artinian, we are led in Chapter 15 to the general theory of Artinian rings, which is described elegantly by means of Jacobson's structure theory.

The material described above provides the structural basis for most of our applications. Nevertheless, there are many important algebras which are not Artinian. Some of the general structure-theoretical results are given in Appendix 15A, and the Noetherian theory is developed in Chapter 16.

One general method of studying algebraic structures, prevalent in much of group theory, is through generators and relations; in Chapter 17 we take an excursion through this avenue, leading to presentations of free algebras and groups, and various questions as to the efficacy of such presentations. Finally, in Chapter 18 we bring in another basic tool, the tensor product, which has a myriad of applications.

Part V

Representations of Groups and Lie Algebras

Introduction to Representations of Groups and Lie Algebras

Noncommutative algebra often involves the investigation of a mathematical structure in terms of matrices. Having described a finite-dimensional representation of an algebra R as an algebra homomorphism $\hat{\rho}: R \to M_n(F)$, for suitable n, we would like to utilize tools from matrix theory (such as the trace) to study the original algebra.

The point of Part V is that the same ideas hold for other algebraic structures. In Chapter 19, we establish the basic framework for studying group representations, which in Chapter 20 leads us to **character theory**, the study of representations through traces. One of the fundamental results is Maschke's Theorem, which implies that every linear representation is a direct sum of irreducible representations. The same idea of proof enables us also to prove the parallel result for compact topological groups, leading us to the study of Lie groups (Appendix 19A) and algebraic groups (Appendix 19B).

Both of these structures can be explored using a derivative structure, that of Lie algebra, which is treated together with its own representation theory in Chapter 21. The corresponding associative structure, the enveloping algebra, is discussed briefly in Appendix 21A. (Later, in Appendix 23B, we consider an interplay between groups and Lie algebras, which leads to a sketch of Zelmanov's solution of the Restricted Burnside Problem.)

Since multiplication in Lie algebras is not associative, we take the occasion in Appendix 21B to study other nonassociative structures such as alternative algebras and Jordan algebras.

Part of the Lie algebra theory (namely, the theory of Dynkin diagrams) spills over into Chapter 22, because of its connections to other structures. In Chapter 22, we also study Coxeter groups, which are certain groups of reflections whose study is motivated by the Weyl group of a semisimple Lie algebra.

One idea that pervades the various approaches is the correspondence of Remark 16.37, which transforms representation theory into the theory of modules; it lurks beneath the surface throughout Chapter 19 (especially in Proposition 19.12) and Chapter 21 and finally breaks through in Appendix 25C.

Part VI

Representable Algebras

Introduction to Representable Algebras

Part V dealt with the theory of representations of various algebraic structures into matrix algebras $M_n(\bar{F})$ over an algebraically closed field. This leads us to consider subrings of matrix algebras (over a larger field); such rings are called *representable*. Since such rings are often algebras over a subfield F of $\bar{F}$ that is not algebraically closed, we encounter some of the intricacies involved in dealing with non-algebraically closed fields.

In Chapter 23 we view the general question of representability of a ring in terms of a simple and natural condition called a *polynomial identity*, which also provides a beautiful structure theory.

The most important class of representable rings is that of f.d. division algebras, which have arisen already in the Wedderburn-Artin Theorem; these are studied in Chapter 24.

Viewing representations of algebraic structures from a categorical point of view, our target algebra, $M_n(\bar{F})$, the endomorphism algebra of a f.g. free module, should be replaced by a structure defined in categorical language, leading us in Chapter 25 to the notion of **projective** (and **injective**) module, which provides the setting for the powerful theories of homology and cohomology that underlie many of our earlier theorems. In Appendix 25A we study equivalences of categories of modules via Morita theory.

In Appendix 25B we introduce **Azumaya algebras**, the natural generalization of central simple algebras; there also is a description of Azumaya algebras in terms of polynomial identities.

Representation theory can be translated to the structure of modules, leading us in Appendix 25B to a remarkable connection with the classification of f.d. Lie algebras.

Various associative algebras (group algebras, enveloping algebras) arise in representation theory, and these are unified in Chapter 26 by means of the theory of Hopf algebras.