Preface

This book is an introduction to the fundamentals of differential geometry (manifolds, flows, Lie groups and their actions, invariant theory, differential forms and de Rham cohomology, bundles and connections, Riemann manifolds, isometric actions, symplectic geometry) which stresses naturality and functoriality from the beginning and is as coordinate-free as possible. The material presented in the beginning is standard — but some parts are not so easily found in text books: Among these are initial submanifolds (2.13) and the extension of the Frobenius theorem for distributions of nonconstant rank (the Stefan-Sussman theory) in (3.21) - (3.28). A quick proof of the Campbell-Baker-Hausdorff formula for Lie groups is in (4.29). Lie group actions are studied in detail: Palais' results that an infinitesimal action of a finite-dimensional Lie algebra on a manifold integrates to a local action of a Lie group and that proper actions admit slices are presented with full proofs in sections (5) and (6). The basics of invariant theory are given in section (7): The Hilbert-Nagata theorem is proved, and Schwarz's theorem on smooth invariant functions is discussed, but not proved.

In the section on vector bundles, the Lie derivative is treated for natural vector bundles, i.e., functors which associate vector bundles to manifolds and vector bundle homomorphisms to local diffeomorphisms. A formula for the Lie derivative is given in the form of a commutator, but it involves the tangent bundle of the vector bundle. So also a careful treatment of tangent bundles of vector bundles is given. Then follows a standard presentation of differential forms and de Rham cohomoloy including the theorems of de Rham and Poincaré duality. This is used to compute the cohomology of compact Lie groups, and a section on extensions of Lie algebras and Lie groups follows.

The chapter on bundles and connections starts with a thorough treatment of the Frölicher-Nijenhuis bracket via the study of all graded derivations of the algebra of differential forms. This bracket is a natural extension of the Lie bracket from vector fields to tangent bundle valued differential forms; it is one of the basic structures of differential geometry. We begin our treatment of connections in the general setting of fiber bundles (without structure group). A connection on a fiber bundle is just a projection onto the vertical bundle. Curvature and the Bianchi identity are expressed with the help of the Frölicher-Nijenhuis bracket. The parallel transport for such a general connection is not defined along the whole of the curve in the base in general — if this is the case, the connection is called complete. We show that every fiber bundle admits complete connections. For complete connections we treat holonomy groups and the holonomy Lie algebra, a subalgebra of the Lie algebra of all vector fields on the standard fiber. Then we present principal bundles and associated bundles in detail together with the most important examples. Finally we investigate principal connections by requiring equivariance under the structure group. It is remarkable how fast the usual structure equations can be derived from the basic properties of the Frölicher-Nijenhuis bracket. Induced connections are investigated thoroughly — we describe tools to recognize induced connections among general ones. If the holonomy Lie algebra of a connection on a fiber bundle with compact standard fiber turns out to be finite-dimensional, we are able to show that in fact the fiber bundle is associated to a principal bundle and the connection is an induced one. I think that the treatment of connections presented here offers some didactical advantages: The geometric content of a connection is treated first, and the additional requirement of equivariance under a structure group is seen to be additional and can be dealt with later — so the student is not required to grasp all the structures at the same time. Besides that it gives new results and new insights. This treatment is taken from [**146**].

The chapter on Riemann geometry contains a careful treatment of connections to geodesic structures to sprays to connectors and back to connections considering also the roles of the second and third tangent bundles in this. Most standard results are proved. Isometric immersions and Riemann submersions are treated in analogy to each other. A unusual feature is the Jacobi flow on the second tangent bundle. The chapter on isometric actions starts off with homogeneous Riemann manifolds and the beginnings of symmetric space theory; then Riemann G-manifolds and polar actions are treated.

The final chapter on symplectic and Poisson geometry puts some emphasis on group actions, momentum mappings and reductions.

There are some glaring omissions: The Laplace-Beltrami operator is treated only summarily, there is no spectral theory, and the structure theory of Lie algebras is not treated and used. Thus the finer theory of symmetric spaces is outside of the scope of this book.

The exposition is not always linear. Sometimes concepts treated in detail in later sections are used or pointed out earlier on when they appear in a natural way. Text cross-references to sections, subsections, theorems, numbered equations, items in a list, etc., appear in parantheses, for example, section (1), subsection (1.1), theorem (3.16), equation (3.16.3) which will be called (3) within (3.16) and its proof, property (3.22.1).

This book grew out of lectures which I have given during the last three decades on advanced differential geometry, Lie groups and their actions, Riemann geometry, and symplectic geometry. I have benefited a lot from the advise of colleagues and remarks by readers and students. In particular I want to thank Konstanze Rietsch whose write-up of my lecture course on isometric group actions was very helpful in the preparation of this book and Simon Hochgerner who helped with the last section.

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