Contents

Preface xi

Introduction xiii
What is discrete differential geometry xiii
Integrability xv
From discrete to smooth xvii
Structure of this book xxi
How to read this book xxii
Acknowledgements xxiii

Chapter 1. Classical Differential Geometry 1
1.1. Conjugate nets 2
1.1.1. Notion of conjugate nets 2
1.1.2. Alternative analytic description of conjugate nets 3
1.1.3. Transformations of conjugate nets 4
1.1.4. Classical formulation of F-transformation 5
1.2. Koenigs and Moutard nets 7
1.2.1. Notion of Koenigs and Moutard nets 7
1.2.2. Transformations of Koenigs and Moutard nets 9
1.2.3. Classical formulation of the Moutard transformation 10
1.3. Asymptotic nets 11
1.4. Orthogonal nets 12
1.4.1. Notion of orthogonal nets 12
1.4.2. Analytic description of orthogonal nets 14
1.4.3. Spinor frames of orthogonal nets 15
1.4.4. Curvatures of surfaces and curvature line parametrized surfaces 16

1.4.5. Ribaucour transformations of orthogonal nets 17
1.5. Principally parametrized sphere congruences 19
1.6. Surfaces with constant negative Gaussian curvature 20
1.7. Isothermic surfaces 22
1.8. Surfaces with constant mean curvature 26
1.9. Bibliographical notes 28

Chapter 2. Discretization Principles. Multidimensional Nets 31
2.1. Discrete conjugate nets (Q-nets) 32
2.1.1. Notion and consistency of Q-nets 32
2.1.2. Transformations of Q-nets 38
2.1.3. Alternative analytic description of Q-nets 40
2.1.4. Continuous limit 42
2.2. Discrete line congruences 43
2.3. Discrete Koenigs and Moutard nets 47
2.3.1. Notion of dual quadrilaterals 47
2.3.2. Notion of discrete Koenigs nets 49
2.3.3. Geometric characterization of two-dimensional discrete Koenigs nets 54
2.3.4. Geometric characterization of three-dimensional discrete Koenigs nets 56
2.3.5. Function ν and Christoffel duality 58
2.3.6. Moutard representative of a discrete Koenigs net 60
2.3.7. Continuous limit 60
2.3.8. Notion and consistency of T-nets 61
2.3.9. Transformations of T-nets 63
2.3.10. Discrete M-nets 65
2.4. Discrete asymptotic nets 66
2.4.1. Notion and consistency of discrete asymptotic nets 66
2.4.2. Discrete Lelieuvre representation 70
2.4.3. Transformations of discrete A-nets 72
2.5. Exercises 73
2.6. Bibliographical notes 82

Chapter 3. Discretization Principles. Nets in Quadrics 87
3.1. Circular nets 88
3.1.1. Notion and consistency of circular nets 88
3.1.2. Transformations of circular nets 92
3.1.3. Analytic description of circular nets 93
3.1.4. Möbius-geometric description of circular nets 96

3.2. Q-nets in quadrics 99
3.3. Discrete line congruences in quadrics 101
3.4. Conical nets 103
3.5. Principal contact element nets 106
3.6. Q-congruences of spheres 110
3.7. Ribaucour congruences of spheres 113
3.8. Discrete curvature line parametrization in Lie, Möbius and Laguerre geometries 115
3.9. Discrete asymptotic nets in Plücker line geometry 118
3.10. Exercises 120
3.11. Bibliographical notes 123

Chapter 4. Special Classes of Discrete Surfaces 127
4.1. Discrete Moutard nets in quadrics 127
4.2. Discrete K-nets 130
4.2.1. Notion of a discrete K-net 130
4.2.2. Bäcklund transformation 133
4.2.3. Hirota equation 133
4.2.4. Discrete zero curvature representation 139
4.2.5. Discrete K-surfaces 139
4.2.6. Discrete sine-Gordon equation 142
4.3. Discrete isothermic nets 145
4.3.1. Notion of a discrete isothermic net 145
4.3.2. Cross-ratio characterization of discrete isothermic nets 147
4.3.3. Darboux transformation of discrete isothermic nets 151
4.3.4. Metric of a discrete isothermic net 152
4.3.5. Moutard representatives of discrete isothermic nets 155
4.3.6. Christoffel duality for discrete isothermic nets 156
4.3.7. 3D consistency and zero curvature representation 158
4.3.8. Continuous limit 160
4.4. S-isothermic nets 161
4.5. Discrete surfaces with constant curvature 170
4.5.1. Parallel discrete surfaces and line congruences 170
4.5.2. Polygons with parallel edges and mixed area 170
4.5.3. Curvatures of a polyhedral surface with a parallel Gauss map 173
4.5.4. Q-nets with constant curvature 175
4.5.5. Curvature of principal contact element nets 177

4.5.6. Circular minimal nets and nets with constant mean curvature 178
4.6. Exercises 179
4.7. Bibliographical notes 183

Chapter 5. Approximation 187
5.1. Discrete hyperbolic systems 187
5.2. Approximation in discrete hyperbolic systems 190
5.3. Convergence of Q-nets 196
5.4. Convergence of discrete Moutard nets 197
5.5. Convergence of discrete asymptotic nets 199
5.6. Convergence of circular nets 200
5.7. Convergence of discrete K-surfaces 205
5.8. Exercises 206
5.9. Bibliographical notes 207

Chapter 6. Consistency as Integrability 209
6.1. Continuous integrable systems 210
6.2. Discrete integrable systems 213
6.3. Discrete 2D integrable systems on graphs 215
6.4. Discrete Laplace type equations 217
6.5. Quad-graphs 218
6.6. Three-dimensional consistency 220
6.7. From 3D consistency to zero curvature representations and Bäcklund transformations 222
6.8. Geometry of boundary value problems for integrable 2D equations 227
6.8.1. Initial value problem 228
6.8.2. Extension to a multidimensional lattice 231
6.9. 3D consistent equations with noncommutative fields 235
6.10. Classification of discrete integrable 2D systems with fields on vertices. I 239
6.11. Proof of the classification theorem 242
6.11.1. 3D consistent systems, biquadratics and tetrahedron property 242
6.11.2. Analysis: descending from multiaffine Q to quartic r 245
6.11.3. Synthesis: ascending from quartic r to biquadratic h 247

6.11.4. Synthesis: ascending from biquadratics h^{ij} to multiaffine Q 249
6.11.5. Putting equations $Q = 0$ on the cube 251
6.12. Classification of discrete integrable 2D systems with fields on vertices. II 252
6.13. Integrable discrete Laplace type equations 256
6.14. Fields on edges: Yang-Baxter maps 261
6.15. Classification of Yang-Baxter maps 266
6.16. Discrete integrable 3D systems 272
6.16.1. Fields on 2-faces. 272
6.16.2. Fields on vertices. 276
6.17. Exercises 279
6.18. Bibliographical notes 286

Chapter 7. Discrete Complex Analysis. Linear Theory 291
7.1. Basic notions of discrete linear complex analysis 291
7.2. Moutard transformation for discrete Cauchy-Riemann equations 294
7.3. Integrable discrete Cauchy-Riemann equations 297
7.4. Discrete exponential functions 300
7.5. Discrete logarithmic function 302
7.6. Exercises 307
7.7. Bibliographical notes 308

Chapter 8. Discrete Complex Analysis. Integrable Circle Patterns 311
8.1. Circle patterns 311
8.2. Integrable cross-ratio and Hirota systems 313
8.3. Integrable circle patterns 316
8.4. z^a and $\log z$ circle patterns 319
8.5. Linearization 324
8.6. Exercises 326
8.7. Bibliographical notes 327

Chapter 9. Foundations 331
9.1. Projective geometry 331
9.2. Lie geometry 335
9.2.1. Objects of Lie geometry 335
9.2.2. Projective model of Lie geometry 336
9.2.3. Lie sphere transformations 339

9.2.4. Planar families of spheres; Dupin cyclides 340
9.3. Möbius geometry 341
9.3.1. Objects of Möbius geometry 341
9.3.2. Projective model of Möbius geometry 344
9.3.3. Möbius transformations 348
9.4. Laguerre geometry 350
9.5. Plücker line geometry 353
9.6. Incidence theorems 357
9.6.1. Menelaus' and Ceva's theorems 357
9.6.2. Generalized Menelaus' theorem 360
9.6.3. Desargues' theorem 361
9.6.4. Quadrangular sets 362
9.6.5. Carnot's and Pascal's theorems 364
9.6.6. Brianchon's theorem 366
9.6.7. Miquel's theorem 367

Appendix. Solutions of Selected Exercises 369
A.1. Solutions of exercises to Chapter 2 369
A.2. Solutions of exercises to Chapter 3 376
A.3. Solutions of exercises to Chapter 4 377
A.4. Solutions of exercises to Chapter 6 381

Bibliography 385

Notation 399

Index 401