Chapter 1

Category $\mathcal{O}$: Basics

We begin by laying the foundations for study of the BGG category $\mathcal{O}$, starting with the axioms and their immediate consequences in 1.1. This category encompasses all finite dimensional $U(\mathfrak{g})$-modules, the traditional starting point for representation theory of semisimple Lie algebras. In 1.6 the finite dimensional simple modules are identified as those having dominant integral highest weights. Then in Chapter 2 we shall recover the classical theorems of Weyl and others in the category $\mathcal{O}$ setting. But our main concern is the study of $\mathcal{O}$ itself, which has wider implications in representation theory.

The most accessible infinite dimensional modules in $\mathcal{O}$ are the Verma modules (1.3), which arise first as an auxiliary tool for the construction of simple modules. Later their study deepens considerably, as we shall see in Chapters 4–5.

Some basic tools introduced in this chapter are central characters and related subcategories $\mathcal{O}_\chi$ (1.7–1.13), along with formal characters of weight modules (1.14–1.16).

One important aspect of category $\mathcal{O}$ is deferred until Chapter 9, largely for pedagogical reasons: the study of subcategories $\mathcal{O}^{\mathfrak{p}}$ attached to arbitrary parabolic subalgebras $\mathfrak{p}$ (not just the Borel subalgebra $\mathfrak{b}$). On one hand, $\mathcal{O}^{\mathfrak{p}}$ requires a cumbersome extra layer of notation; on the other hand, the results there are sometimes easier to derive once the corresponding results for $\mathcal{O}$ are in hand.

1.1. Axioms and Consequences

The **BGG category** $\mathcal{O}$ is defined to be the full subcategory of $\operatorname{Mod} U(\mathfrak{g})$ whose objects are the modules satisfying the following three conditions.

($\mathcal{O}1$) M is a finitely generated $U(\mathfrak{g})$-module.

($\mathcal{O}2$) M is $\mathfrak{h}$-semisimple, that is, M is a weight module: $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$.

($\mathcal{O}3$) M is locally $\mathfrak{n}$-finite: for each $v \in M$, the subspace $U(\mathfrak{n}) \cdot v$ of M is finite dimensional.

As a matter of notation, we usually write $\mathrm{Hom}_{\mathcal{O}}(M, N)$ rather than $\mathrm{Hom}_{U(\mathfrak{g})}(M, N)$ or $\mathrm{Hom}_{\mathfrak{g}}(M, N)$ to emphasize that M and N lie in $\mathcal{O}$.

From 0.7 it is clear that all finite dimensional modules lie in $\mathcal{O}$. Before constructing further examples in 1.3, we explore the implications of the axioms. To begin, we deduce from the axioms two features of the weight structure of an arbitrary M in $\mathcal{O}$:

($\mathcal{O}4$) All weight spaces of M are finite dimensional.

($\mathcal{O}5$) In the notation of 0.7, the set $\Pi(M)$ of all weights of M is contained in the union of finitely many sets of the form $\lambda - \Gamma$, where $\lambda \in \mathfrak{h}^*$ and Γ is the semigroup in Λ_r generated by Φ^+.

First, it is obvious from ($\mathcal{O}2$) that a finite generating set in ($\mathcal{O}1$) can always be taken to consist of weight vectors. To verify ($\mathcal{O}4$) and ($\mathcal{O}5$) it will suffice to let M be generated by a single weight vector v of weight λ. Thanks to the PBW Theorem (0.5), we can write $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})$. Applying a factor from $U(\mathfrak{n})$ to v we get a finite dimensional subspace V of M (thanks to ($\mathcal{O}3$)) spanned by weight vectors having weights of type λ + sum of positive roots. Now V is stable under $U(\mathfrak{h})$, while the action of $U(\mathfrak{n}^-)$ on V produces only weights lower than these. Moreover, only a finite number of standard basis monomials $y_1^{i_1} \dots y_m^{i_m}$ in $U(\mathfrak{n}^-)$ can yield the same weight when applied to a weight vector in V.

It is easy to derive a few further consequences of the axioms.

Theorem. *Category $\mathcal{O}$ satisfies:*

(a) *$\mathcal{O}$ is a noetherian category, i.e., each $M \in \mathcal{O}$ is a noetherian $U(\mathfrak{g})$-module.*

(b) *$\mathcal{O}$ is closed under submodules, quotients, and finite direct sums.*

(c) *$\mathcal{O}$ is an abelian category.*

(d) *If $M \in \mathcal{O}$ and L is finite dimensional, then $L \otimes M$ also lies in $\mathcal{O}$. Thus $M \mapsto L \otimes M$ defines an exact functor $\mathcal{O} \to \mathcal{O}$.*

(e) *If $M \in \mathcal{O}$, then M is $Z(\mathfrak{g})$-finite: for each $v \in M$, the span of $\{z \cdot v \mid z \in Z(\mathfrak{g})\}$ is finite dimensional.*

(f) *If $M \in \mathcal{O}$, then M is finitely generated as a $U(\mathfrak{n}^-)$-module.*

Proof. (a) Since $U(\mathfrak{g})$ is a noetherian ring (0.5), this follows from ($\mathcal{O}1$).

(b) Closure under taking quotients or finite direct sums is immediate. Since $U(\mathfrak{g})$ is noetherian, any submodule of a finitely generated $U(\mathfrak{g})$-module is also finitely generated, whereas ($\mathcal{O}2$) and ($\mathcal{O}3$) are automatic for a submodule of a module in $\mathcal{O}$.

(c) Since $\operatorname{Mod} U(\mathfrak{g})$ itself is an abelian category, we only need to check that $\mathcal{O}$ is closed under finite direct sums and under taking kernels and cokernels of homomorphisms. This follows from (b).

(d) It is clear that $L \otimes M$ satisfies axioms ($\mathcal{O}2$) and ($\mathcal{O}3$). To check finite generation, let $v_1, \dots, v_n$ be a basis of L and let $w_1, \dots, w_p$ generate the module M. We claim that the $v_i \otimes w_j$ generate $L \otimes M$. If N is the submodule they generate, then certainly all $v \otimes w_j$ with $v \in L$ lie in N. In turn, for each $x \in \mathfrak{g}$ we have $x \cdot (v \otimes w_j) = x \cdot v \otimes w_j + v \otimes x \cdot w_j \in N$, and then also $v \otimes x \cdot w_j \in N$. Iteration shows that $v \otimes u \cdot w_j \in N$ for all PBW monomials $u \in U(\mathfrak{g})$. Thus $L \otimes M \subset N$.

(e) Since each $v \in M$ is a sum of weight vectors, we may assume that $v \in M_\lambda$ for some $\lambda \in \mathfrak{h}^*$. The fact that $z \in Z(\mathfrak{g})$ commutes with the action of $\mathfrak{h}$ then implies that $z \cdot v \in M_\lambda$. Weight spaces of M being finite dimensional by ($\mathcal{O}4$), the span of $\{z \cdot v \mid z \in Z(\mathfrak{g})\}$ is therefore finite dimensional.

(f) The axioms imply that M is generated by a finite dimensional $U(\mathfrak{b})$-submodule N. Thanks to the PBW Theorem, a basis of N then generates M as a $U(\mathfrak{n}^-)$-module. □

For some applications it is enough to work in the subcategory $\mathcal{O}_{\text{int}}$ whose objects all have *integral* weights: this encompasses for example all finite dimensional modules (1.6). In any case, proofs are often easier to carry out initially in $\mathcal{O}_{\text{int}}$. Going in the opposite direction, we sometimes consider the larger category $\mathcal{C}$ of $U(\mathfrak{g})$-modules whose objects are the weight modules with finite dimensional weight spaces.

Exercise. (a) If $M \in \mathcal{O}$ and $[\lambda] := \lambda + \Lambda_r$ is any coset of $\mathfrak{h}^*$ modulo Λ_r, let $M^{[\lambda]}$ be the sum of all weight spaces M_μ for which $\mu \in [\lambda]$. Prove that $M^{[\lambda]}$ is a $U(\mathfrak{g})$-submodule of M and that M is the direct sum of (finitely many!) such submodules.

(b) Deduce that all weights of an *indecomposable* module $M \in \mathcal{O}$ lie in a single coset of $\mathfrak{h}^*$ modulo Λ_r.

1.2. Highest Weight Modules

Still lacking is a concrete construction of modules in $\mathcal{O}$, apart from familiar finite dimensional examples such as the adjoint module $\mathfrak{g}$. In this direction we introduce some helpful terminology. First define a nonzero vector v^+ in a $U(\mathfrak{g})$-module M to be a **maximal vector** of weight $\lambda \in \mathfrak{h}^*$ if $v^+ \in M_\lambda$

and $\mathfrak{n} \cdot v^+ = 0$. (Such a vector is also referred to in the literature as a **primitive vector** of weight λ.) Thanks to axioms ($\mathcal{O}2$) and ($\mathcal{O}3$), *every nonzero module in* $\mathcal{O}$ *has at least one maximal vector.*

Next define a $U(\mathfrak{g})$-module M to be a **highest weight module** of weight λ if there is a maximal vector $v^+ \in M_\lambda$ such that $M = U(\mathfrak{g}) \cdot v^+$. By the PBW Theorem, such a module satisfies $M = U(\mathfrak{n}^-) \cdot v^+$. It is easy to spell out the basic properties of an arbitrary highest weight module by specializing the discussion in the previous section:

Theorem. *Let M be a highest weight module of weight $\lambda \in \mathfrak{h}^*$, generated by a maximal vector v^+. Fix an ordering of the positive roots as $\alpha_1, \dots, \alpha_m$ and choose corresponding root vectors y_i in $\mathfrak{g}_{-\alpha_i}$. Then:*

(a) *M is spanned by the vectors $y_1^{i_1} \dots y_m^{i_m} \cdot v^+$ with $i_j \in \mathbb{Z}^+$, having respective weights $\lambda - \sum i_j \alpha_j$. Thus M is a semisimple $\mathfrak{h}$-module.*

(b) *All weights μ of M satisfy $\mu \leq \lambda$: $\mu = \lambda -$ (sum of positive roots).*

(c) *For all weights μ of M, we have $\dim M_\mu < \infty$, while $\dim M_\lambda = 1$. Thus M is a weight module, locally finite as $\mathfrak{n}$-module, and $M \in \mathcal{O}$.*

(d) *Each nonzero quotient of M is again a highest weight module of weight λ.*

(e) *Each submodule of M is a weight module. A submodule generated by a maximal vector of weight $\mu < \lambda$ is proper; in particular, if M is simple, its maximal vectors are all multiples of v^+.*

(f) *M has a unique maximal submodule and unique simple quotient. In particular, M is indecomposable.*

(g) *All simple highest weight modules of weight λ are isomorphic. If M is one of these, $\dim \operatorname{End}_{\mathcal{O}} M = 1$.*

Proof. Part (a) follows from the standard commutation relations involving elements of $\mathfrak{h}$ and $\mathfrak{n}^-$, together with the fact that weight vectors having distinct weights are linearly independent. Then (b) is immediate.

For (c), it is clear that only finitely many choices of $i_1, \dots, i_m$ yield the same weight, by expressing sums of positive roots in terms of simple roots. Local finiteness of the action of $\mathfrak{n}$ follows from the fact that $\mathfrak{g}_\alpha$ maps M_μ into $M_{\mu+\alpha}$, coupled with (b).

Part (d) is clear, while part (e) follows from the fact that M lies in $\mathcal{O}$ (forcing its submodules to lie in $\mathcal{O}$) along with part (b).

Thanks to (e), each *proper* submodule of M is a weight module. It cannot have λ as a weight, since the 1-dimensional space M_λ generates M. Therefore the sum of all proper submodules is still proper, whence (f).

For (g), suppose M_1 and M_2 are simple highest weight modules of weight λ, with respective maximal vectors v_1^+, v_2^+. Set $M_0 := M_1 \oplus M_2$ and $v^+ := (v_1^+, v_2^+)$. Evidently v^+ is a maximal vector in M_0, so the submodule N it generates is a highest weight module of weight λ. In turn, the two projections $N \to M_1$ and $N \to M_2$ are surjective. As simple quotients of a highest weight module, M_1 and M_2 must then be isomorphic, thanks to (f). [In fact, they are both isomorphic to N under the projections, since N is semisimple.]

Finally, since M is simple, any nonzero endomorphism φ must be an isomorphism and also take v^+ to a multiple cv^+. Since v^+ generates M, it follows that φ is just multiplication by c. (This property is analogous to Schur's Lemma, even though M need not be finite dimensional.) □

In a loose sense, highest weight modules are the building blocks for all objects in $\mathcal{O}$. Since we cannot expect modules here to be semisimple, we look instead for a finite **filtration**: a chain of submodules $0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ for which the quotients M_{i+1}/M_i have a known structure.

Corollary. *Let M be any nonzero module in $\mathcal{O}$. Then M has a finite filtration $0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ with nonzero quotients each of which is a highest weight module.*

Proof. Since M can be generated by finitely many weight vectors and is moreover locally $\mathfrak{n}$-finite, the $\mathfrak{n}$-submodule V generated by such a generating family of weight vectors is finite dimensional. If $\dim V = 1$, it is clear that M itself is a highest weight module. Otherwise proceed by induction on $\dim V$.

Start with a nonzero weight vector $v \in V$ of weight λ which is maximal among all weights of V and is therefore a maximal vector in M. It generates a submodule M_1, while the quotient $\overline{M} := M/M_1$ again lies in $\mathcal{O}$ and is generated by the image $\overline{V}$ of V. Since $\dim \overline{V} < \dim V$, the induction hypothesis can be applied to $\overline{M}$, yielding a chain of highest weight submodules whose pre-images in M are the desired $M_2, \ldots, M_n$. □

1.3. Verma Modules and Simple Modules

We can construct a large family of highest weight modules by exploiting the technique of **induction**. The idea, which arises in similar ways in other types of representation theory, is to start with an easily constructed family of modules for a subalgebra and then "induce" to $\mathfrak{g}$.

Here we start with the Borel subalgebra $\mathfrak{b}$ corresponding to a fixed choice of positive roots, which in turn has an abelian quotient algebra $\mathfrak{b}/\mathfrak{n}$ isomorphic to $\mathfrak{h}$. Any $\lambda \in \mathfrak{h}^*$ then defines a 1-dimensional $\mathfrak{b}$-module with trivial $\mathfrak{n}$-action, denoted $\mathbb{C}_\lambda$. Now set $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$, which has a natural structure of left $U(\mathfrak{g})$-module. This is called a **Verma module** and may also be written as $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda$ to emphasize the functorial nature of induction. Thanks to the PBW Theorem, $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{b})$, which allows us to write $M(\lambda) \cong U(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda$ as a left $U(\mathfrak{n}^-)$-module. Therefore $M(\lambda)$ is a free $U(\mathfrak{n}^-)$-module of rank one. In particular, the vector $v^+ := 1 \otimes 1$ in the definition of $M(\lambda)$ is nonzero and is acted on freely by $U(\mathfrak{n}^-)$, while $\mathfrak{n} \cdot v^+ = 0$ and $h \cdot v^+ = \lambda(h) v^+$ for all $h \in \mathfrak{h}$. Thus v^+ is a maximal vector and generates the $U(\mathfrak{g})$-module $M(\lambda)$. Moreover, the set of weights of $M(\lambda)$ is visibly $\lambda - \Gamma$. It follows that $M(\lambda)$ lies in $\mathcal{O}$.

Remark. More generally, one can start with an arbitrary finite dimensional $U(\mathfrak{b})$-module N on which $\mathfrak{h}$ acts semisimply and get an induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$ in $\mathcal{O}$. This defines an *exact* functor from such $U(\mathfrak{b})$-modules to $U(\mathfrak{g})$-modules, since (as above) an induced module is free as a $U(\mathfrak{n}^-)$-module.

One can alternatively describe $M(\lambda)$ by generators and relations. From the PBW Theorem, we see that the left ideal I of $U(\mathfrak{g})$ which annihilates v^+ is generated by $\mathfrak{n}$ together with all $h - \lambda(h) \cdot 1$ with $h \in \mathfrak{h}$. Thus $M(\lambda) \cong U(\mathfrak{g})/I$. Obviously I also annihilates a maximal vector of weight λ generating an arbitrary highest weight module M. So $M(\lambda)$ maps naturally onto M and therefore plays the role of **universal highest weight module** of weight λ.

Using Theorem 1.2(f), we can write unambiguously $L(\lambda)$ (resp. $N(\lambda)$) for the unique simple quotient (resp. unique maximal submodule) of $M(\lambda)$. Since every nonzero module in $\mathcal{O}$ has at least one maximal vector, we conclude from this discussion and Theorem 1.2(g):

Theorem. *Every simple module in $\mathcal{O}$ is isomorphic to a module $L(\lambda)$ with $\lambda \in \mathfrak{h}^*$ and is therefore determined uniquely up to isomorphism by its highest weight. Moreover,* $\dim \operatorname{Hom}_{\mathcal{O}} \big(L(\mu), L(\lambda)\big) = \delta_{\lambda\mu}$. □

Exercise. Show that $M(\lambda)$ has the following property: For any M in $\mathcal{O}$,

$$\operatorname{Hom}_{U(\mathfrak{g})} \big(M(\lambda), M\big) = \operatorname{Hom}_{U(\mathfrak{g})} \big(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda, M\big) \cong \operatorname{Hom}_{U(\mathfrak{b})} \big(\mathbb{C}_\lambda, \operatorname{Res}_{\mathfrak{b}}^{\mathfrak{g}} M\big),$$

where $\operatorname{Res}_{\mathfrak{b}}^{\mathfrak{g}}$ is the restriction functor. [Use the universal mapping property of tensor products.] In the context of induced modules for group algebras of finite groups, this adjointness property is known as *Frobenius reciprocity.*

1.4. Maximal Vectors in Verma Modules

In order to explore the submodule structure of a Verma module $M(\lambda)$, it is natural to begin by looking for maximal vectors of weight $\mu < \lambda$. This theme will be developed further in Chapter 4, but for our immediate purposes it is enough to carry out just the most elementary construction. Fix a standard basis (for example, a Chevalley basis) for $\mathfrak{g}$ as in 0.1, consisting of $h_1, \ldots, h_\ell$ together with root vectors $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ for $\alpha \in \Phi^+$ (abbreviated to x_i, y_i for a numbering of simple roots $\alpha_1, \ldots, \alpha_\ell$).

Proposition. *Given $\lambda \in \mathfrak{h}^*$ and a fixed simple root α, suppose $n := \langle \lambda, \alpha^\vee \rangle$ lies in $\mathbb{Z}^+$. If v^+ is a maximal vector of weight λ in $M(\lambda)$, then $y_\alpha^{n+1} \cdot v^+$ is a maximal vector of weight $\mu := \lambda - (n+1)\alpha < \lambda$. Thus there exists a nonzero homomorphism $M(\mu) \to M(\lambda)$ whose image lies in the maximal submodule $N(\lambda)$.*

Here it is essential that α be *simple*; in this case it is easily seen that all weight spaces $M(\lambda)_{\lambda - k\alpha}$ with $k \in \mathbb{Z}^+$ are 1-dimensional and spanned by a vector $y_\alpha^k \cdot v^+$. Otherwise a less direct approach to the existence of such a maximal vector is required (to be discussed in Chapter 4). The problem is that for a nonsimple positive root α with $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}^{>0}$, a weight space $M(\lambda)_{\lambda - k\alpha}$ is typically of dimension > 1 and spanned by the images of v^+ under a number of monomials in $U(\mathfrak{n})^-$ involving powers $y_\beta^{a(\beta)}$ with $\sum a(\beta) = k\alpha$. To write down explicitly a maximal vector in such a weight space is quite challenging.

The proof of the proposition depends on some standard commutation formulas in $U(\mathfrak{g})$ (see for example [**125**, 21.2]):

Lemma. *Let x_i, y_i be standard basis vectors as above, corresponding to simple roots $\alpha_1, \ldots, \alpha_\ell$. Then for all $k \geq 0$ and $1 \leq i, j \leq \ell$, we have:*

(a) $[x_j, y_i^{k+1}] = 0$ *whenever* $j \neq i$.

(b) $[h_j, y_i^{k+1}] = -(k+1)\alpha_i(h_j) y_i^{k+1}$.

(c) $[x_i, y_i^{k+1}] = -(k+1) y_i^k (k \cdot 1 - h_i)$.

Proof. We outline the steps, which are elementary, leaving the details as an exercise for the reader. When $k = 0$, (a) follows from the fact that $\alpha_j - \alpha_i$ is not a root if $j \neq i$, while (b) just expresses the fact that $[h_j, y_i] = -\alpha_i(h_j) y_i$ and (c) is obvious. Now proceed by induction. For example, in (c) rewrite the left side in $U(\mathfrak{g})$:

$$[x_i, y_i^{k+1}] = x_i y_i^{k+1} - y_i^{k+1} x_i = [x_i, y_i] y_i^k + y_i [x_i, y_i^k] = h_i y_i^k + y_i [x_i, y_i^k].$$

Then apply the induction hypothesis, together with (b) (replacing $k+1$ there by k). □

Proof of Proposition. Recalling the isomorphism of $U(\mathfrak{n}^-)$-modules between $U(\mathfrak{n}^-)$ and $M(\lambda)$, part (c) of the lemma translates into the statement of the proposition in view of part (b). □

Corollary. *With λ as in the proposition, let v^+ be instead a maximal vector of weight λ in $L(\lambda)$. Then $y_\alpha^{n+1} \cdot v^+ = 0$.*

Proof. Since $\lambda - (n+1)\alpha < \lambda$, no maximal vector of this weight can exist in the simple module $L(\lambda)$ (by Theorem 1.2(e)). So the maximal vector of this weight in $M(\lambda)$ constructed in the proposition must lie in the maximal submodule and map to 0 in the quotient $L(\lambda)$. □

1.5. Example: $\mathfrak{sl}(2, \mathbb{C})$

The description of Verma modules and simple modules in the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ serves as a (much simplified) prototype of the general theory to be developed in the following sections. The reader is encouraged to fill in the following outline now, although some steps will become considerably easier later on. (The finite dimensional case has already been reviewed in 0.9.)

Fix a standard basis (h, x, y) for $\mathfrak{g}$, with $[hx] = 2x, [hy] = -2y, [xy] = h$. Since $\dim \mathfrak{h} = 1$, weights $\lambda \in \mathfrak{h}^*$ may be identified with complex numbers. In turn, the integral weight lattice Λ is identified with $\mathbb{Z}$ and Λ_r with $2\mathbb{Z}$.

- $M(\lambda)$ has weights $\lambda, \lambda–2, \lambda–4, \dots$, each with multiplicity one. Basis vectors v_i $(i \geq 0)$ for $M(\lambda)$ can be chosen so that (setting $v_{-1} = 0$):

$$\begin{aligned} h \cdot v_i &= (\lambda - 2i)v_i, \\ x \cdot v_i &= (\lambda - i + 1)v_{i-1}, \\ y \cdot v_i &= (i+1)v_{i+1}. \end{aligned}$$

  (This is not the only way to choose a basis, but is convenient when working with integral weights and then reducing modulo a prime.)
- $\dim L(\lambda) < \infty$ if and only if $\lambda \in \mathbb{Z}^+$. In this case, the maximal submodule of $M(\lambda)$ is isomorphic to $L(-\lambda - 2)$.
- $M(\lambda)$ is simple if and only if λ is not in $\mathbb{Z}^+$.

Exercise. When $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, show that $M(\lambda) \otimes M(\mu)$ cannot lie in $\mathcal{O}$.

1.6. Finite Dimensional Modules

Relying on the representation theory of $\mathfrak{sl}(2, \mathbb{C})$, we can now determine precisely which of the simple modules $L(\lambda)$ with $\lambda \in \mathfrak{h}^*$ are finite dimensional. For each $\alpha \in \Phi^+$, let $\mathfrak{s}_\alpha$ be the copy of $\mathfrak{sl}(2, \mathbb{C})$ in $\mathfrak{g}$ spanned by $h_\alpha, x_\alpha, y_\alpha$. Given an enumeration of simple roots as $\alpha_1, \dots, \alpha_\ell$, abbreviate $\mathfrak{s}_{\alpha_i}$ by $\mathfrak{s}_i$, h_{α_i} by h_i, and s_{α_i} by s_i.

For use in the proof below, observe that if $M \in \mathcal{O}$ and $N \subset M$ is a finite dimensional $\mathfrak{s}_i$-submodule of M generated by a weight vector, then $\mathfrak{h}$ automatically stabilizes N. Indeed, if $v \in N$ has weight μ, then the commutation relations in $\mathfrak{g}$ force $h \cdot (x_i \cdot v) = x_i \cdot (h \cdot v) + \alpha_i(h) x_i \cdot v$. Since $h \cdot v = \mu(h) v \in N$, the left side also lies in N. (Similarly for $y_i \cdot v$.)

Theorem. *The simple module $L(\lambda)$ in $\mathcal{O}$ is finite dimensional if and only if $\lambda \in \Lambda^+$. This is the case if and only if $\dim L(\lambda)_\mu = \dim L(\lambda)_{w\mu}$ for all $\mu \in \mathfrak{h}^*$ and all $w \in W$.*

Proof. Proceed in steps.

(1) In one direction, assume that $\dim L(\lambda) < \infty$. For each fixed i, the structure of the $\mathfrak{s}_i$-module $L(\lambda)$ is then transparent. In particular, $\lambda(h_i) \in \mathbb{Z}^+$, forcing $\lambda \in \Lambda^+$ (0.7).

(2) In the other direction, assume that $\lambda \in \Lambda^+$. To show that $\dim L(\lambda) < \infty$ requires more work. Again we look at the $\mathfrak{s}_i$-module structure of $L(\lambda)$ for each $1 \leq i \leq \ell$. Since $n := \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}^+$, Corollary 1.4 implies that the $\mathfrak{s}_i$-submodule generated by v^+ is finite dimensional (and is in fact the unique simple module of dimension $n+1$).

(3) In turn, we claim that for each fixed i, $L(\lambda)$ is the sum of all its finite dimensional $\mathfrak{s}_i$-submodules; call this sum M. By step (2), $M \neq 0$. On the other hand, take any *finite dimensional* $\mathfrak{s}_i$-submodule N of $L(\lambda)$; so $N \subset M$. Then $\mathfrak{g} \otimes N$ is a finite dimensional $\mathfrak{s}_i$-module, where we use the adjoint action of $\mathfrak{s}_i$ on $\mathfrak{g}$. It is routine to check that the map $\mathfrak{g} \otimes N \to L(\lambda)$ sending $x \otimes v$ to $x \cdot v$ is an $\mathfrak{s}_i$-module homomorphism, so its image lies in M. Thus M is a nonzero $\mathfrak{g}$-submodule of the simple module $L(\lambda)$, forcing $L(\lambda) = M$.

(4) Since each $v \in L(\lambda)$ lies in a finite dimensional $\mathfrak{s}_i$-submodule, by (3), it follows that each x_i or y_i acts on $L(\lambda)$ as a *locally nilpotent* operator. If we denote the representation afforded by $L(\lambda)$ as $\varphi : U(\mathfrak{g}) \to \operatorname{End} L(\lambda)$, it now makes sense to patch together the unipotent operators $\exp \varphi(x_i) \exp \varphi(-y_i) \exp \varphi(x_i)$ which act on finite dimensional $\mathfrak{s}_i$-submodules. This yields a well-defined automorphism r_i of $L(\lambda)$.

(5) If μ is any weight of $L(\lambda)$, we claim that $r_i(L(\lambda)_\mu) = L(\lambda)_{s_i\mu}$ for all i. Indeed, the weight space $L(\lambda)_\mu$ lies in a finite dimensional $\mathfrak{s}_i$-submodule by step (3). This in turn is a weight module, thanks to the observation just before the theorem. Now the claim follows from the $\mathfrak{sl}(2, \mathbb{C})$ theory in 0.9.

(6) Because W is generated by the simple reflections s_i, step (5) implies that all weight spaces $L(\lambda)_{w\mu}$ have the same dimension. Since λ is dominant integral, by assumption, all weights of $L(\lambda)$ are therefore W-conjugates of dominant integral weights $\mu \leq \lambda$. But there exist only finitely many of these (0.6). It follows that $L(\lambda)$ has only finitely many weights, thus is finite dimensional.

(7) From the proof it also follows that $\dim L(\lambda)_\mu = \dim L(\lambda)_{w\mu}$ for all $w \in W$ whenever $\dim L(\lambda) < \infty$. □

Variants of this proof are found in numerous texts, including Bourbaki [**46**, Chap. VIII, §7], Carter [**60**, Chap. 10], Humphreys [**125**, 21.2].

Exercise. Let M be a highest weight module of weight $\lambda \in \Lambda^+$. If all x_i and y_i $(1 \leq i \leq \ell)$ act locally nilpotently on M (which is automatic for the x_i), rework steps in the proof above to show that $\dim M < \infty$. Conclude that $M \cong L(\lambda)$. (This situation comes up in 2.6 below.)

Example. It is easy to recover from step (2) of the proof the familiar fact that $L(0)$ is the trivial one-dimensional $\mathfrak{g}$-module: the subalgebras $\mathfrak{s}_i$ (which generate $\mathfrak{g}$) act trivially on v^+.

Coupled with Weyl's Complete Reducibility Theorem (0.7), the theorem gives a complete parametrization of the finite dimensional modules in $\mathcal{O}$. Of course, there are many concrete features yet to be investigated: notably, formal characters and dimensions. These will be worked out in Chapter 2 after we assemble more information about $\mathcal{O}$. Here we just note a few standard facts for later use in the case $\lambda \in \Lambda^+$:

- Let $\mu := w\lambda$ with $w \in W$. For any $\alpha \in \Phi$, not both $\mu - \alpha$ and $\mu + \alpha$ can occur as weights of $L(\lambda)$. [Recall that W permutes the weights of $L(\lambda)$. If $\beta := w^{-1}\alpha$, not both $\lambda + \beta$ and $\lambda - \beta$ can be $< \lambda$.]
- If μ and $\mu + k\alpha$ (with $k \in \mathbb{Z}, \alpha \in \Phi$) are weights of $L(\lambda)$, then so are all intermediate weights $\mu + i\alpha$. [View $L(\lambda)$ as an $\mathfrak{sl}(2, \mathbb{C})$-module and use the theory in 0.9.]
- The dual space $L(\lambda)^*$, with the standard action $(x \cdot f)(v) = -f(x \cdot v)$ for $x \in \mathfrak{g}, v \in L(\lambda), f \in L(\lambda)^*$, is isomorphic to $L(-w_\circ\lambda)$ (where $w_\circ \in W$ is the longest element). [Observe that $L(\lambda)^*$ is again simple; its weights relative to $\mathfrak{h}$ are the negatives of those for $L(\lambda)$. On the other hand, $w_\circ\lambda$ is the lowest weight of $L(\lambda)$.]

1.7. Action of the Center

Our discussion of Verma modules has shown the existence of a unique maximal submodule and simple quotient, but beyond this (and Proposition 1.4) we have no insight into the submodule structure. Without using deeper information about the universal enveloping algebra we cannot say much more. Indeed, our arguments so far have relied mainly on the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ and the resulting PBW Theorem. But this kind of set-up is found much more generally in infinite dimensional Kac–Moody algebras (see Carter [**60**], Kac [**165**], Moody–Pianzola [**223**]); here there is a good analogue of the category $\mathcal{O}$, but for example "Verma modules" typically fail to have finite Jordan–Hölder length.

To prove a finite length theorem in our situation, we have to look closer at the action of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. By contrast, the study of finite dimensional modules (including the proof of Weyl's Complete Reducibility Theorem), requires only the action of the *Casimir element* of $Z(\mathfrak{g})$ (0.5). Recall from Theorem 1.1(e) that every $M \in \mathcal{O}$ is locally finite as a $Z(\mathfrak{g})$-module. The situation simplifies when M is a highest weight module, generated by a maximal vector v^+ of weight λ. If $z \in Z(\mathfrak{g})$ and $h \in \mathfrak{h}$, we have

$$h \cdot (z \cdot v^+) = z \cdot (h \cdot v^+) = z \cdot (\lambda(h) v^+) = \lambda(h) z \cdot v^+.$$

Since $\dim M_\lambda = 1$, this forces $z \cdot v^+ = \chi_\lambda(z) v^+$ for some scalar $\chi_\lambda(z) \in \mathbb{C}$. In turn, z acts on an arbitrary element $u \cdot v^+$ of M (with $u \in U(\mathfrak{n}^-)$) by the same scalar, since $zu = uz$.

For fixed λ, the function $z \mapsto \chi_\lambda(z)$ defines an algebra homomorphism $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$, whose kernel is a maximal ideal of $Z(\mathfrak{g})$. We call it the **central character** associated with λ. More generally, we call any algebra homomorphism $Z(\mathfrak{g}) \to \mathbb{C}$ a central character. The set of all these is in natural bijection with the set $\operatorname{Max} Z(\mathfrak{g})$ of maximal ideals.

To describe χ_λ more concretely, write $z \in Z(\mathfrak{g})$ as a linear combination of PBW monomials based on the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Any monomial having a nonzero factor from $\mathfrak{n}$ will kill v^+. In turn, factors from $\mathfrak{h}$ just multiply v^+ by scalars, while $\mathfrak{n}^-$ then takes v^+ to vectors of lower weight. The upshot is that $z \cdot v^+$ depends just on the PBW monomials with factors in $\mathfrak{h}$. Denoting by $\mathrm{pr} : U(\mathfrak{g}) \to U(\mathfrak{h})$ the projection onto the subspace $U(\mathfrak{h})$ given by setting all other monomials equal to 0, we see that

$$\chi_\lambda(z) = \lambda(\mathrm{pr}(z)) \text{ for all } z \in Z(\mathfrak{g}).$$

Here the linear function λ is extended canonically to an algebra homomorphism $\lambda : U(\mathfrak{h}) \to \mathbb{C}$.

Using the obvious fact that $\bigcap_{\lambda \in \mathfrak{h}^*} \operatorname{Ker} \lambda = 0$, it follows that *the restriction of* pr *to* $Z(\mathfrak{g})$ *is an algebra homomorphism.* Denote this restriction by ξ; it is called the **Harish-Chandra homomorphism**.

Remark. To understand more intrinsically why ξ is a homomorphism (as in the original study of the center by Harish-Chandra), without resort to representation theory, look at the centralizer $U(\mathfrak{g})_0$ of $\mathfrak{h}$ in the algebra $U(\mathfrak{g})$. This subalgebra obviously includes $Z(\mathfrak{g})$ as well as $U(\mathfrak{h})$. As the notation suggests, it is actually the 0-graded subspace of $U(\mathfrak{g})$ relative to the natural grading induced by the extension to $U(\mathfrak{g})$ of the adjoint representation of $\mathfrak{g}$ (0.5). In turn, $L := U(\mathfrak{g})\mathfrak{n} \cap U(\mathfrak{g})_0$ is seen to coincide with $\mathfrak{n}^- U(\mathfrak{g}) \cap U(\mathfrak{g})_0$. This is a *two-sided* ideal of $U(\mathfrak{g})$, complementary to the subspace $U(\mathfrak{h})$ of

$U(\mathfrak{g})_0$, and the resulting projection homomorphism $U(\mathfrak{g})_0 \to U(\mathfrak{h})$ restricts to $\xi : Z(\mathfrak{g}) \to U(\mathfrak{h})$.

Two natural questions arise at this point: (1) Is the Harish-Chandra homomorphism injective? (2) What is the image of ξ in $U(\mathfrak{h})$?

1.8. Central Characters and Linked Weights

Since $\chi_\lambda = \chi_\mu$ whenever $L(\mu)$ occurs as a subquotient of $M(\lambda)$, it is important to determine precisely when this equality of central characters occurs. It is also natural to ask whether there are any other central characters besides the χ_λ. To deal with both of these issues we have to formalize better the relationship here between λ and μ, then bring the Harish-Chandra homomorphism into the picture.

A valuable clue is provided by Proposition 1.4: If $n := \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}^+$ for some $\alpha \in \Delta$, then $M(\lambda)$ has a maximal vector of weight $\lambda - (n+1)\alpha < \lambda$. In this case, we have $\chi_\lambda = \chi_\mu$ where $\mu := \lambda - (n+1)\alpha$. To express this more suggestively, recall from 0.6 that $s_\alpha \rho = \rho - \alpha$ whenever α is simple. Thus $s_\alpha(\lambda + \rho) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha + \rho - \alpha = \mu + \rho$, or $s_\alpha(\lambda+\rho) - \rho = \mu$.

Definition. For $w \in W$ and $\lambda \in \mathfrak{h}^*$, define a shifted action of W (called the **dot action**) by $w \cdot \lambda = w(\lambda + \rho) - \rho$. If $\lambda, \mu \in \mathfrak{h}^*$, we say that λ and μ are **linked** (or W-linked) if for some $w \in W$, we have $\mu = w \cdot \lambda$. Linkage is clearly an equivalence relation on $\mathfrak{h}^*$. The orbit $\{w \cdot \lambda \mid w \in W\}$ of λ under the dot action is called the **linkage class** (or W-linkage class) of λ.

For example, the fixed point $-\rho$ under the dot action lies in a linkage class by itself (and no other weight does). The usual notion of *regular weight* for $\lambda \in \mathfrak{h}^*$, requiring that the isotropy group of λ in W be trivial, or $|W\lambda| = |W|$, is replaced in the setting of linkage classes by a shifted notion of **regular weight** (which might also be called **dot-regular**): the weight $\lambda \in \mathfrak{h}^*$ is regular if $|W \cdot \lambda| = |W|$, or in other words, $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Phi$. Weights which are not regular may be called **singular**. In this sense, 0 is regular while $-\rho$ is singular.

In general, each $\lambda + \rho \in \Lambda$ is W-conjugate (by 0.6) to a unique element in the closure $\overline{C}$ of the Weyl chamber

$$C := \{\mu \in E \mid \langle \mu, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \Delta\}.$$

So the linkage class of λ has a unique element in $\overline{C} - \rho$; this weight is then the unique maximal element in its linkage class. (For arbitrary $\lambda \in \mathfrak{h}^*$, we shall fine-tune the parametrization in Chapter 3.)

Exercise. Unlike the usual action of W on $\mathfrak{h}^*$, the dot action is not additive. If $\lambda, \mu \in \mathfrak{h}^*$ and $w \in W$, verify that

$$\begin{aligned} w \cdot (\lambda + \mu) &= w \cdot \lambda + \mu, \\ w \cdot \lambda - w \cdot \mu &= w(\lambda - \mu). \end{aligned}$$

This language allows us to reformulate Proposition 1.4:

> *If $\alpha \in \Delta$, $\lambda \in \mathfrak{h}^*$ and $\langle \lambda + \rho, \alpha^\vee \rangle = 0$, then $M(s_\alpha \cdot \lambda) = M(\lambda)$. If $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}^{>0}$, then there exists a nonzero homomorphism $M(s_\alpha \cdot \lambda) \to M(\lambda)$ with image in $N(\lambda)$.*

Now we can strengthen somewhat our earlier conclusion:

Proposition. *If $\lambda \in \Lambda$ and μ is linked to λ, then $\chi_\lambda = \chi_\mu$.*

Proof. Start as before with a *simple* root α. Since $\lambda \in \Lambda$, $n := \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$. In case $n \in \mathbb{Z}^+$, we already saw that $\chi_\lambda = \chi_\mu$ with $\mu = s_\alpha \cdot \lambda$. Suppose on the other hand $n \in \mathbb{Z}^{<0}$. If $n = -1$, then $s_\alpha \cdot \lambda = \lambda$ and there is nothing to prove. If $n < -1$, then setting $\mu := s_\alpha \cdot \lambda$ (which forces $s_\alpha \cdot \mu = \lambda$), we have $\langle \mu, \alpha^\vee \rangle = -n - 2 \geq 0$. So by the first case, $\chi_\lambda = \chi_\mu$.

Now W is generated by the simple reflections (0.3), while the linkage relation is transitive, so the proposition follows by induction on $\ell(w)$. □

1.9. Harish-Chandra Homomorphism

In order to remove the restriction in Proposition 1.8 that λ be an *integral* weight, we can invoke a simple density argument in affine algebraic geometry. Here we view $\mathfrak{h}^*$ as the affine space $\mathbb{A}^\ell$ over $\mathbb{C}$, while $U(\mathfrak{h}) = S(\mathfrak{h})$ (identified in turn with $P(\mathfrak{h}^*)$) is the algebra of polynomial functions in ℓ variables acting naturally on $\mathbb{A}^\ell$. To make this concrete, take for example the fundamental weights $\varpi_1, \dots, \varpi_\ell$ as a basis for $\mathfrak{h}^*$, along with generators $h_1, \dots, h_\ell$ for $S(\mathfrak{h})$.

Whereas Λ is a discrete subspace of $\mathfrak{h}^*$ in the usual topology, it becomes *dense* in the Zariski topology; here the closed sets are the zero sets of finite sets of polynomials. Recall the argument: In $\mathbb{A}^\ell$ we can identify Λ with $\mathbb{Z}^\ell$. Now use induction on ℓ to see that a polynomial function f on $\mathbb{A}^\ell$ vanishing on $\mathbb{Z}^\ell$ must be zero. If $\ell = 1$, use the fact that a nonzero polynomial can have only finitely many roots. If $\ell > 1$, write f as a polynomial in the last variable. Substituting fixed integers for the first $\ell - 1$ variables produces a polynomial in one variable vanishing on $\mathbb{Z}$ (therefore zero). So the induction hypothesis for $\ell - 1$ can be applied, showing that $f = 0$. From this argument we conclude that $\mathbb{Z}^\ell$ is dense in $\mathbb{A}^\ell$.

We know that $\chi_\lambda = \chi_{w \cdot \lambda}$ for all $w \in W$ when $\lambda \in \Lambda$. Since $\chi_\lambda(z) = \lambda(\xi(z))$ for $z \in Z(\mathfrak{g})$, this translates into the statement that the polynomial functions $\xi(z)$ and $w \cdot \xi(z)$ agree on Λ. By density, this forces them to agree everywhere on $\mathfrak{h}^*$. In other words, *Proposition* 1.8 *is true for all* $\lambda \in \mathfrak{h}^*$.

Using this we can say more about the image of ξ. To take account of the ρ-shift involved in the dot action, first compose ξ with the algebra automorphism of $S(\mathfrak{h})$ induced by the substitution $p(\lambda) \mapsto p(\lambda-\rho)$. Call this composite homomorphism $\psi : Z(\mathfrak{g}) \to S(\mathfrak{h})$ the **twisted Harish-Chandra homomorphism**. Now we have

$$\chi_\lambda(z) = (\lambda+\rho)(\psi(z)) \text{ for all } z \in Z(\mathfrak{g}).$$

We have shown that the polynomial functions on $\mathfrak{h}^*$ in the image of ψ are constant on W-orbits (linkage classes), in other words lie in the algebra $S(\mathfrak{h})^W$ of W-invariants. To summarize:

Theorem. (a) *Whenever* $\lambda, \mu \in \mathfrak{h}^*$ *are* W*-linked,* $\chi_\lambda = \chi_\mu$.

(b) *The image of the twisted Harish-Chandra homomorphism* $\psi : Z(\mathfrak{g}) \to U(\mathfrak{h}) = S(\mathfrak{h})$ *lies in the subalgebra* $S(\mathfrak{h})^W$ *of* W*-invariant polynomials.* □

Example. Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ and identify $\mathfrak{h}^*$ as usual with $\mathbb{C}$ via $\lambda \mapsto \lambda(h)$. Recall from 0.5 that a Casimir element in $Z(\mathfrak{g})$ is given explicitly by $c = 2xy + h^2 + 2yx = h^2 + 2h + 4yx$. Recalling that $\rho(h) = 1$, we get $\psi(c) = (h-1)^2 + 2(h-1) = h^2 - 1$, which visibly lies in $S(\mathfrak{h})^W$. In turn, $\chi_\lambda(c) = (\lambda+\rho)(h^2-1) = (\lambda+1)^2 - 1$. It follows that $\chi_\lambda = \chi_\mu$ if and only if λ and μ are linked: $\mu = \lambda$ or $\mu = -\lambda - 2$.

Exercise. Show that the homomorphism ψ is independent of the choice of a simple system in Φ. [Any simple system has the form $w\Delta$ for some $w \in W$.]

1.10. Harish-Chandra's Theorem

What we really need is the converse of Theorem 1.9(a), in order to get a workable necessary condition for $L(\lambda)$ and $L(\mu)$ to occur as composition factors of a Verma module. This is contained in the following basic theorem.

Theorem (Harish-Chandra). *Let* $\psi : Z(\mathfrak{g}) \to S(\mathfrak{h})$ *be the twisted Harish-Chandra homomorphism.*

(a) *The homomorphism* ψ *is an isomorphism of* $Z(\mathfrak{g})$ *onto* $S(\mathfrak{h})^W$.

(b) *For all* $\lambda, \mu \in \mathfrak{h}^*$*, we have* $\chi_\lambda = \chi_\mu$ *if and only if* $\mu = w \cdot \lambda$ *for some* $w \in W$.

(c) *Every central character* $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ *is of the form* χ_λ *for some* $\lambda \in \mathfrak{h}^*$.

(a) This is the key point. Its proof depends mainly on arguments involving the enveloping algebra, independent of category $\mathcal{O}$. Here we just outline the basic ideas, referring in the Notes below to several thorough discussions in the literature.

We have seen that ψ maps $Z(\mathfrak{g})$ into the algebra $S(\mathfrak{h})^W$. Instead of trying to prove injectivity and surjectivity directly, it is better to make a comparison with a similar but somewhat more transparent map. Consider the algebra $P(\mathfrak{g})$ of polynomial functions on the vector space $\mathfrak{g}$, which is naturally isomorphic to $S(\mathfrak{g}^*)$. The restriction map $\theta : P(\mathfrak{g}) \to P(\mathfrak{h})$ is an algebra homomorphism. Next introduce the **adjoint group** $G \subset \operatorname{Aut}\mathfrak{g}$ generated by the operators $\exp \operatorname{ad} x$ with $x \in \mathfrak{g}$ nilpotent. (Since $\operatorname{ad} x$ is nilpotent, the exponential power series is just a polynomial.) The group G is a Lie group (or a semisimple algebraic group if $\mathbb{C}$ is replaced by another algebraically closed field of characteristic 0). It acts naturally on $P(\mathfrak{g})$ as a group of automorphisms, with algebra of fixed points denoted $P(\mathfrak{g})^G$. Similarly, W acts on $P(\mathfrak{h})$.

In this setting Chevalley proved a fundamental **Restriction Theorem**:

$$\theta \textit{ maps } P(\mathfrak{g})^G \textit{ isomorphically onto } P(\mathfrak{h})^W.$$

Since the Killing form is nondegenerate on $\mathfrak{g}$ and restricts to a nondegenerate form on $\mathfrak{h}$, we can further identify $P(\mathfrak{g})$ with $S(\mathfrak{g})$ and $P(\mathfrak{h})$ with $S(\mathfrak{h})$. The proof of Chevalley's theorem involves some use of finite dimensional representation theory: $P(\mathfrak{g})^G$ is generated by traces of powers of operators representing elements of $\mathfrak{g}$ in such representations.

At this point the rewritten isomorphism $S(\mathfrak{g})^G \to S(\mathfrak{h})^W$ somewhat resembles the Harish-Chandra homomorphism ξ. In fact, $S(\mathfrak{g})^G$ can even be identified naturally as a *vector space* with $Z(\mathfrak{g})$, though not as an algebra. Even though the Chevalley map θ does not coincide with ξ (as illustrated when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ in [**125**, 23.3]), one gets enough information from the comparison to see that ξ is *bijective* and thus to complete the proof. The trick is to see that the various maps here are compatible with the natural filtrations of the algebras, then to observe that the induced *graded* versions of ξ and the Chevalley isomorphism agree.

(b) Suppose that for some $\lambda, \mu \in \mathfrak{h}^*$, the linkage classes of λ and μ are disjoint. Using Lagrange interpolation, find a polynomial function f on $\mathfrak{h}^*$ which takes value 1 on $W \cdot \lambda$ and 0 on $W \cdot \mu$. Then replace f by its "average"

$$\frac{1}{|W|} \sum_{w \in W} wf$$

to get a W-*invariant* g having the same values on these dot orbits. Using the assumption about ψ, take any pre-image $z \in Z(\mathfrak{g})$ of g under ψ. Then check that $\chi_\lambda(z) = \lambda(\psi(z)) = g(\lambda) = 1$, whereas $\chi_\mu(z) = \mu(\psi(z)) = g(\mu) = 0$. This forces $\chi_\lambda \neq \chi_\mu$.