Chapter 1

Second-order elliptic equations in $W_2^2(\mathbb{R}^d)$

One of the most important elliptic operators acting on functions $u(x), x \in \mathbb{R}^d$, is Laplace's operator given by the formula

$$\Delta u = u_{x^1x^1} + ... + u_{x^dx^d}.$$

The function Δu is called the Laplacian of u.

Given a domain $D \subset \mathbb{R}^d$ and a function $f(x), x \in \mathbb{R}^d$, one may try to follow what is usually done in the theory of ordinary differential equations in one space dimension and set up the goal of finding explicitly the general solution of the equation

$$\Delta u(x) = f(x), \quad x \in D.$$

It turns out that generally this is impossible.

On few occasions, however, one can find a solution explicitly. For instance, if $d = 2$, $f \equiv -1$, $D = (-\pi/2, \pi/2)^2$ and we are interested in finding explicitly a solution vanishing on the boundary of D, then, as is known from undergraduate school, the solution is given by the formula

$$u(x, y) = \sum_{n,m=1}^{\infty} c_{nm} \cos(2n-1)x \, \cos(2m-1)y,$$

where

$$c_{nm} = 64\pi^{-4}(-1)^{n+m}(2n-1)^{-1}(2m-1)^{-1}[(2n-1)^2 + (2m-1)^2]^{-1}.$$

Now one may try to answer some simple questions about the solution like: Is it true that $u \geq 0$? Is u continuously differentiable in D and how many derivatives does it possess?

Looking at this formula, it is either impossible or very hard to answer these questions. Later, by using different means, we will see that indeed $u \geq 0$ and u is infinitely differentiable in D.

Thus, there are no explicit solutions in the general case and often, even if one can find an explicit formula, some fundamental properties of the solution are still hard to establish.

These are the main reasons why other approaches to studying partial differential equations were developed.

The goal of this chapter is to show how some simple and natural computations lead to the necessity of introducing Sobolev spaces and investigating their properties and then to an $\mathcal{L}_2$ theory of elliptic equations. The general scheme set out in this chapter will be used a few times in the future.

1. The simplest equation $\lambda u - \Delta u = f$

To explain how and why Sobolev spaces naturally appear, we start with investigating the solvability of the simplest equation

$$\lambda u - \Delta u = f \tag{1}$$

in $\mathbb{R}^d$ with an "arbitrary" right-hand side f and fixed $\lambda > 0$.

Notice a few simple properties of equation (1).

1. Lemma. *Let $u \in C_0^2$ be a solution of* (1) *in $\mathbb{R}^d$. Then*

$$\lambda^2 \|u\|^2_{\mathcal{L}_2} + 2\lambda \sum_{j=1}^d \|u_{x^j}\|^2_{\mathcal{L}_2} + \sum_{j,k=1}^d \|u_{x^j x^k}\|^2_{\mathcal{L}_2} = \|f\|^2_{\mathcal{L}_2}. \tag{2}$$

Proof. We use some properties of the Fourier transform, which is defined by

$$F(u)(\xi) = \tilde{u}(\xi) = c_d \int_{\mathbb{R}^d} e^{-i\xi\cdot x} u(x)\,dx, \quad c_d = (2\pi)^{-d/2}.$$

By using integration by parts, one easily proves that

$$i\xi^k \tilde{u}(\xi) = c_d \int_{\mathbb{R}^d} e^{-i\xi\cdot x} u_{x^k}(x)\,dx, \quad -\xi^k \xi^l \tilde{u}(\xi) = c_d \int_{\mathbb{R}^d} e^{-i\xi\cdot x} u_{x^k x^l}(x)\,dx.$$

Hence, (1) implies that $\tilde f = (\lambda + |\xi|^2)\tilde u$ and $|\tilde f|^2 = (\lambda + |\xi|^2)^2|\tilde u|^2$. The latter means that

$$\lambda^2|\tilde u(\xi)|^2 + 2\lambda\sum_{j=1}^{d}|i\xi^j\tilde u(\xi)|^2 + \sum_{j,k=1}^{d}|-\xi^j\xi^k\tilde u(\xi)|^2 = |\tilde f(\xi)|^2.$$

By integrating this over $\mathbb{R}^d$ and using Parseval's identity ($\|\tilde g\|_{\mathcal{L}_2} = \|g\|_{\mathcal{L}_2}$), we arrive at (2). The lemma is proved.

2. Exercise. Prove (2) by raising (1) to the second power and integrating by parts. (Be careful: u is only in C_0^2.)

The following exercise will be mentioned in the hint to Exercise 4.8, that concerns the solvability of elliptic equations with measurable coefficients in $\mathbb{R}^2$.

3. Exercise. By using the Fourier transform, prove that, if $d = 2$ and $u \in C_0^2$, then

$$\int_{\mathbb{R}^d}\det u_{xx}\,dx = 0.$$

It might be of interest that this result holds in any dimension (see Exercise 2.1).

4. Exercise*. Let $\ell_1, ..., \ell_d$ form an orthonormal basis in $\mathbb{R}^d$ and let u be a twice continuously differentiable function. Prove that

$$\Delta u = \sum_{k,i,j} u_{x^ix^j}\ell_k^i\ell_k^j,$$

which is to say that the operator Δ is invariant under orthogonal transformations.

5. Exercise. Let B be the open ball of radius one centered at the origin and let u be a three times continuously differentiable function in $\bar B$. Assume that $u = 0$ on the boundary ∂B of B and prove that

$$\int_B(|\Delta u|^2 - \sum_{i,j}|u_{x^ix^j}|^2)\,dx = (d-1)\int_{\partial B}|u_x\cdot n|^2\,dS,$$

where n is the unit outer normal to ∂B ($n(x) = x$) and dS is the element of the surface measure.

In this chapter we use the following theorem only for $p = 2$ and for this case a different proof is outlined in Exercise 12.

6. Theorem. *Let $\lambda > 0$. Then the set $L := (\lambda - \Delta)C_0^\infty$ is everywhere dense in $\mathcal{L}_p$ for any $p \in [1,\infty)$.*

The proof is based on the following.

7. Lemma. *Let $\lambda > 0$ and let u be a bounded from above twice continuously differentiable function on $\mathbb{R}^d$ satisfying*

$$\Delta u - \lambda u \geq 0$$

in $\mathbb{R}^d$. Then $u \leq 0$. In particular, if u is bounded and $\Delta u - \lambda u = 0$, then $\pm u \leq 0$, so that $u \equiv 0$.

Proof. Assume that $u > 0$ at some points. Set $\zeta(x) = \cosh(\varepsilon|x|)$, where $\varepsilon > 0$ is a small constant. Bearing in mind Taylor's series, one easily proves that ζ is infinitely differentiable. Next the function $v = u/\zeta$ satisfies $(\Delta - \lambda)(\zeta v) \geq 0$, that is,

$$\zeta\Delta v + 2\zeta_{x^j} v_{x^j} + cv \geq 0, \tag{3}$$

where

$$c = \Delta\zeta - \lambda\zeta = \zeta\Big\{\varepsilon^2 + (d-1)|x|^{-1}\varepsilon \tanh(\varepsilon|x|) - \lambda\Big\}.$$

Since $\tanh|x| \leq |x|$, we have $c < 0$ for all small ε (say $\varepsilon^2 < \lambda/d$). Actually, the above computation makes perfect sense only if $x \neq 0$. But the conclusion that $c = \Delta\zeta - \lambda\zeta < 0$ is valid for all x due to the continuity of ζ and its derivatives.

By assumption, $v > 0$ at some points and the positive part of v tends to zero as $|x| \to \infty$. Therefore, v attains its maximum value at a point $x_0 \in \mathbb{R}^d$ and $v(x_0) > 0$. The first derivatives of v vanish at x_0 and the pure second-order derivatives are nonpositive at this point. This definitely contradicts (3) since $v(x_0) > 0$. We have the desired contradiction and the lemma is proved.

8. Exercise. Give an example of unbounded twice continuously differentiable function $u > 0$ such that $\Delta u - u \geq 0$ in $\mathbb{R}^d$.

9. Remark. In the case that $\lambda = 0$ it turns out that if u is a bounded from above twice continuously differentiable function on $\mathbb{R}^d$ satisfying $\Delta u = 0$ in $\mathbb{R}^d$, then u is a constant. The same is true if we replace $\Delta u = 0$ with $\Delta u \geq 0$ and assume that $d = 1$ or $d = 2$. However if $d \geq 3$, there are smooth bounded functions u such that $\Delta u \geq 0$.

Sometimes one needs the first part of the following version of Lemma 7 in the form of *the maximum principle*. Recall that

$$\mathbb{R}^d_+ = \{x = (x^1, x') : x^1 > 0, x' \in \mathbb{R}^{d-1}\}.$$

10. Exercise. Let Ω be a bounded or unbounded domain in $\mathbb{R}^d$ and let u be bounded and continuous in $\bar{\Omega}$ and twice continuously differentiable in Ω. Let $\Delta u - \lambda u \geq 0$ in Ω, where the constant $\lambda > 0$. Prove that under these conditions

$$u \leq \sup_{\partial\Omega} u_+ \quad \text{in} \quad \bar{\Omega} \quad (\sup_{\emptyset} ... := 0).$$

Also prove Hopf's lemma: For $\Omega = \mathbb{R}^d_+$ under the above conditions, if $x_0 \in \partial\Omega$ is such that $u(x_0) = \sup_{\partial\Omega} u_+ > 0$ and $u_{x^1}(x_0)$ exists, then $u_{x^1}(x_0) \leq -\sqrt{\lambda} u(x_0) < 0$.

The following exercise can be used in investigating uniqueness of solutions to some boundary-value problems in $\mathbb{R}^d_+$.

11. Exercise. Let $\Omega = \mathbb{R}^d_+$, a constant $\lambda \geq 0$, and let u be a bounded and twice continuously differentiable function given on $\bar{\Omega}$. Assume that $\Delta u - u \geq 0$ in Ω and

$$u_{x^1} + \sum_{j \geq 2} \beta^j u_{x^j} + \sum_{j,k \geq 2} \alpha^{jk} u_{x^j x^k} - \lambda u \geq 0$$

on $\partial\Omega$, where β^j and α^{jk} are constant and (α^{jk}) is a nonnegative symmetric matrix. Prove that $u \leq 0$ in $\bar{\Omega}$.

Proof of Theorem 6. If the assertion is wrong, then by the Hahn-Banach theorem there is a linear functional on $\mathcal{L}_p$ vanishing on $(\lambda - \Delta)C_0^\infty$. Then by Riesz's representation theorem there is a function $g \in \mathcal{L}_q$ with $q = p/(p-1)$ and $\|g\|_{\mathcal{L}_q} \neq 0$ such that

$$\int_{\mathbb{R}^d} g(x)(\lambda u(x) - \Delta u(x))\, dx = 0$$

for all $u \in C_0^\infty$. Since, for any $y \in \mathbb{R}^d$, the function $u(y - x)$ belongs to C_0^∞, we have

$$\int_{\mathbb{R}^d} g(x)(\lambda u(y - x) - \Delta u(y - x))\, dx = 0 \tag{4}$$

for all $u \in C_0^\infty$ and $y \in \mathbb{R}^d$. Here the left-hand side happens to be

$$(\lambda - \Delta)(g * u)(y),$$

which follows from the well-known rules of differentiating integrals with respect to parameters. These rules also imply that $g * u$ is infinitely differentiable for any locally integrable g, in particular, for $g \in \mathcal{L}_q$. For $g \in \mathcal{L}_q$ by using Hölder's inequality, we see that $g * u$ is bounded. Thus, by Lemma 7 we conclude $g * u = 0$ and

$$\int_{\mathbb{R}^d} gu\, dx = 0 \tag{5}$$

for any $u \in C_0^\infty$. Finally, we use the well-known fact from integration theory that if g is locally integrable and (5) holds for any $u \in C_0^\infty$, then $g = 0$ (a.e.). This is the desired contradiction and the theorem is proved.

Now imagine that we have to solve (1) with an $f \in \mathcal{L}_2$. Then one can proceed as follows. Since $\bar{L} = \mathcal{L}_2$ (Theorem 6), there is a sequence $u^n \in C_0^2$ such that, for

$$f^n := \lambda u^n - \Delta u^n \tag{6}$$

we have $\|f - f^n\|_{\mathcal{L}_2} \to 0$. Furthermore (6) and Lemma 1 imply that

$$\lambda^2 \|u^n - u^m\|_{\mathcal{L}_2}^2 + 2\lambda \sum_{j=1}^d \|u^n_{x^j} - u^m_{x^j}\|_{\mathcal{L}_2}^2$$
$$+ \sum_{j,k=1}^d \|u^n_{x^j x^k} - u^m_{x^j x^k}\|_{\mathcal{L}_2}^2 = \|f^n - f^m\|_{\mathcal{L}_2}^2.$$

Since the sequence f_n converges in $\mathcal{L}_2$, it is a Cauchy sequence:

$$\|f^n - f^m\|_{\mathcal{L}_2} \to 0$$

as $n, m \to \infty$. It follows that $u^n, u^n_{x^j}, u^n_{x^j x^k}$ are also Cauchy sequences in $\mathcal{L}_2$. The completeness of $\mathcal{L}_2$ implies that there exist the $\mathcal{L}_2$ limits

$$v = \lim_{n\to\infty} u^n, \quad v_j = \lim_{n\to\infty} u^n_{x^j}, \quad v_{jk} = \lim_{n\to\infty} u^n_{x^j x^k}.$$

Now set by definition

$$v_{x^j x^k} = v_{jk}$$

putting aside for a while the justification of that notation. Then, from (6) we conclude that (a.e.)

$$\lambda v - \Delta v = f. \tag{7}$$

This is a natural way to come to the necessity of introducing Sobolev spaces and generalized solutions of (7).

12. Exercise. The above proof of Theorem 6 uses Lemma 7, the proof of which is based on considering maximum points of certain functions. This way does not work for higher-order or nonelliptic operators with constant coefficients. In connection with this take $p = 2$ in Theorem 6. Use the fact that the Fourier transform can be defined preserving Parseval's identity for all $u \in \mathcal{L}_2$ as the limit in $\mathcal{L}_2$ of $\tilde{u}_n$, where $u_n \in C_0^\infty$ and $u_n \to u$ in $\mathcal{L}_2$.

(i) Prove that if $f \in \mathcal{L}_2$ and $h \in \mathcal{L}_1$, then $f * h \in \mathcal{L}_2$ and $c_d F(f * h) = \tilde{f}\tilde{h}$.

(ii) Derive that the left-hand side in (4) belongs to $\mathcal{L}_2$ and its Fourier transform is

$$c_d^{-1}\tilde{g}(\lambda+|\xi|^2)\tilde{u}.$$

Conclude that $\tilde{g}=0$, $g=0$.

13. Exercise. Let $m\geq 1$ be an integer and let a^α be some (complex) numbers, not all of which are zero, given for any multi-indices α such that $|\alpha|\leq m$. Consider the operator

$$L=\sum_{|\alpha|\leq m}a^\alpha D^\alpha$$

and prove that the set LC_0^∞ is everywhere dense in $\mathcal{L}_p$ for any $p\in[2,\infty)$. You may like to first prove the following fact: If v is an infinitely differentiable function such that v and each of its derivatives of any order are bounded and belong to $\mathcal{L}_2$, then $F(D^\alpha v)(\xi)=i^{|\alpha|}\xi^\alpha\tilde{v}(\xi)$ for any multi-index α.

14. Remark. As is easy to see from its proof, Lemma 1 remains valid for any $\lambda\in\mathbb{R}$. The same is true for Theorem 6 at least if $p\geq 2$ as is seen from Exercise 12 for $p=2$ and from Exercise 13 for more general p.

2. Integrating the determinants of Hessians (optional)

The following three exercises do not have much to do with the main subject of these lectures. They are related to the remarkable properties of Jacobians and two very powerful tools often used in the theory of *nonlinear* PDEs. The author could not resist the temptation of discussing them once Exercise 1.3 has been proposed.

1. Exercise. Let Ω be a connected bounded domain with smooth boundary and let $F,G:\Omega\to\mathbb{R}^d$ be $C^1(\bar{\Omega})$ mappings such that

$$F=G\quad\text{on}\quad\partial\Omega.$$

Observe that for large $t>0$ the mappings $F_t=F(x)+tx$ and $G_t=G(x)+tx$ are one-to-one, map $\partial\Omega$ into the boundary of $F_t(\Omega)$ and $G_t(\Omega)$, and therefore

$$\operatorname{Vol}F_t(\Omega)=\operatorname{Vol}G_t(\Omega).$$

Express this equality in terms of $\partial F_t/\partial x$ and $\partial G_t/\partial x$, then use that the determinants of these matrices are polynomials in t, and conclude that

$$\int_\Omega\det\frac{\partial F}{\partial x}\,dx=\int_\Omega\det\frac{\partial G}{\partial x}\,dx.$$

For Ω being a large ball, $F=u_x$, and $G=0$, this exercise explains without computations why the result of Exercise 1.3 is true.

2. Exercise. For the domain from the preceding exercise show that there is no $C^1(\bar{\Omega})$ function $G:\bar{\Omega}\to\partial\Omega$ such that $G(x)=x$ on $\partial\Omega$.

Extend the result to $C(\bar{\Omega})$ functions G if $\Omega=B_1:=\{x\in\mathbb{R}^d:|x|<1\}$.

3. Exercise (Brouwer's fixed point theorem)**.** Let $f : \bar{B}_1 \to \bar{B}_1$ be a continuous mapping. We suggest the reader prove that f has fixed points in $\bar{B}_1$ (where $f(x) = x$) by using Exercise 2. You may like to start by assuming the contrary and on the basis of considering the line passing through x and $f(x)$, construct a mapping G as in Exercise 2.

Upon mapping each convex closed bounded set onto $\bar{B}_1$, extend the result to all continuous mappings of convex closed bounded sets.

4. Exercise (Fan Ky minimax theorem)**.** Let X, Y be closed bounded convex subsets of $\mathbb{R}^d$ and $f(x,y)$ a real-valued function defined on $X \times Y$ such that

(i) $f(x,y)$ is concave with respect to $y \in Y$ for each $x \in X$,

(ii) $f(x,y)$ is convex with respect to $x \in X$ for each $y \in Y$.

Prove that then

$$\min_{x\in X}\max_{y\in Y} f(x,y) = \max_{y\in Y}\min_{x\in X} f(x,y). \tag{1}$$

3. Sobolev spaces $W_p^k(\Omega)$

Recall that if Ω is a domain (open set) in $\mathbb{R}^d$ and $k \in \{0,1,2,...\}$, we denote by $C^k(\bar{\Omega})$ the subset of $C^k(\Omega)$ consisting of functions u such that u and $D^\alpha u$ extend to functions continuous in $\bar{\Omega}$ (the closure of Ω) whenever $|\alpha| \le k$. For these extensions we keep the same notation u and $D^\alpha u$, respectively.

1. Definition. Let $p \in [1,\infty)$, $k \in \{0,1,2,...\}$ and let Ω be a domain in $\mathbb{R}^d$. For functions $u \in C^k(\bar{\Omega})$ define

$$\|u\|_{W_p^k(\Omega)} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{\mathcal{L}_p(\Omega)}.$$

For a function $u \in \mathcal{L}_p(\Omega)$ we write $u \in W_p^k(\Omega)$ if there exists a sequence $u^n \in C^k(\bar{\Omega})$ such that $\|u^n - u\|_{\mathcal{L}_p(\Omega)} \to 0$,

$$\|u^n\|_{W_p^k(\Omega)} < \infty, \quad \|D^\alpha u^n - D^\alpha u^m\|_{\mathcal{L}_p(\Omega)} \to 0 \quad \forall |\alpha| \le k$$

as $n, m \to \infty$. We call any such sequence u^n *a defining sequence for* u. Denote $W_p^k = W_p^k(\mathbb{R}^d)$. The spaces $W_p^k(\Omega)$ are called *Sobolev spaces*.

Obviously the function v constructed before (1.7) belongs to W_2^2.

2. Definition. If $u \in W_p^k(\Omega)$ and u^n is its defining sequence, define *the generalized derivatives* $D^\alpha u$ for multi-indices α with $|\alpha| \le k$ by

$$D^\alpha u = \mathcal{L}_p\text{-}\lim_{n\to\infty} D^\alpha u^n. \tag{1}$$

Of course, we have to show that Definition 2 makes sense.

3. Lemma. *If $u \in W^k_p(\Omega)$ and $|\alpha| \leq k$, then $D^\alpha u$ exists and is independent of the choice of defining sequence.*

Proof. Since $D^\alpha u^n$ is a Cauchy sequence in $\mathcal{L}_p(\Omega)$, the limit in (1) exists. To prove that it is unique, we take a test function $\phi \in C^\infty_0(\Omega)$ (that is, ϕ is an infinitely differentiable function with compact support in Ω) and notice that integrating by parts yields

$$\int_\Omega \phi D^\alpha u^n \, dx = (-1)^{|\alpha|} \int_\Omega u^n D^\alpha \phi \, dx.$$

We pass to the limit noticing that by Hölder's inequality, if $g^n \to g$ in $\mathcal{L}_p(\Omega)$ and $h \in \mathcal{L}_q(\Omega)$ with $q = p/(p-1)$, then

$$\Big| \int_\Omega g^n h \, dx - \int_\Omega gh \, dx \Big| \leq \|g^n - g\|_{\mathcal{L}_p(\Omega)} \|h\|_{\mathcal{L}_q(\Omega)} \to 0.$$

Then by virtue of (1) for any defining sequence, we have

$$\int_\Omega \phi D^\alpha u \, dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi \, dx. \tag{2}$$

Now if u^n_1 and u^n_2 are two defining sequences for u and v_1 and v_2 are the corresponding right-hand sides of (1), then by (2)

$$\int_\Omega \phi v_1 \, dx = \int_\Omega \phi v_2 \, dx, \quad \int_\Omega \phi(v_1 - v_2) \, dx = 0$$

for all $\phi \in C^\infty_0(\Omega)$. Since $v_1 - v_2 \in \mathcal{L}_p$, it follows that $v_1 - v_2 = 0$ (a.e.), which is exactly what is asserted. The lemma is proved.

The last part of the above proof shows that the following definition is unambiguous.

4. Definition. Let α be a multi-index and let v and h be locally integrable functions on Ω such that

$$\int_\Omega \phi h \, dx = (-1)^{|\alpha|} \int_\Omega v D^\alpha \phi \, dx \tag{3}$$

for all $\phi \in C^\infty_0(\Omega)$. Then we call the function h the D^α *derivative of v in the sense of distributions*, or the D^α *Sobolev derivative*, or else the D^α *generalized derivative* and write $h = D^\alpha v$.

Observe that Sobolev derivatives are the usual locally integrable functions and not just distributions.

The fact that (2) holds for any $\phi \in C_0^\infty(\Omega)$ means that $D^\alpha u$ is the derivative of u in the sense of distributions and the notions of generalized derivatives introduced in Definitions 2 and 4 agree if $u \in W^k_p(\Omega)$. It is also worth noting that as we have seen, (3) defines $D^\alpha v$ uniquely (up to almost everywhere).

Since for any smooth function u the usual derivative $D^\alpha u$ satisfies (2), we have the following.

5. Corollary. *If u is a k times continuously differentiable function and $u \in W^k_p(\Omega)$, then, for any multi-index α with $|\alpha| \le k$, the usual derivative $D^\alpha u$ coincides with the Sobolev one (a.e.).*

In the following exercise and many times in the future we use local versions of various function spaces. Unless specified otherwise, they are introduced in the same way as, say, $W^k_{p,loc}(\Omega)$, which is the space of functions u on Ω such that $u\zeta \in W^k_p(\Omega)$ for any $\zeta \in C_0^\infty(\Omega)$. In particular, $C^\infty_{loc}(\Omega)$ is the set of functions u defined on Ω and such that $u\zeta \in C_0^\infty(\Omega)$ for any $\zeta \in C_0^\infty(\Omega)$.

6. Exercise. Let $g \in \mathcal{L}_p(\Omega)$, $\phi_n \in C^\infty_{loc}(\Omega)$,

$$\|\phi_n - g\|_{\mathcal{L}_p(\Omega)} \to 0 \quad \text{and} \quad \|D_1\phi_n - D_1\phi_m\|_{\mathcal{L}_p(\Omega)} \to 0$$

as $n, m \to \infty$. Prove that the generalized derivative $D_1 g$ belongs to $\mathcal{L}_p(\Omega)$ and $\|D_1\phi_n - D_1 g\|_{\mathcal{L}_p(\Omega)} \to 0$. (Warning: The function g need not belong to $W^1_p(\Omega)$.)

7. Definition. For $u \in W^k_p(\Omega)$ define

$$\|u\|_{W^k_p(\Omega)} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{\mathcal{L}_p(\Omega)}, \quad [u]_{W^k_p(\Omega)} = \sum_{|\alpha|=k} \|D^\alpha u\|_{\mathcal{L}_p(\Omega)}.$$

Notice that if u^n is a defining sequence for $u \in W^k_p(\Omega)$, then

$$\|u - u^n\|_{W^k_p(\Omega)} \to 0, \quad \|u^n\|_{W^k_p(\Omega)} \to \|u\|_{W^k_p(\Omega)}.$$

8. Exercise*. Let $u \in W^k_p(\Omega)$ and $|\alpha| + |\beta| \le k$. Prove that

$$D^\alpha u \in W^{k-|\alpha|}_p(\Omega), \quad D^\beta(D^\alpha u) = D^{\alpha+\beta}u, \quad \|D^\alpha u\|_{W^{k-|\alpha|}_p(\Omega)} \le \|u\|_{W^k_p(\Omega)}.$$

9. Exercise*. Let $u \in W^k_p(\Omega)$, $a \in C^n(\bar\Omega)$, and $|\alpha| \le k - n$. Prove that $aD^\alpha u \in W^n_p(\Omega)$ and

$$\|aD^\alpha u\|_{W^n_p(\Omega)} \le N(d,n)\|a\|_{C^n(\bar\Omega)}\|u\|_{W^k_p(\Omega)}.$$

10. Exercise*. Let $\psi : \Omega \to \Omega'$ be a one-to-one mapping of a domain $\Omega \subset \mathbb{R}^d$ onto a domain $\Omega' \subset \mathbb{R}^d$. Assume that $k \geq 1$, $\psi \in C^k(\bar{\Omega})$, and $\psi^{-1} \in C^k(\bar{\Omega}')$ and prove that

$$u \in W^k_p(\Omega) \Longleftrightarrow u(\psi^{-1}(\cdot)) \in W^k_p(\Omega').$$

We remind the reader that by $C^k_0(\bar{\Omega})$ we mean the subset of $C^k(\bar{\Omega})$ consisting of all functions u vanishing for $x \in \Omega$ with sufficiently large $|x|$. This notation will be used most often for $\Omega = \mathbb{R}^d$ and $\Omega = \mathbb{R}^d_+$, where

$$\mathbb{R}^d_+ = \{(x^1, x') : x^1 > 0, x' = (x^2, ..., x^d) \in \mathbb{R}^{d-1}\}$$

with an obvious modification of this notation if $d = 1$. Assertion (ii) of the following theorem will be improved in Theorem 8.5 (iii).

11. Theorem. *(i) The space $W^k_p(\Omega)$ with norm $\|\cdot\|_{W^k_p(\Omega)}$ is a Banach space. (ii) Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$. Then the set $C^k_0(\bar{\Omega})$ is dense in $W^k_p(\Omega)$.*

Proof. (i) As usual we need only prove that $W^k_p(\Omega)$ is complete. Let u^n be a Cauchy sequence in $W^k_p(\Omega)$ and, for each n, let v^{nj} be defining sequences for u^n. Then by definition

$$\|u^n - v^{nj}\|_{W^k_p(\Omega)} \to 0$$

as $j \to \infty$. Hence, for each n, there exists $j(n)$ such that

$$\|u^n - v^n\|_{W^k_p(\Omega)} \leq 1/n,$$

where $v^n = v^{nj(n)}$.

Observe that

$$\|v^n - v^m\|_{W^k_p(\Omega)} \leq \|v^n - u^n\|_{W^k_p(\Omega)} + \|u^n - u^m\|_{W^k_p(\Omega)}$$

$$+ \|u^m - v^m\|_{W^k_p(\Omega)} \leq 1/n + 1/m + \|u^n - u^m\|_{W^k_p(\Omega)} \to 0 \qquad (4)$$

as $n, m \to \infty$. In particular,

$$\|v^n - v^m\|_{\mathcal{L}_p(\Omega)} \to 0$$

and there is a function $v \in \mathcal{L}_p(\Omega)$ such that $v^n \to v$ in $\mathcal{L}_p(\Omega)$. This and (4) mean by definition that $v \in W^k_p(\Omega)$. Furthermore, $v^n \to v$ in $W^k_p(\Omega)$ by the definition of $D^\alpha v$, $|\alpha| \leq k$, and

$$\lim_{n\to\infty} \|v - u^n\|_{W^k_p(\Omega)} \leq \lim_{n\to\infty} (\|v - v^n\|_{W^k_p(\Omega)} + 1/n) = 0.$$

This proves (i).

Since $C^k(\bar\Omega)\cap W_p^k(\Omega)$ is dense in $W_p^k(\Omega)$ by definition, to prove (ii), it suffices to show that any

$$u\in C^k(\bar\Omega)\cap W_p^k(\Omega)$$

can be approximated in W_p^k by elements of $C_0^k(\bar\Omega)$. Take any $\zeta\in C_0^\infty$ such that $\zeta(x)=1$ for $|x|\le 1$ and let $u^n(x)=u(x)\zeta(x/n)$. Notice that, by the Leibnitz rule for $|\alpha|\le k$,

$$D^\alpha u^n(x)=\zeta(x/n)D^\alpha u(x)+\sum_{\substack{|\beta+|\gamma|=|\alpha|\\|\gamma|\ge 1}}c^{\beta\gamma}(D^\beta u(x))n^{-|\gamma|}(D^\gamma\zeta)(x/n),$$

where $c^{\beta\gamma}$ are certain constants. Hence by the triangle inequality and the dominated convergence theorem, for a constant N independent of n,

$$\|D^\alpha u^n-D^\alpha u\|_{\mathcal{L}_p(\Omega)}\le\|(1-\zeta(\cdot/n))D^\alpha u\|_{\mathcal{L}_p(\Omega)}+Nn^{-1}\|u\|_{W_p^k(\Omega)},$$

$$\|(1-\zeta(\cdot/n))D^\alpha u\|_{\mathcal{L}_p(\Omega)}^p=\int_\Omega|1-\zeta(x/n)|^p|D^\alpha u(x)|^p\,dx\to 0.$$

It follows that $u^n\to u$ in $W_p^k(\Omega)$. The theorem is proved.

12. Exercise. Take $d=1$ and prove that $C_0^\infty((0,1))$ is *not* dense in $W_2^1((0,1))$.

13. Exercise*. Let u^n be a Cauchy sequence in $W_p^k(\Omega)$, $u\in\mathcal{L}_p(\Omega)$ and let $u^n\to u$ in $\mathcal{L}_p(\Omega)$. Prove that $u\in W_p^k(\Omega)$ and $u^n\to u$ in $W_p^k(\Omega)$.

A version of the following corollary of Theorem 11 (i) will be stated in Corollary 8.1.3, which in turn is used in the hint to Exercise 11.1.10.

14. Corollary. *Let $1<p<\infty$, $k\in\{0,1,2,...\}$, and let U be a bounded subset of $W_p^k(\Omega)$. Then for any sequence $u_n\in U$ there exist a subsequence $u_{n'}$ and a function $u\in W_p^k(\Omega)$ such that the sequence $D^\alpha u_{n'}$ converges weakly in $\mathcal{L}_p(\Omega)$ to $D^\alpha u$ whenever $|\alpha|\le k$.*

Proof. Let $\mathcal{L}_p(\Omega,\mathbb{R}^m)$ be the space of $\mathbb{R}^m$-valued functions with natural norm. Take $m=1+d+...+d^k$ and observe that by Theorem 11 (i) the set

$$M:=\{(D^\alpha u,|\alpha|\le k):u\in W_p^k(\Omega)\}$$

is a closed linear subset of $\mathcal{L}_p(\Omega,\mathbb{R}^m)$. Then recall that, if F is a bounded set in $\mathcal{L}_p(\Omega,\mathbb{R}^m)$ and $f_n\in F$, $n=1,2,...$, then there exists a subsequence $f_{n'}$ and $f\in\mathcal{L}_p(\Omega,\mathbb{R}^m)$ such that $f_{n'}\to f$ weakly in $\mathcal{L}_p(\Omega,\mathbb{R}^m)$. We also know that linear subspaces of Banach spaces are closed in the weak topology.

These three facts combined imply that there is a subsequence

$$(D^\alpha u_{n'}, |\alpha| \le k) \in \{(D^\alpha v, |\alpha| \le k) : v \in U\}$$

converging weakly in $\mathcal{L}_p(\Omega, \mathbb{R}^m)$ to a $(D^\alpha u, |\alpha| \le k) \in M$. This is exactly what is asserted.

15. Remark. For $1 < p < \infty$, the set M in the above proof is actually isomorphic to $W_p^k(\Omega)$ and as a linear subspace of a reflexive space it is reflexive itself. It follows that $W_p^k(\Omega)$ is reflexive too. This observation is not used in the future.

Our discussion before (1.7) proves the existence part in the following result in which $\lambda > 0$ is a fixed number.

16. Theorem. *For any $f \in \mathcal{L}_2$ there exists a unique $u \in W_2^2$ such that $\lambda u - \Delta u = f$ (a.e.). In addition, for any $u \in W_2^2$ we have*

$$\lambda^2 \|u\|_{\mathcal{L}_2}^2 + 2\lambda \sum_{j=1}^d \|u_{x^j}\|_{\mathcal{L}_2}^2 + \sum_{j,k=1}^d \|u_{x^j x^k}\|_{\mathcal{L}_2}^2 = \|\lambda u - \Delta u\|_{\mathcal{L}_2}^2, \tag{5}$$

where by continuity one can take $\lambda = 0$ as well.

The uniqueness follows from (5), which in turn follows from Lemma 1.1 and Theorem 11 (ii).

17. Exercise. Let $\Omega = \mathbb{R}_+^d$, $u \in W_p^1(\Omega)$, $j \ge 2$, and $\phi \in C_0^\infty$ (not $C_0^\infty(\Omega)$). Prove that

$$\int_\Omega u D_j \phi \, dx = -\int_\Omega \phi D_j u \, dx.$$

The following exercise is used in the proof of Theorem 10.5.1.

18. Exercise*. Prove that if $u \in W_p^1(\Omega)$ and $p \ge 1$, then $u_\pm, |u| \in W_p^1(\Omega)$ and

$$(u_+)_x = u_x I_{u>0}, \quad (|u|)_x = u_x \operatorname{sign} u$$

(a.e.). Conclude that if $u, v \in W_p^1(\Omega)$, then $\max(u, v) \in W_p^1(\Omega)$.

19. Exercise. Prove that if $u \in W_p^1(\Omega)$, then $u_x I_{u=0} = 0$ (a.e.). Conclude that $u_x I_{u=c} = 0$ (a.e.) for any constant c.

20. Exercise*. Prove that if $u \in W_p^k(\mathbb{R}_+^d)$, then for almost any $x^1 > 0$ we have

$$u(x^1, \cdot) \in W_p^k(\mathbb{R}^{d-1}).$$

21. Exercise*. By using defining sequences, prove that if $u \in W^k_p(\mathbb{R}^d_+)$, α is a multi-index with $\alpha_1 = 0$, and $v(x) := u(|x^1|, x')$, then

$$D^\alpha v(x) = (D^\alpha u)(|x^1|, x')$$

in $\mathbb{R}^d$, where $D^\alpha v$ is the derivative in the sense of Definition 4.

22. Exercise (Hard if $1 < p < 2$)**.** Prove that if $d = 1$, $u \in W^2_p\big((a,b)\big)$ and $p > 1$, then $|u|^p \in W^2_1\big((a,b)\big)$.

23. Exercise. (i) Let B be the open unit ball centered at the origin and let u be a twice continuously differentiable function on $\bar{B}$. Assume that $u = 0$ on ∂B. Set $f = \Delta u$ and prove that

$$\|u\|^2_{\mathcal{L}_2(B)} + \sum_i \|u_{x^i}\|^2_{\mathcal{L}_2(B)} \leq 4\|f\|^2_{\mathcal{L}_2(B)}.$$

(ii) Given an integer n, denote by P_n the set of polynomials of x of degree $\leq n$ and let A be the operator $A : P_n \to P_n$ given by the formula

$$Ap = \Delta[(1 - |x|^2)p].$$

Conclude from (i) that A is invertible.

(iii) Use (i), (ii), and Exercise 1.5 to show that for any $f \in \mathcal{L}_2(B)$ there is a function $u \in W^2_2(B)$ such that $\Delta u = f$ in B and

$$\|u\|^2_{\mathcal{L}_2(B)} + \sum_i \|u_{x^i}\|^2_{\mathcal{L}_2(B)} + \sum_{i,j} \|u_{x^i x^j}\|^2_{\mathcal{L}_2(B)} \leq 5\|f\|^2_{\mathcal{L}_2(B)}.$$

4. Second-order elliptic differential operators

In this section we show how to prove the solvability of general second-order elliptic differential equations in $\mathbb{R}^d$ by using the method of continuity and the method of a priori estimates both introduced by S.N. Bernstein in the beginning of the 20th century.

1. Definition. Let $a^{ij}(x), b^i(x), c(x)$ be *real-valued* measurable functions on $\mathbb{R}^d$ defined for $i, j = 1, ..., d$. Assume that $a^{ij} = a^{ji}$. The expression

$$L = a^{ij} D_{ij} + b^i D_i + c$$

is called a *second-order elliptic differential operator* if there is a constant $\kappa > 0$ called *a constant of ellipticity* such that, for all $x, \xi \in \mathbb{R}^d$, we have

$$\kappa^{-1}|\xi|^2 \geq a^{ij}(x)\xi^i\xi^j \geq \kappa|\xi|^2. \tag{1}$$

2. Exercise*. Derive from (1) and the symmetry of a^{ij} that the a^{ij} are bounded.

3. Exercise*. Assuming that a, b, c are bounded prove that $L : u \to Lu$ is a continuous operator from W^2_p to $\mathcal{L}_p$.

Now we present the method of continuity.

4. Theorem. *Let L be a second-order elliptic differential operator with bounded coefficients. Assume that there are constants $\lambda, N_0 \in (0, \infty)$ such that, for any $u \in C^2_0$ and $t \in [0, 1]$, we have*

$$\|u\|_{W^2_2} \leq N_0 \|L_t u\|_{\mathcal{L}_2}, \tag{2}$$

where

$$L_t = (1 - t)(\Delta - \lambda) + tL.$$

Then for any $f \in \mathcal{L}_2$ there exists a unique $u \in W^2_2$ such that $Lu = f$. Furthermore, (2) *holds for any $u \in W^2_2$ and $t \in [0, 1]$.*

Proof. That (2) holds for any $u \in W^2_2$ and $t \in [0, 1]$ follows directly from Exercise 3 and Theorem 3.11. The uniqueness in W^2_2 of the solution of $Lu = f$ and even $L_t u = f$ for any $t \in [0, 1]$ follows from (2).

To prove the existence, take a point $t \in [0, 1]$ and call it "good" if for any $f \in \mathcal{L}_2$ there exists a unique $u \in W^2_2$ such that $L_t u = f$. Let T be the set of all good points. Notice that $0 \in T$ by Theorem 3.16.

Obviously, we only need to prove that $1 \in T$ and to do that, it suffices to prove that there is an $\varepsilon > 0$ such that, for any $t_0 \in T$,

$$[t_0 - \varepsilon, t_0 + \varepsilon] \cap [0, 1] \subset T. \tag{3}$$

Fix a $t_0 \in T$ and $f \in \mathcal{L}_2$. For $u \in W^2_2$ and $t \in \mathbb{R}$ consider the equation

$$L_{t_0} v = f + (t_0 - t)(Lu - \Delta u + \lambda u). \tag{4}$$

Since $t_0 \in T$, there is a unique $v \in W^2_2$ satisfying equation (4). Introduce an operator Q_t by defining

$$v = Q_t u.$$

Then Q_t is an operator mapping W^2_2 into itself. It turns out that, if $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ and ε is chosen appropriately, then Q_t is a contraction in W^2_2.

Indeed, if $u^1, u^2 \in W^2_2$, then

$$L_{t_0} Q_t u^i = f + (t_0 - t)(Lu^i - \Delta u^i + \lambda u^i),$$

$$L_{t_0}(Q_t u^1 - Q_t u^2) = (t_0 - t)(L - \Delta + \lambda)(u^1 - u^2),$$

which owing to (2) implies that

$$\|Q_t u^1 - Q_t u^2\|_{W^2_2} \le N_0|t_0 - t| \cdot \|(L - \Delta + \lambda)(u^1 - u^2)\|_{\mathcal{L}_2}$$

$$\le N_0|t_0 - t|N_1\|u^1 - u^2\|_{W^2_2},$$

where N_1 is independent of u^i and t. By taking $\varepsilon = (2N_0N_1)^{-1}$, we get that Q_t is a 1/2 contraction in W^2_2 if $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. By Banach's fixed point theorem, for each such t, there exists a $u \in W^2_2$ such that $Q_t u = u$, which means

$$L_{t_0}u = f + (t_0 - t)(Lu - \Delta u + \lambda u), \quad L_t u = f.$$

Thus for any $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ and $f \in \mathcal{L}_2$ there is a solution $u \in W^2_2$ of $L_t u = f$. We know that the solution is unique if in addition $t \in [0, 1]$. This finishes proving (3) and the theorem.

5. Remark. One can restate assumption (2) in the following way. We require the estimate

$$\|u\|_{W^2_2} \le N\|f\|_{\mathcal{L}_2}$$

to hold whenever $u \in W^2_2$ is a solution of $L_t u = f$. Since the existence of such u is not known a priori, estimate (2) is usually called *an a priori* estimate.

6. Exercise. One could take $\lambda = 0$ in Theorem 4 if one knew that the equation $\Delta u = f$ is uniquely solvable in W^2_2 for any $f \in \mathcal{L}_2$. In connection with this prove that there is no finite constant N such that for any $u \in C^2_0$ we have $\|u\|_{\mathcal{L}_2} \le N\|\Delta u\|_{\mathcal{L}_2}$.

Theorem 4 reduces proving the solvability of elliptic equations to obtaining a priori estimates. In the future we will see that it is possible to obtain them if the coefficients of L are uniformly *continuous* or, more generally, are of class VMO. There is a remarkable exception when $d \le 2$. The following set of exercises is aimed at proving the a priori estimate of type (2) for $d = 2$ and *measurable* coefficients.

7. Exercise. Let $A = (a^{ij})$ and $U = (u_{ij})$ be 2×2 symmetric matrices. Assume that

$$\mu|\xi|^2 \le a^{ij}\xi^i\xi^j \le \nu|\xi|^2 \tag{5}$$

for all $\xi \in \mathbb{R}^2$, where $\mu > 0$ and $\nu > 0$ are some constants. Prove that

$$\frac{1}{2\mu^2}\Big(\sum_{i,j=1}^{2} a^{ij}u_{ij}\Big)^2 \ge \frac{\mu^2}{2\nu^2}\sum_{i,j=1}^{2} u^2_{ij} + \det U.$$

8. Exercise. Let $d = 2$, $a^{ij}(x)$ be measurable functions on $\mathbb{R}^2$ satisfying $a^{ij} = a^{ji}$ and condition (5) for all $x, \xi \in \mathbb{R}^2$, where $\mu > 0$ and $\nu > 0$ are some constants. For a $\lambda > 0$ define

$$Lu = L_\lambda u = a^{ij} u_{x^i x^j} - \lambda(a^{11} + a^{22})u.$$

Prove that, for any $u \in C_0^2$,

$$\lambda^2 \|u\|_{\mathcal{L}_2}^2 + 2\lambda \sum_{j=1}^{2} \|u_{x^j}\|_{\mathcal{L}_2}^2 + \sum_{j,k=1}^{2} \|u_{x^j x^k}\|_{\mathcal{L}_2}^2 \le \frac{\nu^2}{\mu^4} \|Lu\|_{\mathcal{L}_2}^2. \tag{6}$$

9. Exercise (Two-dimensional case)**.** Under the conditions of Exercise 8 prove that for any $f \in \mathcal{L}_2$ there is a unique $u \in W_2^2$ satisfying $Lu = f$.

The reader can find a continuation of this series in Exercise 6.7 where we add lower-order terms, Exercise 8.2.11 for equations in half spaces, Exercise 11.5.5 with $\lambda \mathrm{tr}\, a$ replaced with λ and equations in $\mathbb{R}^2$ for large λ or in smooth bounded domains with any $\lambda \ge 0$, Exercise 11.6.5 for equations in $\mathbb{R}^2$ with λ small, and Exercise 8.2.6 concerning the Neumann problem.

5. Multiplicative inequalities

Theorem 3.16 settles the issue of solvability of equations $\lambda u - \Delta u = f$ in W_2^2. To get prepared for treating more general elliptic equations by using the ideas from Section 4, we need the following multiplicative inequalities. Everywhere in this section

$$\Omega = \mathbb{R}^d \quad \text{or} \quad \Omega = \mathbb{R}^d_+.$$

1. Theorem. *For any $p \in [1, \infty)$ and $u \in W_p^2(\Omega)$ we have*

$$\|u_x\|_{\mathcal{L}_p(\Omega)} \le N \|u\|_{\mathcal{L}_p(\Omega)}^{1/2} \|u_{xx}\|_{\mathcal{L}_p(\Omega)}^{1/2}, \tag{1}$$

where N is independent of u.

Proof. By virtue of Theorem 3.11 we need only prove (1) for $u \in C_0^2(\bar{\Omega})$. Let $\zeta \in C_0^\infty(\mathbb{R})$ be a function of one variable with $\zeta(0) = 1$, $\zeta'(0) = 0$. Take $u \in C_0^2(\bar{\Omega})$ and denote by e_1 the first basis vector. Notice that

$$\big(u_{x^1}(x+te_1)\zeta(t)\big)_t = u_{x^1x^1}(x+te_1)\zeta(t) + \big(u(x+te_1)\zeta'(t)\big)_t - u(x+te_1)\zeta''(t),$$

$$u_{x^1}(x) = -\int_0^\infty u_{x^1x^1}(x+te_1)\zeta(t)\,dt + \int_0^\infty u(x+te_1)\zeta''(t)\,dt.$$

Hence by Minkowski's inequality

$$\|u_{x^1}\|_{\mathcal{L}_p(\Omega)} \leq \int_0^\infty (|\zeta(t)|+|\zeta''(t)|)\big(\|u_{xx}(\cdot+te_1)\|_{\mathcal{L}_p(\Omega)}+\|u(\cdot+te_1)\|_{\mathcal{L}_p(\Omega)}\big)\,dt,$$

where the right-hand side equals a constant times

$$\|u_{xx}\|_{\mathcal{L}_p(\Omega)} + \|u\|_{\mathcal{L}_p(\Omega)}$$

if $\Omega = \mathbb{R}^d$ and less than this quantity if $\Omega = \mathbb{R}^d_+$. Thus,

$$\|u_{x^1}\|_{\mathcal{L}_p(\Omega)} \leq N(\|u_{xx}\|_{\mathcal{L}_p(\Omega)} + \|u\|_{\mathcal{L}_p(\Omega)})$$

and the same estimate obviously also holds for u_{x^i} with any $i = 2, ..., d$.

Now, take a constant $c > 0$ and put $u(cx)$ in place of u in the just proved inequality:

$$\|u_x\|_{\mathcal{L}_p(\Omega)} \leq N(\|u_{xx}\|_{\mathcal{L}_p(\Omega)} + \|u\|_{\mathcal{L}_p(\Omega)}).$$

Then we get

$$c\|u_x\|_{\mathcal{L}_p(\Omega)} \leq N(c^2\|u_{xx}\|_{\mathcal{L}_p(\Omega)} + \|u\|_{\mathcal{L}_p(\Omega)}).$$

Upon dividing through by c and minimizing with respect to $c > 0$, we arrive at (1). The theorem is proved.

Quite often instead of (1) we need its corollary, which we actually have obtained in the end of the preceding proof.

2. Corollary. *For any $p \in [1, \infty)$ and $\varepsilon > 0$ there exists a constant N such that, if $u \in W^2_p(\Omega)$, then*

$$\|u_x\|_{\mathcal{L}_p(\Omega)} \leq \varepsilon\|u_{xx}\|_{\mathcal{L}_p(\Omega)} + N\|u\|_{\mathcal{L}_p(\Omega)}.$$

In addition, one can take $N = N_0\varepsilon^{-1}$, where N_0 is independent of ε.

3. Exercise. Show that if Ω is a bounded domain, then the assertion of Theorem 1 is wrong.

4. Corollary. *Let $b^1, ..., b^d$, and c be bounded measurable functions on $\mathbb{R}^d$. Then there exists $\lambda_0 \geq 1$, depending only on d and bounds of $b^1, ..., b^d$, and c, such that for any $\lambda \geq \lambda_0$ and any $f \in \mathcal{L}_2$ in W^2_2 there is a unique solution of the equation*

$$Lu - \lambda u := \Delta u + b^i u_{x^i} + cu - \lambda u = f.$$

Furthermore, for any $u \in W^2_2$ and $\lambda \geq \lambda_0$

$$\lambda\|u\|_{\mathcal{L}_2} + \lambda^{1/2}\|u_x\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2} \leq N\|Lu - \lambda u\|, \tag{2}$$

where the constant N depends only on d and the bounds of $b^1, ..., b^d$, and c.

Indeed, for $t \in [0,1]$ set

$$L_t u = \Delta u - \lambda u + t(b^i u_{x^i} + cu)$$
$$= (1-t)(\Delta - \lambda)u + t(\Delta u - \lambda u + b^i u_{x^i} + cu).$$

Then by Theorem 4.4 we only need to check that (2) holds with L_t in place of $L - \lambda$ for any $t \in [0,1]$. However, by Lemma 1.1

$$\lambda \|u\|_{\mathcal{L}_2} + \lambda^{1/2} \|u_x\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2} \leq N_1 \|\Delta u - \lambda u\|_{\mathcal{L}_2}$$
$$\leq N_1 \|L_t u\|_{\mathcal{L}_2} + N_2 \|u_x\|_{\mathcal{L}_2} + N_3 \|u\|_{\mathcal{L}_2},$$

where $N_1 = N_1(d)$ and N_2 and N_3 depend only on the bounds of $b^1, ..., b^d$, and c and d. By Corollary 2

$$N_2 \|u_x\|_{\mathcal{L}_2} \leq (1/2) \|u_{xx}\|_{\mathcal{L}_2} + N_4 \|u\|_{\mathcal{L}_2},$$

so that

$$(\lambda - N_3 - N_4) \|u\|_{\mathcal{L}_2} + \lambda^{1/2} \|u_x\|_{\mathcal{L}_2} + (1/2) \|u_{xx}\|_{\mathcal{L}_2} \leq N_1 \|L_t u - \lambda u\|_{\mathcal{L}_2}.$$

This implies our a priori estimate for $\lambda \geq 2(N_3 + N_4) + 1 =: \lambda_0$ since then $\lambda - N_3 - N_4 \geq (1/2)\lambda$.

5. Exercise. Let $d = 1$ and let L be a second-order elliptic operator with bounded measurable coefficients. Prove that there exists $\lambda_0 \geq 1$ such that for any $\lambda \geq \lambda_0$ and any $f \in \mathcal{L}_2$ in W^2_2 there is a unique solution of the equation $Lu - \lambda u = f$.

The following exercise is generalized in Exercise 13.3.20.

6. Exercise*. Prove by induction on n that for any $n \in \{0, 1, 2, ...\}$, $k \in \{0, 1, 2, ..., n\}$, and $u \in W^n_p(\Omega)$

$$[u]_{W^k_p(\Omega)} \leq N \|u\|^{1-\gamma}_{\mathcal{L}_p(\Omega)} [u]^{\gamma}_{W^n_p(\Omega)}, \tag{3}$$

where $\gamma = k/n$ and N is independent of u.

Exercise 6 and Young's inequality

$$a^{1-\gamma} b^{\gamma} \leq (1-\gamma)a + \gamma b, \quad a, b \geq 0, \gamma \in [0,1],$$

imply the following.

7. Corollary. *For any $p \in [1, \infty)$, $n \in \{0, 1, 2, ...\}$, and $u \in W_p^n(\Omega)$,*

$$\|u\|_{W_p^n(\Omega)} \leq N(\|u\|_{\mathcal{L}_p(\Omega)} + [u]_{W_p^n(\Omega)}), \tag{4}$$

where N is independent of u.

It also follows from (3) and (4) that

$$\|u\|_{W_p^k(\Omega)} \leq N(\|u\|_{\mathcal{L}_p(\Omega)} + [u]_{W_p^k(\Omega)})$$

$$\leq N(\|u\|_{\mathcal{L}_p(\Omega)} + \|u\|^{1-\gamma}_{\mathcal{L}_p(\Omega)} \|u\|^{\gamma}_{W_p^n(\Omega)}),$$

and since obviously

$$\|u\|_{\mathcal{L}_p(\Omega)} \leq \|u\|^{1-\gamma}_{\mathcal{L}_p(\Omega)} \|u\|^{\gamma}_{W_p^n(\Omega)},$$

we arrive at the first inequality in (5) below.

8. Corollary. *For any $p \in [1, \infty)$, $n \in \{0, 1, 2, ...\}$, $k \in \{0, 1, 2, ..., n\}$, $\varepsilon > 0$, and $u \in W_p^n(\Omega)$,*

$$\|u\|_{W_p^k(\Omega)} \leq N\|u\|^{1-\gamma}_{\mathcal{L}_p(\Omega)} \|u\|^{\gamma}_{W_p^n(\Omega)}$$

$$\leq N\varepsilon\|u\|_{W_p^n(\Omega)} + N\varepsilon^{-\gamma/(1-\gamma)}\|u\|_{\mathcal{L}_p(\Omega)}, \tag{5}$$

where $\gamma = k/n$ and N are independent of u and ε.

The second inequality for $\gamma = 1$ and $\varepsilon < 1$ is trivial because the right-hand side is infinite unless $u = 0$. For $\gamma = 1$ and $\varepsilon \geq 1$ it is also trivial and in this case we see that one cannot replace $N\varepsilon$ with another ε. If $\gamma \in [0, 1)$, the second inequality follows from Young's inequality and the observation that

$$Na^{1-\gamma}b^{\gamma} = (N^{1/(1-\gamma)}\varepsilon^{-\gamma/(1-\gamma)}a)^{1-\gamma}(\varepsilon b)^{\gamma}, \quad \gamma \in [0, 1), \varepsilon > 0.$$

6. Solvability of elliptic equations with continuous coefficients

We deal with an operator L given on functions defined in $\mathbb{R}^d$. For convenience of future references we collect what we need in the following.

1. Assumption. The operator L is a second-order elliptic differential operator with constant of ellipticity $\kappa > 0$ (see Definition 4.1) independent of x. The coefficients of L are *bounded and measurable and the leading coefficients are uniformly continuous* on $\mathbb{R}^d$. More precisely there is a constant K and an increasing function $\omega(\varepsilon)$, $\varepsilon \geq 0$, such that $\omega(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, and for all $x, y \in \mathbb{R}^d$ and $i, j = 1, ..., d$ we have

$$|a^{ij}(x)|, |b^i(x)|, |c(x)| \leq K, \quad |a^{ij}(x) - a^{ij}(y)| \leq \omega(|x-y|).$$

Throughout the whole section we suppose that Assumption 1 is satisfied.

2. Lemma. *Additionally assume that the a^{ij} are constant. Then there exist constants $\lambda_0 \geq 1$ and N_0, depending only on K, κ, and d, such that, for any $\lambda \geq \lambda_0$ and $u \in C_0^2$, we have*

$$\lambda \|u\|_{\mathcal{L}_2} + \lambda^{1/2} \|u_x\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2} \leq N_0 \|Lu - \lambda u\|_{\mathcal{L}_2}. \tag{1}$$

Proof. By repeating the argument from Corollary 5.4, we reduce the general situation to the one in which $L = a^{ij} D_{ij}$. Then we observe that by Theorem 5.1

$$\lambda^{1/2} \|u_x\|_{\mathcal{L}_2} \leq N \lambda^{1/2} \|u\|_{\mathcal{L}_2}^{1/2} \|u_{xx}\|_{\mathcal{L}_2}^{1/2} \leq N \lambda \|u\|_{\mathcal{L}_2} + N \|u_{xx}\|_{\mathcal{L}_2},$$

so that we only need to prove that

$$\lambda \|u\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2} \leq N_0 \|Lu - \lambda u\|_{\mathcal{L}_2}. \tag{2}$$

Now we use a change of coordinates in order to reduce L to Δ and apply Lemma 1.1. We know that there exists a $d \times d$ symmetric matrix σ such that $a = (a^{ij}) = \sigma^2$. Notice that, since the eigenvalues of a belong to $[\kappa, \kappa^{-1}]$, the eigenvalues of σ belong to $[\kappa^{1/2}, \kappa^{-1/2}]$.

Define $v(x) = u(\sigma x)$. Then

$$v_{x^i}(x) = u_{x^k}(\sigma x) \sigma^{ki}, \quad v_{x^i x^j}(x) = u_{x^k x^r}(\sigma x) \sigma^{ki} \sigma^{rj}, \quad \Delta v(x) = (Lu)(\sigma x).$$

Also observe that

$$\|\Delta v - \lambda v\|_{\mathcal{L}_2}^2 = \det \sigma^{-1} \|Lu - \lambda u\|_{\mathcal{L}_2}^2 \leq N \|Lu - \lambda u\|_{\mathcal{L}_2}^2,$$

and, since there are obvious formulas expressing $u_x(x)$ and $u_{xx}(x)$ in terms of v_x and v_{xx}, also

$$\|u\|_{\mathcal{L}_2} \leq N \|v\|_{\mathcal{L}_2}, \quad \|u_x\|_{\mathcal{L}_2} \leq N \|v_x\|_{\mathcal{L}_2}, \quad \|u_{xx}\|_{\mathcal{L}_2} \leq N \|v_{xx}\|_{\mathcal{L}_2}.$$

This and Lemma 1.1 yield that for $\lambda > 0$

$$\lambda^2\|u\|^2_{\mathcal{L}_2} \leq N\lambda^2\|v\|^2_{\mathcal{L}_2} \leq N\|\Delta v - \lambda v\|^2_{\mathcal{L}_2} \leq N\|Lu - \lambda u\|^2_{\mathcal{L}_2},$$

$$\|u_{xx}\|^2_{\mathcal{L}_2} \leq N\|v_{xx}\|^2_{\mathcal{L}_2} \leq N\|\Delta v - \lambda v\|^2_{\mathcal{L}_2} \leq N\|Lu - \lambda u\|^2_{\mathcal{L}_2}.$$

The lemma is proved.

3. Lemma. *There exists an $\varepsilon = \varepsilon(d, \kappa, K, \omega) > 0$ such that if $u \in C^2_0$ has support in a ball of radius ε, then estimate (1) holds for any $\lambda \geq \lambda_0$ with $2N_0$ in place of N_0, where λ_0 and N_0 are taken from Lemma 2.*

Proof. Without loss of generality we assume that the ball in question is centered at the origin, so that $u(x) = 0$ for $|x| \geq \varepsilon$, where a small $\varepsilon > 0$ is to be specified later. Since ε is small, $Lu(x)$ should be close to $L_0u(x)$, where the operator L_0 is defined by "freezing" the leading coefficients at the origin:

$$L_0u(x) = a^{ij}(0)D_{ij}u(x) + b^i(x)D_iu(x) + c(x)u(x).$$

In fact,

$$|L_0u(x) - Lu(x)| \leq N_1\omega(\varepsilon)|D^2u(x)|,$$

where N_1 depends only on d. Furthermore, by Lemma 2, for $\lambda \geq \lambda_0$,

$$\lambda\|u\|_{\mathcal{L}_2} + \lambda^{1/2}\|u_x\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2} \leq N_0\|L_0u - \lambda u\|_{\mathcal{L}_2}$$

$$\leq N_0\|Lu - \lambda u\|_{\mathcal{L}_2} + N_2\omega(\varepsilon)\|u_{xx}\|_{\mathcal{L}_2}, \tag{3}$$

where N_2 depends only on K, κ, and d. Upon choosing ε so that $N_2\omega(\varepsilon) \leq 1/2$ and collecting like terms in (3), we get our assertion. The lemma is proved.

4. Theorem. *There exist constants $\lambda_0 \geq 1$ and N_0, depending only on K, κ, ω, and d, such that estimate (1) holds true for any $u \in W^2_2$ and $\lambda \geq \lambda_0$. Furthermore, for any $\lambda \geq \lambda_0$ and $f \in \mathcal{L}_2$, there exists a unique $u \in W^2_2$ satisfying $Lu - \lambda u = f$.*

Proof. Having in mind the method of continuity (Theorem 4.4), we first concentrate on a priori estimates for $u \in C^2_0$. As in the proof of Lemma 2 it suffices to prove (2).

Take a $\zeta \in C^\infty_0$ with

$$\|\zeta\|_{\mathcal{L}_2} = 1$$

and with support in the ball $B_\varepsilon = \{x : |x| < \varepsilon\}$, where $\varepsilon > 0$ is taken from Lemma 3. Observe that

$$1 = \int_{\mathbb{R}^d} \zeta^2(x-y)\,dy,$$

$$u^2_{x^ix^j}(x) = \int_{\mathbb{R}^d} u^2_{x^ix^j}(x)\zeta^2(x-y)\,dy. \tag{4}$$

These formulas are usually associated with the term "a partition of unity". In (4)

$$u^2_{x^ix^j}(x)\zeta^2(x-y) = \big[(u(x)\zeta(x-y))_{x^ix^j} - u_{x^i}(x)\zeta_{x^j}(x-y)$$
$$- u_{x^j}(x)\zeta_{x^i}(x-y) - u(x)\zeta_{x^ix^j}(x-y)\big]^2$$
$$\le 2\big[(u(x)\zeta(x-y))_{x^ix^j}\big]^2 + N(|u_x(x)|^2 + |u(x)|^2)\eta(x-y),$$

with

$$\eta = |\zeta_x|^2 + |\zeta_{xx}|^2 \in \mathcal{L}_1.$$

Hence, by integrating through (4) with respect to x, we find

$$\|u_{xx}\|^2_{\mathcal{L}_2} \le N\int_{\mathbb{R}^d} \|(\zeta(\cdot-y)u)_{xx}\|^2_{\mathcal{L}_2}\,dy + N(\|u_x\|^2_{\mathcal{L}_2} + \|u\|^2_{\mathcal{L}_2}).$$

Define λ_0 and N_0 as in Lemma 3. Then by this lemma

$$\|(\zeta(\cdot-y)u)_{xx}\|^2_{\mathcal{L}_2} \le 4N_0^2\|(L-\lambda)(\zeta(\cdot-y)u)\|^2_{\mathcal{L}_2}.$$

Therefore,

$$\|u_{xx}\|^2_{\mathcal{L}_2} \le N\int_{\mathbb{R}^d} \|(L-\lambda)(\zeta(\cdot-y)u)\|^2_{\mathcal{L}_2}\,dy + N(\|u_x\|^2_{\mathcal{L}_2} + \|u\|^2_{\mathcal{L}_2}).$$

We obtained an estimate of u_{xx} through $(L-\lambda)(\zeta(\cdot-y)u)$. Now we get ζ outside L. Notice that

$$(L-\lambda)(u(x)\zeta(x-y)) = \zeta(x-y)(L-\lambda)u(x) + 2a^{ij}(x)u_{x^i}(x)\zeta_{x^j}(x-y)$$
$$+ u(x)(a^{ij}(x)\zeta_{x^jx^j}(x-y) + b^i(x)\zeta_{x^j}(x-y)),$$

$$|(L-\lambda)(u(x)\zeta(x-y))|^2 \le 2|\zeta(x-y)(L-\lambda)u(x)|^2$$
$$+ N(|u_x(x)|^2 + |u(x)|^2)\eta(x-y).$$

It follows that

$$\int_{\mathbb{R}^d} \|(L-\lambda)(\zeta(\cdot - y)u)\|^2_{\mathcal{L}_2}\,dy \le 2\|(L-\lambda)u\|^2_{\mathcal{L}_2} + N(\|u_x\|^2_{\mathcal{L}_2} + \|u\|^2_{\mathcal{L}_2}),$$

$$\|u_{xx}\|^2_{\mathcal{L}_2} \le N\big(\|(L-\lambda)u\|^2_{\mathcal{L}_2} + \|u_x\|^2_{\mathcal{L}_2} + \|u\|^2_{\mathcal{L}_2}\big). \tag{5}$$

Similarly,

$$\begin{aligned}\lambda^2\|u\|^2_{\mathcal{L}_2} &= \lambda^2 \int_{\mathbb{R}^d} \|\zeta(\cdot - y)u\|^2_{\mathcal{L}_2}\,dy \\ &\le 4N_0^2 \int_{\mathbb{R}^d} \|(L-\lambda)(\zeta(\cdot - y)u)\|^2_{\mathcal{L}_2}\,dy \\ &\le N\big(\|(L-\lambda)u\|^2_{\mathcal{L}_2} + \|u_x\|^2_{\mathcal{L}_2} + \|u\|^2_{\mathcal{L}_2}\big).\end{aligned}$$

By combining this with (5) and Corollary 5.2, we find

$$\lambda\|u\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2} \le N_1\|Lu - \lambda u\|_{\mathcal{L}_2} + (1/2)\|u_{xx}\|_{\mathcal{L}_2} + N_2\|u\|_{\mathcal{L}_2},$$

where N_i depend only on K, κ, ω, and d.

To finish proving (2) with $2N_1$ in place of N_0, it only remains to take $\lambda \ge 2N_2$, so that $N_0 \le \lambda/2$ (do not forget that $\lambda \ge \lambda_0$ with λ_0 from Lemma 3).

By inspecting the above argument, one easily sees that λ_0 and N_0 can be taken to be the same for $L_t = (1-t)(\Delta - 1) + tL$ in place of L with any $t \in [0,1]$. Therefore, the method of continuity is applicable and the theorem is proved.

We will see much later in Theorem 11.6.2 that, if $c \le 0$, then one can take any $\lambda_0 > 0$ in Theorem 4 if one allows N_0 to depend on λ. The proof of this fact will be based on an $\mathcal{L}_p$ version of Exercise 6 before which we state the following a priori estimate.

5. Theorem. *There exists a constant N depending only on d, K, κ, and ω such that for any $u \in W^2_2$ and $\lambda \ge 0$ we have*

$$\|u\|_{W^2_2} \le N(\|\lambda u - Lu\|_{\mathcal{L}_2} + \|u\|_{\mathcal{L}_2}).$$

The proof is almost trivial since for $\lambda \ge \lambda_0$ our assertion is contained in Theorem 4, whereas for $\lambda \in [0, \lambda_0]$ we have

$$\|u\|_{W^2_2} \le N\|\lambda_0 u - Lu\|_{\mathcal{L}_2} \le N(\|\lambda u - Lu\|_{\mathcal{L}_2} + (\lambda_0 - \lambda)\|u\|_{\mathcal{L}_2})$$

with $\lambda_0 - \lambda \le \lambda_0$.

6. Exercise*. Assume that, for a particular L (as always satisfying Assumption 1), there exists a constant N such that for any $\lambda \geq 0$ and $u \in W_2^2$ we have

$$\|u\|_{\mathcal{L}_2} \leq N\|\lambda u - Lu\|_{\mathcal{L}_2}$$

with N independent of u and λ. Prove that, for this L, the assertions of Theorem 4 hold true for any $\lambda \geq 0$ rather than $\lambda \geq \lambda_0 \geq 1$.

7. Exercise. Under the conditions of Exercise 4.8 prove that if b^i, $i = 1, 2$, are bounded measurable functions on $\mathbb{R}^2$, then there are constants $\lambda_0 \geq 1$ and N, depending only on μ, ν, and the bounds of b^i, such that for any $\lambda \geq \lambda_0$ and $f \in \mathcal{L}_2$ the equation $L_\lambda u + b^i u_{x^i} = f$ has a unique solution $u \in W_2^2$ and for any $u \in W_2^2$

$$\lambda\|u\|_{\mathcal{L}_2} + \lambda^{1/2}\|u_x\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2} \leq N\|L_\lambda u + b^i u_{x^i}\|_{\mathcal{L}_2}.$$

In Exercise 11.6.5 we will see that one can take any $\lambda_0 > 0$ if we allow N to depend on λ.

8. Exercise. Consider the nonlinear equation

$$a^{ij}(x)u_{x^i x^j}(x) + F(u_x(x), u(x), x) - \lambda u(x) = 0, \tag{6}$$

where $F(\alpha, \beta, x)$ is a measurable function of (α, β, x) on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ such that, for an $f \in \mathcal{L}_2$, we have

$$|F(\alpha, \beta, x)| \leq K(|\alpha| + |\beta| + f(x)),$$

$$|F(\alpha_1, \beta_1, x) - F(\alpha_2, \beta_2, x)| \leq K(|\alpha_1 - \alpha_2| + |\beta_1 - \beta_2|)$$

for all $\alpha, \alpha_i, \beta, \beta_i, x$. Use the method of continuity to prove that there is $\lambda_0 = \lambda_0(d, \kappa, K, \omega)$ such that for any $\lambda \geq \lambda_0$ equation (6) has a unique solution $u \in W_2^2$.

7. Higher regularity of solutions

Here we want to show that if the coefficients of the operator L from Section 6 have bounded derivatives up to order n and the right-hand side f is in W_2^n, then the solutions of $Lu - \lambda u = f$ belong to W_2^{2+n} provided that λ is large enough but yet independently of n. We will again use the method of a priori estimates and the method of continuity.

First of all we need to know that the desired result holds for $L = \Delta$. The simplest way to do this is to use the Fourier transform. However, our goal is to give a presentation which would be applicable to equations in W_p^{2+n} with $p \neq 2$ without much additional effort. Therefore, we prefer an approach based on the Green's function G_λ of $\lambda - \Delta$, which is introduced in the following way.

Let $\lambda > 0$ and as in Section 1 first take $u \in C_0^2$, denote $f = \lambda u - \Delta u$ and write

$$\tilde{u}(\xi) = (\lambda + |\xi|^2)^{-1}\tilde{f}(\xi) = \tilde{f}(\xi)\int_0^\infty e^{-\lambda t}e^{-|\xi|^2 t}\,dt.$$

Then recall that $e^{-|\xi|^2 t}$ is proportional to the Fourier transform of

$$p(t,x) = (4\pi t)^{-d/2}e^{-|x|^2/(4t)} \tag{1}$$

and that the product of the Fourier transforms is proportional to the Fourier transform of the convolution. Then, after observing that the formula

$$\int_{\mathbb{R}^d} p(t,x)\,dx = 1$$

and the computation

$$\int_{\mathbb{R}^d}\int_0^\infty e^{-\lambda t}|p(t,x)|\,dtdx = \int_0^\infty e^{-\lambda t}\int_{\mathbb{R}^d}|p(t,x)|\,dx = \lambda^{-1} < \infty$$

allow us to use Fubini's theorem in finding the Fourier transform of

$$G_\lambda(x) := \int_0^\infty e^{-\lambda t}p(t,x)\,dt\,, \tag{2}$$

we easily see that

$$F(u) = F(G_\lambda * f). \tag{3}$$

Note that

$$G_\lambda(x) \geq 0, \quad \int_{\mathbb{R}^d} G_\lambda(x)\,dx = \lambda^{-1}. \tag{4}$$

Since also $f \in \mathcal{L}_1$, we have that $G_\lambda * f \in \mathcal{L}_1$ and by uniqueness of the Fourier transform we obtain from (3) that

$$u = G_\lambda * f \tag{5}$$

almost everywhere. However, $f \in C_0$ and the dominated convergence theorem implies that

$$G_\lambda * f(x) = \int_{\mathbb{R}^d} f(x-y)G_\lambda(y)\,dy$$

is a continuous function of x, so that (5) holds on $\mathbb{R}^d$ rather than only almost everywhere.

Also observe that if $f \in C_0^n$ for an integer $n \geq 1$, then the rules of differentiating convolutions show that

$$D^n(G_\lambda * f) = G_\lambda * D^n f. \tag{6}$$

In particular, $G_\lambda * f$ is infinitely differentiable if $f \in C_0^\infty$.

1. Theorem. *(i) The formula*

$$R_\lambda f = G_\lambda * f$$

defines a bounded operator in $\mathcal{L}_p$ for any $p \in [1, \infty]$. Furthermore,

$$\lambda \|R_\lambda f\|_{\mathcal{L}_p} \leq \|f\|_{\mathcal{L}_p}. \tag{7}$$

(ii) For any $f \in \mathcal{L}_2$ the unique solution $u \in W_2^2$ of the equation $\lambda u - \Delta u = f$ is given by $u = R_\lambda f$.

(iii) If $n \geq 1$ and $f \in W_2^n$, then $R_\lambda f \in W_2^{2+n}$ and

$$D^n R_\lambda f = R_\lambda D^n f. \tag{8}$$

Proof. Assertion (i) follows from (4) and Minkowski's inequality, which we have already used a few times.

(ii) Equation (5) says that

$$u = R_\lambda(\lambda u - \Delta u)$$

for $u \in C_0^2$. By the above both sides of this equation are continuous $\mathcal{L}_2$-valued functions on W_2^2, so that the formula holds for any $u \in W_2^2$.

(iii) By (6) equation (8) holds if $f \in C_0^n$. If $f \in W_2^n$ and $f_k \in C_0^n$ is its defining sequence, then for $0 \leq i \leq n$,

$$\|D^i R_\lambda f_k - D^i R_\lambda f_m\|_{W_2^2} = \|R_\lambda(D^i f_k - D^i f_m)\|_{W_2^2}$$

$$\leq N \|D^i f_k - D^i f_m\|_{\mathcal{L}_2} \to 0$$

as $k, m \to \infty$, where the constant N is independent of k, m. This shows that the $R_\lambda f_k$ form a Cauchy sequence in W_2^{2+n}. By completeness, there is a $v \in W_2^{2+n}$ such that

$$D^i R_\lambda f_k \to D^i v$$

in $\mathcal{L}_2$ for $i \le 2+n$. In particular, $R_\lambda f_k \to v$ and we discover that $v = R_\lambda f$ due to the continuity of R_λ on $\mathcal{L}_2$. After that it only remains to substitute f_k in place of f in (8) and to pass to the limit. The theorem is proved.

2. Corollary. *Let $n \ge 0$, $\lambda > 0$, and $f \in W^n_2$. Then there exists a unique solution $u \in W^{2+n}_2$ of the equation $\lambda u - \Delta u = f$. Furthermore, for any multi-index α with $|\alpha| \le n$*

$$\lambda^2 \|D^\alpha u\|^2_{\mathcal{L}_2} + 2\lambda \sum_{j=1}^d \|D^\alpha u_{x^j}\|^2_{\mathcal{L}_2} + \sum_{j,k=1}^d \|D^\alpha u_{x^j x^k}\|^2_{\mathcal{L}_2} = \|D^\alpha f\|^2_{\mathcal{L}_2}.$$

In particular, there exists a constant N depending only on d and n such that

$$\lambda \|u\|_{W^n_2} + \|u_{xx}\|_{W^n_2} \le N \|f\|_{W^n_2}.$$

Indeed, uniqueness follows from Theorem 3.16. Furthermore, by Theorem 1 the solution u, found in W^2_2 according to Theorem 3.16, is actually in W^{2+n}_2. After that one can differentiate the equation $\lambda u - \Delta u = f$ and get the estimate again from Theorem 3.16.

3. Exercise. Prove that the norm of λR_λ in any $\mathcal{L}_p$ is 1.

4. Exercise. For $u \in C^2_0$ prove that $|\nabla u| \le 2(\sup |\Delta u|^{1/2})(\sup |u|^{1/2})$ on $\mathbb{R}^d$ and that the estimate is sharp.

5. Theorem. *Take the operator L from Section 6, an integer $n \ge 1$, and assume that the coefficients a, b, c are in C^n and their norms in C^n are bounded by a constant K_1. Take λ_0 from Theorem 6.4. Then there exists a constant N_0, depending only on K, K_1, n, κ, ω, and d, such that, for any $\lambda \ge \lambda_0$ and $u \in W^{n+2}_2$*

$$\lambda \|u\|_{W^n_2} + \|u\|_{W^{n+2}_2} \le N_0 \|Lu - \lambda u\|_{W^n_2}. \tag{9}$$

Furthermore, for any $\lambda \ge \lambda_0$ and $f \in W^n_2$, there exists a unique $u \in W^{n+2}_2$ satisfying $Lu - \lambda u = f$.

Proof. Recall that if $g \in C^n$ and $u \in W^n_p$, then $gu \in W^n_2$ and

$$\|gu\|_{W^n_2} \le N \|g\|_{C^n} \|u\|_{W^n_2},$$

where N depends only on d and n (see Exercise 3.9). Also invoke Corollary 2. With these two tools available, one can repeat the proof of Theorem 4.4, using the method of continuity, taking there W^n_2 and W^{n+2}_2 in place of $\mathcal{L}_2$ and W^2_2, respectively. Then one sees that to prove the theorem, it suffices only to find N_0 such that (9) holds for $u \in C^{n+2}_0$ and $\lambda \ge \lambda_0$.

Take a $u \in C_0^{n+2}$, observe that Lu is n times continuously differentiable, and, for any multi-index α with $|\alpha| \le n$, by the Leibnitz formula write

$$D^\alpha(L-\lambda)u = (L-\lambda)D^\alpha u + \sum_{|\beta|\le|\alpha|+1} c^\beta D^\beta u,$$

where c^β are certain bounded functions. Hence by Theorem 6.4 for $\lambda \ge \lambda_0$

$$\lambda\|D^\alpha u\|_{\mathcal{L}_2} + \|D^\alpha u\|_{W_2^2} \le N\|(L-\lambda)D^\alpha u\|_{\mathcal{L}_2}$$

$$\le N\|D^\alpha(L-\lambda)u\|_{\mathcal{L}_2} + N\sum_{|\beta|\le|\alpha|+1}\|D^\beta u\|_{\mathcal{L}_2}.$$

By summing up over α such that $|\alpha| \le k$, where $k \in \{0,1,...,n\}$, we get

$$\lambda\|u\|_{W_2^k} + \|u\|_{W_2^{k+2}} \le N\|(L-\lambda)u\|_{W_2^n} + N\|u\|_{W_2^{k+1}}.$$

The induction on k leads to

$$\lambda\|u\|_{W_2^n}+\|u\|_{W_2^{n+2}} \le N\|(L-\lambda)u\|_{W_2^n}+N\|u\|_{W_2^1} \le N\|(L-\lambda)u\|_{W_2^n}+N\|u\|_{W_2^2}$$

and to obtain (9) it only remains to use Theorem 6.4 again. The theorem is proved.

Here is a result on *global regularity*.

6. Corollary. *Under the assumptions of Theorem* 5 *take a function* $u \in W_2^2$, $\lambda \ge 0$ *and assume that* $Lu - \lambda u \in W_2^n$. *Then* $u \in W_2^{n+2}$ *and*

$$\lambda\|u\|_{W_2^n} + \|u\|_{W_2^{n+2}} \le N(\|Lu-\lambda u\|_{W_2^n} + \|u\|_{\mathcal{L}_2}),$$

where N *depends only on* $n, d, K, K_1, \kappa,$ *and* ω.

Indeed, denote $\lambda_1 = \lambda_0 + \lambda$ and introduce

$$g := Lu - \lambda_1 u = (Lu - \lambda u) - \lambda_0 u. \tag{10}$$

If, for an $r \in \{0,...,n-1\}$, we have $u \in W_2^{r+2}$, then $g \in W_2^{r+1}$ and by Theorem 5 equation (10) has a solution in $W_2^{(r+1)+2}$, which is unique in W_2^2. Since u is a solution of class W_2^2, it follows that $u \in W_2^{r+3}$. An obvious induction on r proves that $u \in W_2^{n+2}$.

Furthermore, for $r=1,...,n$ by Theorem 5 we obtain

$$A_r := \lambda\|u\|_{W^r_2} + \|u\|_{W^{r+2}_2} \le \lambda_1\|u\|_{W^r_2} + \|u\|_{W^{r+2}_2}$$

$$\le N\|g\|_{W^r_2} \le N(\|Lu-\lambda u\|_{W^n_2} + \|u\|_{W^r_2})$$

$$\le N\|Lu-\lambda u\|_{W^n_2} + NA_{r-1} \le ... \le N\|Lu-\lambda u\|_{W^n_2} + NA_0$$

and referring to Theorem 6.5 finishes the argument.

Corollary 6 can be localized providing a *local regularity result.*

7. Theorem. *Under the assumptions of Theorem* 5 *take two numbers* $0<\rho<R\le\infty$ *and a function* $u\in W^2_2(B_R)$. *Also take a* $\lambda\ge 0$ *and assume that* $Lu-\lambda u\in W^n_2(B_R)$. *Then* $u\in W^{n+2}_2(B_\rho)$ *and*

$$\lambda\|u\|_{W^n_2(B_\rho)} + \|u\|_{W^{n+2}_2(B_\rho)} \le N(\|Lu-\lambda u\|_{W^n_2(B_R)} + \|u\|_{W^1_2(B_R)}),$$

where N *depends only on* n, d, K, K_1, κ, ρ, R, *and* ω.

Proof. Take a $\zeta\in C^\infty_0(B_R)$ and notice that

$$L(u\zeta) - \lambda\zeta u = \zeta(Lu-\lambda u) + u(L\zeta - c\zeta) + 2a^{ij}u_{x^i}\zeta_{x^j} =: g.$$

Since $n\ge 1$, we have that $g\in W^1_2$, so that $u\zeta\in W^3_2$ by Corollary 6. This holds for any $\zeta\in C^\infty_0(B_R)$, implying that for $n\ge 2$ we have $g\in W^2_2$. In that case by Corollary 6 we have $u\zeta\in W^4_2$. Proceeding further in this way, we see that $g\in W^n_2$ and $u\zeta\in W^{n+2}_2$ for any $\zeta\in C^\infty_0(B_R)$. In particular, $u\in W^{n+2}_2(B_\rho)$.

This allows us to apply Corollary 6 to the functions $u\zeta_r$, $r=0,...,n+1$, where $\zeta_r\in C^\infty_0(B_R)$ are chosen in such a way that $\zeta_0=1$ on B_ρ and $\zeta_{r+1}=1$ on the support of ζ_r. Then for $r=0,...,n$ we obtain

$$\lambda\|u\zeta_r\|_{W^{n-r}_2} + \|u\zeta_r\|_{W^{n-r+2}_2} \le NF + N\|u\zeta_{r+1}\|_{W^{n-r+1}_2}, \tag{11}$$

where

$$F := \|Lu-\lambda u\|_{W^n_2(B_R)}.$$

In particular,

$$\|u\zeta_r\|_{W^{n-r+2}_2} \le NF + N\|u\zeta_{r+1}\|_{W^{n-r+1}_2}, \quad \|u\zeta_r\|_{W^{n-r+2}_2} \le NF + N\|u\zeta_{n+1}\|_{W^1_2}.$$

We take here $r=1$ and use (11) with $r=0$. Then we get

$$\lambda\|u\zeta_0\|_{W^n_2} + \|u\zeta_0\|_{W^{n+2}_2} \le NF + N\|u\zeta_1\|_{W^{n+1}_2} \le NF + N\|u\zeta_{n+1}\|_{W^1_2}.$$

The theorem is proved.

This result will be substantially improved in Theorems 2.4.7 and 9.4.1.

Corollary 6 implies, in particular, that if $\lambda \geq \lambda_0$ and $f \in W_2^n$ for any $n \geq 0$ and if each derivative of a, b, c is bounded, then the unique solution $u \in W_2^2$ of equation $\lambda u - Lu = f$ belongs to W_2^n also for all $n \geq 0$. It turns out that in this case u is infinitely differentiable with each derivative bounded on $\mathbb{R}^d$, or, to be more precise, admits a function that equals u almost everywhere and is infinitely differentiable with each derivative bounded on $\mathbb{R}^d$. In this way we get the classical solvability of the equation $\lambda u - Lu = f$.

This fact is obtained immediately from one of the Sobolev embedding theorems that reads as follows.

8. Theorem. *Let integers $n > k \geq 0$ and assume that $2(n-k) > d$. Then for any $u \in W_2^n$ there exists a unique function $v \in C^k$ such that $v = u$ almost everywhere. Furthermore, there is a constant N independent of u such that*

$$\|v\|_{C^k} \leq N\|u\|_{W_2^n}. \tag{12}$$

Proof. First we claim that it suffices to prove (12) with $v = u \in C_0^n$. Indeed, if (12) is true in this case, then for any $u \in W_2^n$ we take a defining sequence $u_m \in C_0^n$ and observe that by (12) we have

$$\|u_r - u_m\|_{C^k} \leq N\|u_r - u_m\|_{W_2^n} \to 0$$

as $r, m \to \infty$. By the completeness of C^k there exists a $v \in C^k$ such that $u_m \to v$ in C^k. Since $u_n \to u$ in $\mathcal{L}_2$, we have $u = v$ (a.e.). After that it only remains to notice that

$$\|v\|_{C^k} = \lim_{m\to\infty} \|u_m\|_{C^k} \leq N \lim_{m\to\infty} \|u_m\|_{W_2^n} = \|u\|_{W_2^n}.$$

Thus, we assume that $u \in C_0^n$. In that case, for any multi-index α with $|\alpha| \leq n$ we have

$$D^\alpha u(x) = i^{|\alpha|} c_d \int_{\mathbb{R}^d} e^{i\xi\cdot x} \xi^\alpha \tilde{u}(\xi)\, d\xi.$$

Hence

$$\max_{\mathbb{R}^d} |D^\alpha u| \leq c_d \int_{\mathbb{R}^d} |\xi|^{|\alpha|} |\tilde{u}(\xi)|\, d\xi.$$

By using Hölder's inequality and Parseval's identity, we see that the square of the last integral is dominated by the product of

$$\int_{\mathbb{R}^d} |\xi|^{2|\alpha|}(1+\sum_{j=1}^d |\xi^j|^{2n})^{-1}\,d\xi,$$

which is finite if $|\alpha| \le k$, and

$$\int_{\mathbb{R}^d}(1+\sum_{j=1}^d |\xi^j|^{2n})|\tilde{u}(\xi)|^2\,d\xi = \|u\|^2_{\mathcal{L}_2} + \sum_{j=1}^d \|D_j^n u\|^2_{\mathcal{L}_2} \le N\|u\|^2_{W_2^n}.$$

The theorem is proved.

Theorem 8 will be generalized to a very large extent in Sections 10.2 and 13.8.

9. Remark. If we have two functions f and g of class $\mathcal{L}_p$ defined in a domain and $f = g$ almost everywhere, then we say that f is a modification of g and vice versa. Speaking about elements of $\mathcal{L}_p$, it is common to say that a function u is, say, continuous if it has a continuous modification. We will always use this stipulation in the future. In this sense Theorem 8 says that under its conditions any $u \in W_2^n$ is continuous.

10. Remark. There is also a "local" version of Theorem 8. Namely, if $2(n-k) > d$, D is a domain in $\mathbb{R}^d$, bounded domain $G \subset \bar{G} \subset D$, and $u \in W_2^n(D)$, then $u \in C^k(G)$ and the norm of u in $C^k(G)$ is less than a constant, independent of u, times the norm of u in $W_2^n(D)$.

This result follows at once from Theorem 8 if one takes a $\zeta \in C_0^\infty(D)$ such that $\zeta = 1$ on G and applies Theorem 8 to $u\zeta$.

11. Exercise*. Let $d = 1$, $\Omega = (0,1)$. For $u \in W_2^1(\Omega)$ prove that

$$|u(x)-u(y)| \le N|x-y|^{1/2}\|u\|_{W_2^1(\Omega)}, \quad |u(x)| \le N\|u\|_{W_2^1(\Omega)}$$

for any $x, y \in \Omega$, where N is independent of u.

8. Sobolev mollifiers

The most important point in Section 7 was that by Theorem 7.1 (iii) the solutions of $\Delta u - \lambda u = f$ are smoother if f is smoother. It turns out that this fact can also be obtained in a different and very general way without using explicit representations of solutions.

In this section we take

$$p \in [1, \infty).$$

By definition a function in W_p^k can be approximated by smooth functions. There is one very powerful unified way to do such approximations by using the Sobolev mollifiers. In particular, this method will allow us to give a criterion for deciding if $u \in W^k_p(\Omega)$ on the basis of knowing its Sobolev derivatives.

We need the following Young's inequalities.

1. Lemma. *(i) Let $g \in \mathcal{L}_p$ and $h \in \mathcal{L}_1$. Then $g * h \in \mathcal{L}_p$ and*

$$\|g * h\|_{\mathcal{L}_p} \leq \|g\|_{\mathcal{L}_p} \|h\|_{\mathcal{L}_1}. \tag{1}$$

(ii) More generally, let $q, r \in [1, \infty)$ and

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{p} + 1. \tag{2}$$

Also let $g \in \mathcal{L}_q$ and $h \in \mathcal{L}_r$. Then

$$\|g * h\|_{\mathcal{L}_p} \leq \|g\|_{\mathcal{L}_r} \|h\|_{\mathcal{L}_q}. \tag{3}$$

Proof. (i) The convolution $g * h(y)$ is

$$\int_{\mathbb{R}^d} g(y - x) h(x)\, dx,$$

which is the "sum" with respect to x of the functions $g(y-x)h(x)$ as functions of y. By Minkowski's inequality (the norm of a sum is less than the sum of norms)

$$\|g * h\|_{\mathcal{L}_p} \leq \|\, |g| * |h| \,\|_{\mathcal{L}_p} \leq \int_{\mathbb{R}^d} |h(x)| \Big(\int_{\mathbb{R}^d} |g(y-x)|^p \, dy \Big)^{1/p} dx,$$

where the last expression is just the right-hand side of (1).

(ii) If $r = 1$, we can use (i). In case that $r > 1$, observe that $1/q > 1/p$ and $p > q$. Similarly, $p \geq r$. Then, by Hölder's inequality

$$|g * h|(x) \leq |g| * |h|(x) = \int_{\mathbb{R}^d} (|g(y-x)|\, |h(y)|^{q/p}) |h(y)|^{1-q/p} \, dy$$

$$\leq \big(|g|^r * |h|^{rq/p} \big)^{1/r}(x) \Big(\int_{\mathbb{R}^d} |h(y)|^{(1-q/p)r/(r-1)} \, dy \Big)^{1-1/r}$$

$$= \big(|g|^r * |h|^{rq/p} \big)^{1/r}(x) \|h\|^{q(1-1/r)}_{\mathcal{L}_q}.$$

Furthermore, by (i)

$$\|\big(|g|^r * |h|^{rq/p}\big)^{1/r}\|_{\mathcal{L}_p} = \|\,|g|^r * |h|^{rq/p}\|^{1/r}_{\mathcal{L}_{p/r}}$$

$$\leq \|\,|g|^r\|^{1/r}_{\mathcal{L}_1}\|\,|h|^{rq/p}\|^{1/r}_{\mathcal{L}_{p/r}} = \|g\|_{\mathcal{L}_r}\|h\|^{q/p}_{\mathcal{L}_q}.$$

Hence,

$$\|g * h\|_{\mathcal{L}_p} \leq \|g\|_{\mathcal{L}_r}\|h\|^{q/p}_{\mathcal{L}_q}\|h\|^{q(1-1/r)}_{\mathcal{L}_q}$$

and this is (3) since $q/p + q(1-1/r) = 1$. The lemma is proved.

Let $\zeta \in C_0^\infty$. For $\varepsilon > 0$ let $\zeta_\varepsilon(x) = \varepsilon^{-d}\zeta(x/\varepsilon)$ and for any locally integrable u define

$$u^{(\varepsilon)}(x) = u * \zeta_\varepsilon(x) = \varepsilon^{-d}\int_{\mathbb{R}^d} u(y)\zeta((x-y)/\varepsilon)\,dy = \int_{\mathbb{R}^d} u(x-\varepsilon y)\zeta(y)\,dy. \tag{4}$$

2. Lemma. *(i) If u is locally integrable in $\mathbb{R}^d$, then $u^{(\varepsilon)}$ is infinitely differentiable in $\mathbb{R}^d$ and for any multi-index α*

$$D^\alpha u^{(\varepsilon)}(x) = u * D^\alpha\zeta_\varepsilon(x) = \varepsilon^{-d-|\alpha|}\int_{\mathbb{R}^d} u(y)(D^\alpha\zeta)((x-y)/\varepsilon)\,dy. \tag{5}$$

In particular, if $u \in \mathcal{L}_p$, then by Hölder's inequality the $D^\alpha u^{(\varepsilon)}$ are bounded for any $\varepsilon > 0$ and α.

(ii) Let α be a multi-index, u locally integrable in $\mathbb{R}^d$, and let $D^\alpha u$ be the Sobolev D^α derivative of u (see Definition 3.4). Then $D^\alpha u^{(\varepsilon)} = (D^\alpha u)^{(\varepsilon)}(x)$.

(iii) If

$$\int_{\mathbb{R}^d} \zeta\,dx = 1$$

and $u \in \mathcal{L}_p$, then $u^{(\varepsilon)} \in W^k_p$ for any k and $u^{(\varepsilon)} \to u$ in $\mathcal{L}_p$ as $\varepsilon \downarrow 0$. In particular, $C_b^\infty \cap W^k_p$ is dense in W^k_p, where $C_b^\infty = \bigcap_n C^n$.

Proof. Assertion (i) is a standard result in integration theory. Assertion (ii) follows from (5) by the definition of $D^\alpha u$.

To prove (iii), notice that for any bounded continuous function ϕ with compact support and $\varepsilon \in (0,1)$ the functions $\phi^{(\varepsilon)}$ are bounded by a constant independent of ε. For $\varepsilon \leq 1$ they also have support in a ball independent of ε. Indeed, if $\zeta(x) = 0$ and $\phi(x) = 0$ for $|x| \geq R$, then it is not hard to see that, for $|x| \geq R + \varepsilon R$, we have $\phi(y)\zeta((x-y)/\varepsilon) = 0$ for all y, so that $\phi^{(\varepsilon)}(x) = 0$. Furthermore, for any x

$$\phi^{(\varepsilon)}(x) = \int_{\mathbb{R}^d} \phi(x - \varepsilon y)\zeta(y)\, dy \to \phi(x)$$

as $\varepsilon \downarrow 0$ by the dominated convergence theorem. Also if $\phi \in C_0^k$ and $|\alpha| \leq k$, then

$$D^\alpha \phi^{(\varepsilon)}(x) = \int_{\mathbb{R}^d} D^\alpha \phi(x - \varepsilon y)\zeta(y)\, dy \to D^\alpha \phi(x)$$

as $\varepsilon \downarrow 0$ with all functions involved uniformly bounded and vanishing outside the same ball.

By the dominated convergence theorem and the above properties we have $\phi^{(\varepsilon)} \to \phi$ in W_p^k if $\phi \in C_0^k$. Also use Lemma 1, which, along with (5), implies in particular that $u^{(\varepsilon)} \in W_p^k$ for any k. Then by inspecting

$$\varlimsup_{\varepsilon\downarrow 0} \|u - u^{(\varepsilon)}\|_{W_p^k} \leq \|u - \phi\|_{W_p^k} + \varlimsup_{\varepsilon\downarrow 0} \|\phi - \phi^{(\varepsilon)}\|_{W_p^k} + \varlimsup_{\varepsilon\downarrow 0} \|(\phi - u)^{(\varepsilon)}\|_{W_p^k}$$

$$\leq (1 + N(k,d)\|\zeta\|_{\mathcal{L}_1})\|u - \phi\|_{W_p^k}$$

and using the fact that C_0^k is dense in W_p^k, we get (iii). The lemma is proved.

In light of assertion (i) of Lemma 2 the operator $u \to u^{(\varepsilon)}$ is called a *Sobolev mollifier*.

3. Exercise*. Prove that if α is a multi-index $u \in \mathcal{L}_2$ and the Sobolev derivative $D^\alpha u \in \mathcal{L}_2$, then

$$\int_{\mathbb{R}^d} |\xi^\alpha|^2 |\tilde{u}(\xi)|^2\, d\xi \leq N \|D^\alpha u\|_{\mathcal{L}_2}^2,$$

where N is independent of u and $\tilde{u}$ is the Fourier transform of u.

New proof of Corollary 7.2. We know that $u \in W_2^2$ and that we need only show that $u \in W_2^{2+n}$.

Take a $\zeta \in C_0^\infty(\mathbb{R}^d)$ with unit integral. Then by assertions (i) and (ii) of Lemma 2 for any multi-index α with $|\alpha| \leq n$ and $\varepsilon > 0$ we have

$$\lambda u^{\alpha\varepsilon} - \Delta u^{\alpha\varepsilon} = f^{\alpha\varepsilon},$$

where

$$u^{\alpha\varepsilon} = D^\alpha u^{(\varepsilon)}, \quad f^{\alpha\varepsilon} = (D^\alpha f)^{(\varepsilon)}.$$

By assertion (iii) we have $u^{\alpha\varepsilon} \in W^2_2$, so that by Theorem 3.16 for any $\delta > 0$

$$\|D^\alpha[u^{(\varepsilon)} - u^{(\delta)}]\|_{W^2_2} \le N\|(D^\alpha f)^{(\varepsilon)} - (D^\alpha f)^{(\delta)}\|_{\mathcal{L}_2},$$

where N is independent of ε, δ. Since this holds for any α with $|\alpha| \le n$,

$$\|u^{(\varepsilon)} - u^{(\delta)}\|_{W^{2+n}_2} \le N \sum_{|\alpha|\le n} \|(D^\alpha f)^{(\varepsilon)} - (D^\alpha f)^{(\delta)}\|_{\mathcal{L}_2}.$$

The right-hand side here tends to zero as $\varepsilon, \delta \downarrow 0$ by Lemma 2 (iii), so that the sequence of smooth functions $u^{(1/j)}$ is Cauchy in W^{2+n}_2. In addition, $u^{(1/j)} \to u$ in $\mathcal{L}_2$ as $j \to \infty$ again by Lemma 2 (iii). By referring to the definition of W^{2+n}_2, we obtain what we needed.

4. Exercise*. Let u be a continuous function on $\mathbb{R}^d$ and let the generalized derivative u_{x^1} also be a continuous function. By inspecting the proof of Lemma 2 (iii), prove that u_{x^1} is the classical derivative of u with respect to x^1.

Assertion (i) of the following theorem turns out to be true for smooth domains (see Exercise 8.4.5). Assertion (iii) generalizes assertion (ii) of Theorem 3.11. We suggest the reader further generalize assertions (i) and (iii) of Theorem 5 for the simplest case of Lipschitz domains, namely Lipschitz half spaces, in Exercise 8. The proofs of assertions (i) through (iii) are based on mollifying the function u extended as zero outside Ω, but since near the boundary the mollified functions may have "bad" behavior, we shift them "outwards".

5. Theorem. *Assume that $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$. Then*

(i) A function $u \in \mathcal{L}_p(\Omega)$ belongs to $W^k_p(\Omega)$ if and only if its Sobolev derivatives $D^\alpha u$ exist and belong to $\mathcal{L}_p(\Omega)$ for any multi-index α with $|\alpha| \le k$.

(ii) For any $\zeta \in C^\infty_0$, which integrates to one and, in case $\Omega = \mathbb{R}^d_+$, additionally satisfies $\zeta(x) = 0$ if $x^1 \ge 0$, we have in the notation before Lemma 2 that

$$(uI_\Omega)^{(\varepsilon)} \to u$$

in $W^k_p(\Omega)$ as $\varepsilon \downarrow 0$ for any $u \in W^k_p(\Omega)$.

(iii) The set of functions which are infinitely differentiable in $\bar\Omega$ and vanish for large $|x|$ is dense in $W^k_p(\Omega)$.

(iv) For ζ as in (ii) there is a constant $N = N(d, k, \zeta)$ such that, for

any $u \in \mathcal{L}_p$ *and constant* M*,*

$$\begin{aligned} u \in W^k_p, \quad \|u\|_{W^k_p} \le M \Longrightarrow \|u^{(\varepsilon)}\|_{W^k_p} \le NM \quad \forall \varepsilon > 0, \\ p > 1, \quad \|u^{(\varepsilon)}\|_{W^k_p} \le M \quad \forall \varepsilon \in (0,1) \Longrightarrow u \in W^k_p, \quad \|u\|_{W^k_p} \le NM. \end{aligned} \tag{6}$$

Proof. (i) The "only if" part follows from Definition 3.2. To prove the "if" part, take $u \in \mathcal{L}_p(\Omega)$ such that $D^\alpha u \in \mathcal{L}_p(\Omega)$ for $|\alpha| \le k$ and define

$$v = uI_\Omega, \quad v^\alpha = I_\Omega D^\alpha u.$$

Also take ζ as in assertion (ii). Then $v \in \mathcal{L}_p$ and the $v^{(\varepsilon)}$ are infinitely differentiable, have bounded derivatives and belong to W^m_p for any m by Lemma 2. In case $\Omega = \mathbb{R}^d$, the same lemma shows that $D^\alpha v^{(\varepsilon)} \to D^\alpha u$ in $\mathcal{L}_p$ if $|\alpha| \le k$. Hence $v^{(1/n)}$ is a Cauchy sequence in W^k_p and $v^{(1/n)} \to u$ in $\mathcal{L}_p$, so that by definition $u \in W^k_p$.

In the remaining case that $\Omega = \mathbb{R}^d_+$ the same argument is applicable if we restrict our attention to $x^1 > 0$. Indeed, for those x and $|\alpha| \le k$, due to the fact that $\zeta((x-y)/\varepsilon) \ne 0$ only if $y^1 > x^1$ and, in particular, $\zeta((x-\cdot)/\varepsilon) \in C^\infty_0(\Omega)$, we have

$$\begin{aligned} D^\alpha v^{(\varepsilon)}(x) &= \varepsilon^{-d-|\alpha|} \int_{\mathbb{R}^d} v(y)(D^\alpha \zeta)((x-y)/\varepsilon)\, dy \\ &= \varepsilon^{-d-|\alpha|} \int_{\Omega} v(y)(D^\alpha \zeta)((x-y)/\varepsilon)\, dy \\ &= \varepsilon^{-d} \int_{\Omega} \zeta((x-y)/\varepsilon) D^\alpha u(y)\, dy = v^\alpha * \zeta_\varepsilon(x). \end{aligned}$$

In addition $v^\alpha \in \mathcal{L}_p$, so that $v^\alpha * \zeta_\varepsilon \to v^\alpha$ in $\mathcal{L}_p$ and $v^\alpha * \zeta_{1/n}$ is a Cauchy sequence in $\mathcal{L}_p$ whereas $D^\alpha v^{(1/n)}$ ($= v^\alpha * \zeta_{1/n}$ on Ω) is a Cauchy sequence in $\mathcal{L}_p(\Omega)$. Finally, $v^{(1/n)} \to v$ in $\mathcal{L}_p$; in particular, $v^{(1/n)} \to v = u$ in $\mathcal{L}_p(\Omega)$. By definition $u \in W^k_p$ and assertion (i) is proved.

Assertion (ii) has actually been proved in the above argument. Assertion (iii) also follows easily from the above. Indeed $(uI_\Omega)^{(\varepsilon)}$ is infinitely differentiable in $\bar{\Omega}$, has bounded derivatives, belongs to W^k_p by Lemma 2, and converges to u in $W^k_p(\Omega)$. In addition, by using cut-off functions as in the proof of Theorem 3.11, we can approximate in $W^k_p(\Omega)$ the smooth functions $(uI_\Omega)^{(\varepsilon)}$ by smooth ones vanishing for large $|x|$.

To prove (iv), notice that the first implication in (6) follows from the formula $D^\alpha(u^{(\varepsilon)}) = (D^\alpha u)^{(\varepsilon)}$ and Lemma 1 estimating norms of convolutions. To prove the second one, observe that, for $|\alpha| \le k$, the $D^\alpha u^{(1/n)}$ are bounded sequences in $\mathcal{L}_p$. Since $p > 1$, there is a subsequence $n' \to \infty$ and functions $u^\alpha \in \mathcal{L}_p$ such that $D^\alpha u^{(1/n')} \to u^\alpha$ weakly in $\mathcal{L}_p$. In particular, $\|u^\alpha\|_{\mathcal{L}_p} \le M$ and for any $\phi \in C_0^\infty$

$$(-1)^{|\alpha|}\int_{\mathbb{R}^d} u^{(1/n')} D^\alpha \phi\, dx = \int_{\mathbb{R}^d} \phi D^\alpha u^{(1/n')}\, dx \to \int_{\mathbb{R}^d} \phi u^\alpha\, dx.$$

Since $u^{(1/n')} \to u$ in $\mathcal{L}_p$, the first expression tends to

$$(-1)^{|\alpha|}\int_{\mathbb{R}^d} u D^\alpha \phi\, dx.$$

Hence by definition $u^\alpha = D^\alpha u$, which along with the above estimate $\|u^\alpha\|_{\mathcal{L}_p} \le M$ and assertion (i), proves (iv). The theorem is proved.

6. Exercise*. Let $u \in W^k_p$ be even (or odd) with respect to x^1. Prove that the derivatives of order $\le k$ which do not contain differentiations in x^1 are even (respectively, odd) with respect to x^1.

7. Exercise*. Let $u(x) = 0$ for $|x| \ge 1$ and let u be Lipschitz continuous: $|u(x) - u(y)| \le \alpha|x - y|$, where α is a constant. Prove that $u \in W^1_p$ for any p and $\|u_x\|_{\mathcal{L}_p} \le N(d)\alpha$.

8. Exercise. Let $\Omega = \{x : x^1 > f(x')\}$ where f is a Lipschitz continuous function: $|f(x') - f(y')| \le \alpha|x' - y'|$, where α is a constant. Prove that in this situation assertions (i) and (iii) of Theorem 5 are still valid.

9. Singular-integral representation of u_{xx}

By Theorem 7.1 if $u \in C^2_0$, then for any $\lambda > 0$ we have

$$u(x) = \int_{\mathbb{R}^d} (\lambda u(x-y) - \Delta u(x-y)) G_\lambda(y)\, dy, \tag{1}$$

where G_λ is given by an explicit formula from which it follows that, as $\lambda \downarrow 0$,

$$G_\lambda(y) \uparrow G_0(y) := \frac{1}{(4\pi)^{d/2}} \int_0^\infty t^{-d/2} e^{-|y|^2/4t}\, dt = N(d)\frac{1}{|y|^{d-2}},$$

where the last equality is obtained by a change of variables. As is easy to see, $N(d) < \infty$ if and only if $d \ge 3$. In that case G_0 is locally summable and by the dominated convergence theorem we obtain from (1) that for any $u \in C_0^2$

$$u(x) = N(d) \int_{\mathbb{R}^d} \frac{1}{|y|^{d-2}} f(x-y)\, dy, \tag{2}$$

where $f = -\Delta u$. This is a representation of u by means of the classical *Newtonian potential* of f.

In the sequel we take $d \geq 3$. Again owing to the dominated convergence theorem, one can differentiate (2) and see that if $u \in C_0^3$, then $f \in C_0^1$ and

$$u_{x^1}(x) = N(d) \int_{\mathbb{R}^d} \frac{1}{|y|^{d-2}} f_{x^1}(x-y)\, dy.$$

We represent $\mathbb{R}^d$ as $\{y = (y^1, y') : y^1 \in \mathbb{R}, y' \in \mathbb{R}^{d-1}\}$ and write

$$\int_{\mathbb{R}^d} \frac{1}{|y|^{d-2}} f_{x^1}(x-y)\, dy = \int_{\mathbb{R}^{d-1}} \left(\int_{-\infty}^{\infty} \frac{1}{|y-x|^{d-2}} f_{y^1}(y)\, dy^1 \right) dy'.$$

If $y' \neq x'$, the function $|y-x|^{-(d-2)}$ is an infinitely differentiable function of y^1 and one can transform the inside integral by integrating by parts, thus finding that for almost all y' it equals

$$(d-2) \int_{-\infty}^{\infty} \frac{y^1 - x^1}{|y-x|^d} f(y)\, dy^1.$$

Observe that

$$\int_{\mathbb{R}^d} \left| \frac{y^1 - x^1}{|y-x|^d} f(y) \right| dy \leq \int_{\mathbb{R}^d} \frac{1}{|y-x|^{d-1}} |f(y)|\, dy < \infty$$

since f has compact support and

$$\int_{|y-x|<R} \frac{1}{|y-x|^{d-1}}\, dy < \infty \tag{3}$$

for any R. This allows us to write the repeated integral which we obtain after integrating by parts as a usual integral over $\mathbb{R}^d$ and leads to the formula

$$u_{x^1}(x) = N(d)(d-2) \int_{\mathbb{R}^d} \frac{y^1 - x^1}{|y-x|^d} f(y)\, dy = N_1(d) \int_{\mathbb{R}^d} \frac{y^1}{|y|^d} f(y+x)\, dy.$$

Similarly for any $i = 1, ..., d$

$$u_{x^i}(x) = N_1(d) \int_{\mathbb{R}^d} \frac{y^i}{|y|^d} f(y+x)\, dy.$$

For the same reasons as above one can differentiate this formula with respect to x one more time and find

$$u_{x^ix^j}(0) = N_1(d)\int_{\mathbb{R}^d}\frac{y^i}{|y|^d}f_{y^j}(y)\,dy.$$

However, if we now try to go the same way as above integrating by parts, we will not be able to rewrite this expression as a usual integral over $\mathbb{R}^d$ because this time we will have d in place of $d-1$ in (3) and the integral will diverge. This is the reason why we take a $\zeta \in C_0^\infty$ depending only on $|y|$ and such that $\zeta(0)=1$ and write

$$u_{x^ix^j}(0) = N_1(d)\int_{\mathbb{R}^d}\frac{y^i}{|y|^d}(f(y)-f(0)\zeta(y))_{y^j}\,dy + f(0)N_{ij}, \tag{4}$$

where

$$N_{ij} := N_1(d)\int_{\mathbb{R}^d}\frac{y^i}{|y|^d}\zeta_{y^j}(y)\,dy.$$

Now we integrate by parts with respect to y^j and notice that

$$-\frac{\partial}{\partial y^j}\frac{y^i}{|y|^d} = \frac{y^iy^jd-\delta^{ij}|y|^2}{|y|^{d+2}} =: K_{ij}(y),$$

$$|K_{ij}(y)| \le \frac{N}{|y|^d}, \quad |f(y)-f(0)\zeta(y)| = |f(y)-f(0)+f(0)(1-\zeta(y))| \le N|y|.$$

We also use the fact that $f(y)-f(0)\zeta(y)$ has compact support. Then we easily transform (4) into

$$u_{x^ix^j}(0) = N_1(d)\int_{\mathbb{R}^d}K_{ij}(y)(f(y)-f(0)\zeta(y))\,dy + f(0)N_{ij}, \tag{5}$$

with the integral converging in the usual sense. In particular,

$$\int_{\mathbb{R}^d}K_{ij}(y)(f(y)-f(0)\zeta(y))\,dy = \lim_{r\downarrow 0}\int_{|y|\ge r}K_{ij}(y)(f(y)-f(0)\zeta(y))\,dy.$$

It turns out that for any $r>0$

$$\int_{|y|\ge r}K_{ij}(y)\zeta(y)\,dy = 0.$$

By using the fact that ζ is radially symmetric, to prove this, it suffices to show that the integral of K_{ij} over spheres centered at the origin vanishes. If

$i \neq j$, this is obvious because K_{ij} is anti-symmetric in y^i. If $i = j$, then the said integrals are obviously independent of i and their sum is zero because

$$\sum_i (y^i)^2 d - |y|^2 d = 0.$$

Thus,

$$u_{x^i x^j}(0) = N_1(d) \lim_{r \downarrow 0} \int_{|y| \geq r} K_{ij}(y) f(y)\, dy + f(0) N_{ij}.$$

Similarly, for any x, if $u \in C_0^3$,

$$u_{x^i x^j}(x) = N_1(d) \mathcal{K}_{ij} f(x) + f(x) N_{ij}, \tag{6}$$

where

$$\mathcal{K}_{ij} f(x) = \lim_{r \downarrow 0} \int_{|y| \geq r} K_{ij}(y) f(x+y)\, dy$$

is one of the so-called singular-integral operators. That it is singular is reflected, in particular, in the fact that the usual estimate

$$\|k * f\|_{\mathcal{L}_p} \leq \|k\|_{\mathcal{L}_1} \|f\|_{\mathcal{L}_p}$$

does not allow us to estimate the $\mathcal{L}_p$ norm of $\mathcal{K}_{ij} f$ through the $\mathcal{L}_p$ norm of f since the K_{ij} are not integrable near the origin, let alone near infinity.

A quite discouraging fact is that (6) does not seem to allow us to get the estimate even if $p = 2$, when we know that the estimate exists from Theorem 3.16.

There is a theory of singular-integral operators (see, for instance, [**19**]) in which one shows that they are well defined and bounded on $\mathcal{L}_p$ for any $p \in (1, \infty)$ and this along with (6) yields the estimate

$$\|u_{xx}\|_{\mathcal{L}_p} \leq N(d, p) \|\Delta u\|_{\mathcal{L}_p}.$$

We are going to avoid using the theory of singular-integral operators and obtain this estimate in a somewhat different way.

1. Exercise. Show that $N_{ij} = -\delta^{ij} d^{-1}$.

To give the reader some feeling about certain issues arising in the theory of singular integrals, we give the following.

2. Exercise. Take $k \neq j$ and for $\varepsilon \in (0, 1)$ define $K_{kj\varepsilon}(x) = K_{kj}(x)$ if $\varepsilon < |x| < \varepsilon^{-1}$ and $K_{kj\varepsilon}(x) = 0$ otherwise. Take a $u \in C_0^\infty$, set $f = -\Delta u$, and prove that in the $\mathcal{L}_2$ sense $N_1(d) K_{kj\varepsilon} * f \to u_{x^k x^j}$.

3. Exercise*. Take $d \geq 3$, an $f \in C_0^\infty$, define u by (2) and show that u is infinitely differentiable, $\Delta u = -f$ in $\mathbb{R}^d$, and $\|u_{xx}\|_{\mathcal{L}_2} \leq N(d)\|f\|_{\mathcal{L}_2}$.

10. Hints to exercises

1.5. The integrand on the left is the divergence of the vector-field F given by

$$F^i = u_{x^i}\Delta u - u_{x^j}u_{x^j x^i}$$

and on the boundary we have $u_x = (u_x \cdot n)n$,

$$(F, n) = (u_x, n)(\Delta u - (u_{xx}n) \cdot n).$$

To compute $\Delta u - (u_{xx}n)\cdot n$ at a point $x_0 \in \partial B$, take $d-1$ unit mutually orthogonal vectors $\ell_1, ..., \ell_{d-1}$ orthogonal to $n(x_0)$, so that owing to Exercise 1.4

$$\Delta u - (u_{xx}n) \cdot n = \sum_{k=1}^{d-1} u_{x^i x^j}\ell^i_k \ell^j_k$$

at x_0. Then, after differentiating the relation $u_x = (u_x \cdot n)n$ at x_0 along ℓ_k, use the equality $u_x \cdot \ell_k = 0$ and show that at this point

$$u_{x^i x^j}\ell^j_k = \ell^i_k(u_x \cdot n) + n^i(u_{x^j x} \cdot n)\ell^j_k, \quad u_{x^i x^j}\ell^i_k\ell^j_k = u_x \cdot n.$$

By the way, the latter identity is known as the Meusnier theorem.

1.8. Try e^{x^1}.

1.10. To prove the first assertion, show that the points where u/ζ takes its positive maximum value can only belong to $\partial\Omega$, and then let $\varepsilon \downarrow 0$. To prove the second, either consider the function

$$u(x) - u(x_0)\exp(-x^1\sqrt{\lambda})$$

or, what will be used in the hint to Exercise 1.11, observe that Lemma 1.7 holds true even if we allow $|u|$ to grow exponentially at infinity and then consider $ue^{\delta x^1} - u(x_0)$ with appropriate $\delta > 0$.

1.11. Follow the hint to Exercise 1.10 to make λ strictly positive and preserve $u \leq \sup_{\partial\Omega} u_+$. Then work with $u/\cosh(\varepsilon|x'|)$ and use Exercise 1.10.

1.12. (i) Use a simple corollary of Minkowski's inequality: $\|f*h\|_{\mathcal{L}_p} \leq \|f\|_{\mathcal{L}_p}\|h\|_{\mathcal{L}_1}$ (see Lemma 8.1); (ii) observe that for $u_\varepsilon(x) = \varepsilon^{-d}u(x/\varepsilon)$, we have $\tilde{u}_\varepsilon(\xi) \to \int u\,dx$ as $\varepsilon \downarrow 0$.

1.13. To prove the fact mentioned in the statement, consider $F(\phi_n D^\alpha v)$, where $\phi_n = \phi(x/n)$, $\phi \in C_0^\infty$, $\phi(x) = 1$ for $|x| \leq 1$. Then send n to infinity and use the continuity of the Fourier transform in $\mathcal{L}_2$. Then while proving the denseness, use part of the proof of Theorem 1.6 and the facts that $v := g * u$ and all its derivatives of any order are in $\mathcal{L}_2$ and are bounded if $g \in \mathcal{L}_q$, $q = p/(p-1)$ (≤ 2), and $u \in C_0^\infty$. Then you will arrive at $\sigma\tilde{v} = 0$, where

$$\sigma(\xi) = \sum_{|\alpha| \leq m} a^\alpha i^{|\alpha|}\xi^\alpha$$

is the so-called *characteristic polynomial of* L. Finally, observe that $\sigma(\xi) \neq 0$ (a.e.) since at least one of $a^\alpha \neq 0$ (the set where $\sigma = 0$ cannot have density points).

2.2. Show that the vectors $G_{x^1}, ..., G_{x^d}$ are linearly dependent, set $F(x) = x$ and use Exercise 2.1. For the second part use approximations by polynomials and observe that $g(x)/|g(x)| \in \partial B_1$, whenever $g(x) \neq 0$.

2.3. If there are no fixed points, then introduce a mapping G, which sends $x \in \bar{B}_1$ into the point $G(x)$ on ∂B_1 such that x lies between $G(x)$ and $f(x)$ on the straight segment joining those points.

2.4. Reduce the general situation to the one where f is *strictly* concave in y and strictly convex in x. Then consider the function bringing $x \in X$ to the unique point $y(x) \in Y$ maximizing $f(x, y)$ over Y. From the uniqueness of $y(x)$ deduce its continuity in x. Similarly introduce $x(y)$ by minimizing $f(x, y)$ over X. Then from Exercise 2.3 obtain that the mapping $x \to x(y(x))$ has a fixed point $x_0 \in X$. Set $y_0 = y(x_0)$ and prove that $x_0 = x(y_0)$,

$$f(x_0, y_0) = \min_X f(x, y_0) = \max_Y f(x_0, y),$$

$$\max_Y \min_X f(x, y) \geq f(x_0, y_0) \geq \min_X \max_Y f(x, y).$$

This yields a one-sided inequality in (2.1). Show that the opposite inequality is true regardless of whether the requirements (i) and (ii) are imposed or not.

3.6. Observe that there exists a $g_1 \in \mathcal{L}_p$ such that $\|D_1\phi_n - g_1\|_{\mathcal{L}_p} \to 0$ and use the fact that

$$\int_\Omega g D_1\psi \, dx = \lim_{n\to\infty} \int_\Omega \phi_n D_1\psi \, dx = -\lim_{n\to\infty} \int_\Omega (D_1\phi_n)\psi \, dx = \int_\Omega g_1\psi \, dx$$

to conclude that $D_1 g = g_1$.

3.13. There is a $v \in W^k_p(\Omega)$ such that $u^n \to v$ in $W^k_p(\Omega)$. Find $\|v - u\|_{\mathcal{L}_p(\Omega)}$.

3.17. Use the definition of $W^1_p(\Omega)$.

3.18. First prove that if $u \in W^1_p(\Omega)$, then for any $\varepsilon > 0$

$$u_\varepsilon := F_\varepsilon(u) \in W^1_p(\Omega),$$

where $F_\varepsilon(t) = ([t^2+\varepsilon^2]^{1/2} - \varepsilon)I_{t>0}$, and compute $(u_\varepsilon)_x$. After that use Exercise 3.13. You may also like to notice that the functions $F_\varepsilon(t)$ are continuously differentiable on $\mathbb{R}$, $0 \leq F_\varepsilon(t) \uparrow t_+$ as $\varepsilon \downarrow 0$.

3.19. What is $(u_-)_x$?

3.20. If $\|\phi_n\|_{\mathcal{L}_p} \to 0$, then for a subsequence n' we have that $\phi_{n'}(x^1, \cdot) \to 0$ in $\mathcal{L}_p(\mathbb{R}^{d-1})$ for almost all x^1.

3.23. (i) For $d \geq 2$ multiply both parts of $f = \Delta u$ by $(2 - |x|^2)u$ and integrate by parts. (ii) P_n is a linear finite-dimensional space. (iii) Use approximations of f by polynomials.

4.6. Use dilations.

4.7. Notice that $a^{ij}u_{ij} = \operatorname{tr} AU$ and the trace of a matrix is invariant under orthogonal transformations.

4.8. Notice that $Lu = a^{ij}v_{ij}$, where $v_{ij} = u_{x^ix^j} - \lambda\delta^{ij}u$ and use Exercises 1.3 and 4.7.

4.9. Consider $(1-t)(\Delta - 2\lambda) + tL$.

6.6. Use the method of continuity moving λ and Theorems 6.4 and 6.5.

7.3. Prove that the norm is independent of λ and use that $\lambda R_\lambda f = f + R_\lambda \Delta f$ for smooth f.

7.4. Use (7.2) and (7.5) to show that $|\nabla u| \le \lambda^{-1/2} \sup|\lambda u - \Delta u|$. Then minimize with respect to λ. To prove the sharpness, take $d = 1$.

8.3. From Lemma 8.2 and Parseval's identity derive that

$$\int_{\mathbb{R}^d} |\xi^\alpha|^2 |\widetilde{u^{(\varepsilon)}}(\xi)|^2 \, d\xi = N\|D^\alpha u^{(\varepsilon)}\|^2_{\mathcal{L}_2} \le N\|D^\alpha u\|^2_{\mathcal{L}_2},$$

where

$$\widetilde{u^{(\varepsilon)}}(\xi) = N\tilde{u}(\xi)\tilde{\zeta}(\varepsilon\xi).$$

You may first like to prove part of the above for $u \in C_0^\infty$.

8.6. Use mollifiers.

8.7. Use mollifiers.

8.8. Take ζ with support in $2\alpha|x'| < -x^1$.

9.1. What is $\sum_i N_{ii}$?

9.2. First show that

$$K_{kj\varepsilon} * f(x) = \int_{\varepsilon^{-1} > |y| > \varepsilon} K_{ij}(y)[f(x+y) - f(x)\zeta(y)] \, dy$$

and conclude that $N_1(d)K_{kj\varepsilon} * f \to u_{x^kx^j}$ on $\mathbb{R}^d$. Then introduce R as a number such that $f(x) = 0$ for $|x| > R$ and show that $|K_{kj\varepsilon} * f(x)|$ is bounded independently of ε if $|x| \le R+1$ by repeating the argument which led to (9.5). To prove the uniform boundedness of $|K_{kj\varepsilon} * f(x)|$ for $|x| \ge R+1$, show that $|K_{kj\varepsilon} * f(x)| \le N|x|^{-d}$.

9.3. First work with $G_\lambda f$.