CHAPTER 1

# Introduction

Jeremy J. Gray and Karen Hunger Parshall
Open University, United Kingdom and University of Virginia, United States

## Algebra: What? When? Where?

It is an interesting, and difficult, problem even to define algebra; to begin to give it a history presents yet another challenge. Some historians of mathematics identify the abstract and general features of Mesopotamian mathematics with algebra, and so consider it the oldest branch of written mathematics, dating from at least 2000 B.C.E.[1] Others see its origins around 150 C.E. in Diophantus's *Arithmetica* with its solutions of general, indeterminate problems [Klein, 1968]. Still others find its more modern roots in medieval Islam, noting that the very word originated only in the ninth century in al-Khwārizmī's text entitled *al-Kitāb al-mukhtasar fī hisāb al-jabr wa'l-muqābala* or *The Compendious Book on Calculation by Completion and Balancing*. There, "al-jabr"—which was ultimately Latinized into "algebra"—translates as "completion" or "restoration." It referred to adding the same thing to both sides of an equation or restoring all of the terms to a standard form, and al-Khwārizmī himself regarded it as different from—although not unrelated to—the kind of geometrical procedures that Euclid had utilized in the *Elements* (300 B.C.E.) and that had held sway over the course of the intervening millennium [van der Waerden, 1985]. Those historians who insist on algebra as something distinct from geometry, however, see its emergence in the sixteenth-century, European texts of mathematicians such as Girolamo Cardano, Niccolò Tartaglia, and Rafael Bombelli on cubic and quartic equations as well as on the introduction of complex numbers,[2] while others require the presence of the characteristic features of high school algebra today, namely, the familiar signs for the arithmetic operations and the use of letters for constants and unknowns. For the latter, algebra came about in the late sixteenth and seventeenth centuries in the work of François Viète, Thomas Harriot, Pierre de Fermat, and René Descartes.[3] Finally, mathematicians tend to share yet another view of algebra's origins. For

[1]For a very sophisticated and highly nuanced interpretation of Babylonian algebra, see [Høyrup, 2002]. Many histories of algebra begin their story in ancient Mesopotamia. See, for example, [van der Waerden, 1983], [Scholz, 1990], [Sesiano, 1999], and [Bashmakova and Smirnova, 2000].

[2]See, for example, [Cardano, 1545/1968] for a sense of this sixteenth-century Italian work.

[3]For a sense of the seventeenth-century scene, see [Bos, 2001]. In [Nový, 1973], the history of algebra "begins" some time after 1770.

them, algebra is modern algebra, namely, the subject that was put into book form and given its name by Bartel van der Waerden beginning in 1930.[4]

All of these definitions have something to recommend them, and, indeed, there is a degree of continuity between these types of algebra that suggests a coherence to the topic across the centuries. The ancient Mesopotamians possessed systematic methods for solving numerical questions about numbers and their reciprocals and about numbers and squares. Interest in this topic passed to medieval Islamic mathematicians and eventually from them to the Renaissance Italians. When, in the late eighteenth and early nineteenth centuries, the quintic equation proved a stumbling block to the production of algorithms or formulas for solving polynomial equations of successive degrees, a series of mathematicians took up structural investigations of the problem. Although the list of structurally minded algebraists could justifiably begin with Harriot,[5] it most certainly includes the names of the eighteenth- and early nineteenth-century mathematicians, Alexandre Vandermonde, Joseph-Louis Lagrange, Niels Henrik Abel, and Évariste Galois.[6]

If the solution of equations has long engaged mathematical minds, so too has the problem of providing integer—and, indeed, natural number—solutions to mathematical questions. Whether this began and continued in a culture that found it natural to restrict the range of solutions in this fashion—so natural it went without comment—or whether it was a deliberately imposed restriction, a long-standing tradition connects algebra with number theory, and specifically with the branch today called algebraic number theory. Fermat may have labored in vain to interest his contemporaries in the subject, and such was his melancholy opinion, but with Leonhard Euler and Lagrange the subject burgeoned in the eighteenth century and became one of the key areas of interest to the German mathematical community in the nineteenth.[7]

Algebra has also established a close connection with geometry. While some have argued that Euclid was doing "geometrical algebra" as early as 300 B.C.E., the modern connection between the two fields is much more recent, not extending back chronologically much before the time of Fermat and Descartes.[8] This, however, became a vital connection, with each discipline at times turning to the other for central insights and methods. Euler was the first to *define* the conic sections algebraically, rendering it obvious that the same $b^2 - 4ac$ test that distinguished ellipses from hyperbolas distinguished the real and complex roots of a polynomial equation. In the nineteenth century, mathematicians from George Boole, Arthur Cayley, and James Joseph Sylvester in Great Britain to Charles Hermite, Paul Gordan, and David Hilbert on the Continent pursued this clue energetically and, in so doing, developed the new field of invariant theory.[9] Algebraic and, especially, projective geometry was one half of a subject, invariant theory the other, and so

---

[4]On the rise of the structural approach to algebra that culminated, in some sense, in van der Waerden's text, see [Corry, 1996].

[5]On Harriot's mathematical work, see [Stedall, 2003].

[6]For an analysis of these developments, see, for example, [Edwards, 1984].

[7]See [Euler, 1770-1771/1984] with its additions by Lagrange.

[8]The debate over "geometrical algebra" flared famously in an exchange between Bartel van der Waerden and Sabatei Unguru on the pages of the *Archive for History of Exact Sciences*. See [Unguru, 1975] and [van der Waerden, 1976].

[9]On these developments, see, for example, [Parshall, 1989].

began an amicable and long-running family dispute between those who stressed geometry and those who emphasized algebra.

Algebra even has a claim to being if not the core of mathematics then at least its language. Most papers in any branch of mathematics have an algebraic cast to them determined by the very symbolism used, even if the intellectual difficulties are in another field. This fact led some eighteenth-century philosophers to regard mathematics—that is, algebraic mathematics—as the best language because it generated arguments having the highest degree of certainty. Two hundred years later, mathematicians turned to syntax and formal languages to seek assurances about the nature of mathematics, while linguists such as Noam Chomsky defined language itself as a purely syntactic object expressed in mathematical terms. Algebras, languages, complexity, and issues in computation and the theory of algorithms are central to such topics as the P versus NP question.

Algebra, then, has not only an ever-evolving and distinct identity but also close connections to arithmetic and number theory, to geometry, and to topics in linguistics and computer science. It is a vast and vastly complicated area to try to pin down from an historical point of view. As noted in the acknowledgments, this book grew out of a conference hosted by the Mathematical Sciences Research Institute, Berkeley in 2003. It thus explores a number of *episodes* in the modern period of algebra's development, specifically episodes that stretch from the time of the French Revolution in the late eighteenth century to the arrival of modern algebra in the early twentieth century to the advent of category theory as an algebraic tool in the mid-twentieth century.

## Episodes in the History of Modern Algebra

The volume opens with a chapter by Eduardo Ortiz on "Babbage and French *Idéologie*: Functional Equations, Language, and the Analytical Method" that explores eighteenth- and early nineteenth-century conceptions of mathematics as a language. As Ortiz shows, this was a particular concern of the Abbé de Condillac, who advocated inventing and adapting languages in the light of the experience provided by various sciences, taking algebra as a model. This revived a debate that had flourished in the seventeenth century about the relationship between logic and grammar, but Condillac reversed the old view and argued that mastery of grammar should come before the study of logic. As this may suggest, Condillac was interested in the idea that mental processes involved in understanding, such as analogy and association of ideas, should feature more prominently in the language in which thought is expressed. Charles Babbage read Condillac's work as a young man and applied it to his own research on functional equations. The idea of analogy in a context that, for Babbage, was much more algebraic than analytic, struck him forcefully, and he devised a method for constructing general solutions when special cases are given. The method, Ortiz argues, seems capable of further use even today, but has never before been adequately described in the historical literature.

The themes of language, the operational calculus, and broader forces at play in the history of mathematics are continued in the chapter by Sloan Despeaux on Duncan Gregory and the *Cambridge Mathematical Journal*. The emerging generation of British mathematicians in the 1820s and 1830s was largely indifferent to the strictures of Augustin-Louis Cauchy and other continental Europeans about the lack of rigor in the formal methods of the operational calculus. They applied it cavalierly, but with some success, to various kinds of differential equations, and while they had to admit they could not make it work for equations with variable coefficients, they were nonetheless led to interesting reflections about algebraic processes. Moreover, although they did not succeed in their goal of creating a single, unified field that embraced polynomial equations, differential equations, and functional equations, they did create and sustain a journal for mathematics in Britain that served as a key line of communication for a developing community of British mathematicians. Among the journal's notable publications was George Boole's work on linear differential equations with constant coefficients, a paper which highlighted not only the potential strength of the Cambridge-style, formal methods but also the journal's effectiveness in conveying new mathematical ideas to a self-taught Boole with no real Cambridge connections other than access to its pages.

The episodes in the history of modern algebra that Ortiz and Despeaux detail seem far from the modern algebra that has often been seen as having sprung with all the inevitability of a great invention in the early decades of the twentieth century from the group around Emmy Noether. It is as hard for mathematicians to imagine the twentieth century without it as it is to imagine the world without Hollywood, and this has hindered historians' ability to describe the transformation from the older algebra to the new. It has thus seemed easier to look at the precursors in the Noether line, such as Richard Dedekind, than to explore the work of those who stand outside it. The next two chapters here, however, take on that harder task.

Olaf Neumann traces the concept of divisibility from its origins in (and even before) the eighteenth century to the twentieth-century work of Emmy Noether and Wolfgang Krull. The idea was to establish, over a variety of domains, theories of divisibility that mimicked that of elementary number theory. The development of this idea, as Neumann shows, was not unilateral. Following Ernst Kummer's invention of ideal complex numbers in the mid-nineteenth century, two rather distinct approaches evolved. One—championed by Karl Weierstrass, Egor Zolotarev, Kurt Hensel, Krull, and others—centered on the idea of embedding the various domains either in principal ideal domains or in direct products of principle ideal domains. The other grew out of Dedekind's theory of ideals and engaged a string of mathematicians from David Hilbert through Adolf Hurwitz and Francis Macaulay to Emmy Noether. Thus, while the "direct" line to Noether is explored, so is another, equally important, intertwining line. The two lines taken together played a key role in the development of the modern concepts of commutative ring theory.

Another key nineteenth-century figure who stands largely off of the "direct" line to Noether is Leopold Kronecker. In his chapter, "Kronecker's Fundamental Theorem of General Arithmetic," Harold Edwards explores just what Kronecker meant by this theorem, which is a replacement for the fundamental theorem of algebra. The question is how to compute with roots of a polynomial equation, or, equivalently, how to factor polynomials with coefficients in an algebraic number field. Kronecker, in keeping with his constructivist approach to mathematics, advocated not regarding these roots as real numbers, and therefore had to develop a method for working with them. He succeeded in his task and saw his approach as lying at the heart of what he called the "precious" Galois principle. This led Kronecker to his new fundamental theorem, and Edwards here not only gives his own account of it but also uses this account to shed light on his own understanding of some of Kronecker's murkier arguments.

The next four chapters in this volume deal with the way in which the structural theory of algebras and hypercomplex number systems was brought to bear on the theory of numbers. Key figures here are Leonard Eugene Dickson and A. Adrian Albert on the American side, as Della Fenster discusses in her chapter, and Richard Brauer, Helmut Hasse, and Emmy Noether on the German side, as Günther Frei, Joachim Schwermer, and Charles Curtis discuss in theirs. The picture that emerges is one of fruitful and competitive interaction between the Americans and the Germans.

Günther Frei takes us back to the mid-nineteenth-century origins of the structure theory of algebras in the British ideas of William Rowan Hamilton, Charles Graves, and Arthur Cayley and traces the development of that theory through the work of Joseph H. M. Wedderburn on the general structure theory and Dickson's research on associative and cyclic algebras in the early decades of the twentieth century. In his 1923 book, *Algebras and Their Arithmetics*, Dickson laid out many of these ideas as well as work at the interface of the theory of algebras and algebraic number theory on the so-called arithmetic of algebras. The German translation of his book, which appeared four years later in 1927, sparked rapid developments in the arithmetic-algebraic school of research that centered on the work of Brauer, Hasse, and Noether. In particular, Frei traces in this broader context the development and extension of the Reciprocity Law from Hilbert's interpretation of it as an infinite product over all primes of norm residue symbols, to Hasse's success in providing explicit expression of these symbols in the quadratic case, to later breakthroughs in treating higher-order cases. As he demonstrates, Dickson's book was critical in linking the results of the American and German schools and, thereby, in giving rise to powerful, new techniques in modern algebra.

Another key technique in the theory of algebras *per se* is the local-global principle for algebras. Its origins are associated with Hasse's work in number theory, the subject of Joachim Schwermer's chapter. Schwermer grounds his analysis in the work of Gauss on quadratic forms, takes it through the novel ideas Hermann Minkowski introduced in his study of quadratic forms in several variables, and culminates in Hasse's introduction of $p$-adic methods in the formulation of a local-global principle for number fields. As Schwermer shows, quoting a letter from Hasse to Hermann Weyl, Hasse seems to have gotten this idea from his reading of Minkowski, and Schwermer traces how Hasse brought together ideas from Minkowski, Kurt Hensel, and Emmy Noether in formulating his own theory.

Della Fenster's chapter also treats an aspect of Hasse's work, focusing on the history of the often-so-called Brauer-Hasse-Noether Theorem but highlighting Adrian Albert's work in that direction. She carefully unpacks the story of how Brauer, Hasse, and Noether's joint paper of 1931 showed that, indeed, a central division algebra is cyclic, only to be followed two months later by a joint paper of Albert and Hasse that gave a new proof of the same result and an account of what had preceded it. Fenster shows that Albert's work had its origins in a vigorous, American research effort that had stemmed from Wedderburn's reduction in 1907 of the study of associative algebras over a field to the classification of division algebras (also called skew fields) and from his suggestion that these algebras are best regarded as algebras over their centers. Wedderburn had also noted that the dimension of a central division algebra over its base field must be a perfect square, while Dickson had introduced (in 1906) the concept of a cyclic division algebra. The question thus naturally arose whether a central division algebra is necessarily cyclic, since those of small dimension were shown to be, although not without some effort. Albert, a student of Dickson, not surprisingly approached the Brauer-Hasse-Noether Theorem from this direction.

Although as Frei's and Fenster's chapters show, the work of the Americans and the Germans intersected in the solution of key, open questions, their chapters as well as Schwermer's equally underscore the fact that mathematical concerns were different on the two sides of the Atlantic. In his chapter, Charles Curtis explores a fundamental German interest, namely, the use of noncommutative methods in number theory as animated by Emmy Noether's confidence in the fruitfulness of such an approach. Curtis concentrates on Noether's address to the International Congress of Mathematicians in Zürich in 1932, where she spoke on the application of noncommutative methods to commutative algebra and, in particular, to algebraic number theory. As Fenster also discussed, it was the profound immersion of the Germans in number theory—and especially in the theory of $p$-adic numbers—that had given them the edge over Albert, while the Americans had the greater experience in dealing with algebras. Curtis looks at the different techniques of Brauer and Noether, then at Hasse's involvement and the vindication of Noether's hopes for the approach, and finally at Max Deuring's contributions to it.

In different ways and in different contexts what is at issue in the chapters by Frei, Schwermer, Fenster, and Curtis—chapters which deal chronologically with the decades immediately preceding and immediately following the turn of the twentieth century—is the emergence of the mathematician's notion of "modern algebra." What, though, it makes sense to ask, was algebra like immediately before and during the transformation into this "modern" phase? This is precisely the historical question that Leo Corry addresses in his chapter, "From *Algebra* (1895) to *Moderne Algebra* (1930)," by closely examining the changing categorization of algebraic research in the principal abstracting journal of the period, the *Jahrbuch über die Fortschritte der Mathematik*. In the period from the 1890s to 1930, a vast amount of new algebra was created, and this shows up in the content of books by authors such as Dickson. These books nevertheless adhered, to varying degrees, to the overview of algebra that had been presented in 1895 by Heinrich Weber in his definitive three-volume *Lehrbuch der Algebra*. The decisive novelty in Bartel van der Waerden's two-volume *Moderne Algebra* of 1930–1931 was the new organization or direction of the subject. Weber and his immediate successors saw algebra in general and Galois theory in particular as the study of the solution of equations illuminated by the concepts of group and field. Polynomials in several variables form a closely related subject; algebraic number theory lies a little farther away and closer to analytic number theory. Later developments (many of them discussed in the present volume) involving hypercomplex number systems, the structure of algebras, and abstract field theory complicated this picture, as Corry shows with a shrewd use of the *Jahrbuch*. Van der Waerden and the "Noether boys" inverted the picture, so that structural features (groups, rings, fields) became central to what algebra was about, and topics such as the solution of equations and the real numbers became consequences or applications of them.

While van der Waerden's book was critical in the process that defined "modern algebra," his work in algebraic geometry was also important in redefining that field in the first half of the twentieth century. The final three chapters in this volume look precisely at the overlap between the history of algebra and the history of algebraic geometry. In the first, Norbert Schappacher examines and contextualizes van der Waerden's research in algebraic geometry. As he shows, van der Waerden's ideas arose at precisely the historical moment when the standing of contemporary Italian work on algebraic geometry was in decline and the modern structural algebra was on the rise. It is tempting to suppose, and commonly held, that it was van der Waerden's strong identification with the program to bring in modern algebra that drove him to rewrite algebraic geometry. A variant of this unduly simple thesis suggests that it was a lack of rigor among the Italians that impelled van der Waerden to bring in algebra. As Schappacher convincingly argues, neither of these simplistic readings is, in fact, correct. Indeed, many mathematicians found it hard both to grasp the work of Francesco Severi and others and to formulate it in other, perhaps more algebraic, ways, but other factors—publishing practices and politics, among them—also played a role. In any event, André Weil and Oscar Zariski shifted algebraic geometry into a modern algebraic mode, and the subtleties of van der Waerden's evolving position on the shift underscore not only the complexity of the process but also the range of possibilities involved in it. For example, van der Waerden was by no means devoted to rewriting all geometric concepts in terms of ideal theory. He was much more concerned with introducing and exploiting his concept of a generic point on an algebraic variety and with defining intersection multiplicity. He thus stayed closer to the geometric subject matter than did, for example, Zariski.

Silke Slembek's chapter focuses precisely on the evolution of Zariski's algebraic geometric ideas. As Zariski himself famously observed, his book on *Algebraic Surfaces* (1935) forever diverted him from the Italian approach in which he had first matured as a mathematician. Slembek argues that Zariski was, indeed, concerned in the 1930s with creating, through the medium of commutative algebra, a way of making rigorous the Italian insights into geometry. He called his approach the "arithmetization" of algebraic geometry because of his use of Krull's arithmetical ideal theory, but as Slembek shows, he was also profoundly innovative. For example, Zariski concentrated on the treatment—and, if possible, the resolution—of singularities of an algebraic surface. He wanted a proof that was not only rigorous (Robert Walker had provided one in 1935 that met Zariski's standards) but that could claim to be geometric rather than analytic. He succeeded in this in 1939 by exploiting his notion of the arithmetically normal variety. Zariski showed that an algebraic surface can be reduced to an arithmetically normal algebraic surface and that the resolution of singularities for such surfaces can be rigorously established. As Schappacher's and Slembek's chapters underscore, van der Waerden and Zariski may be seen to have taken different paths, the one more geometric and the other more algebraic.

The volume's final chapter moves the story forward in time another two decades to Alexandre Grothendieck and his metaphor of the rising sea. As Colin McLarty explains, the remarkably fruitful collaboration between Grothendieck and Jean-Pierre Serre from the mid-1950s to about 1970 was based on a striking contrast of approaches. Serre was a master of the elegant incisive strike that cracks a nut precisely with a single blow; Grothendieck sought to dissolve the shell slowly in a body of liquid, the "rising sea." One source of their shared interests was the Weil conjectures, which Serre explained to Grothendieck in cohomological terms in

1955. McLarty traces the origins of cohomological studies in France in the preceding decade and shows how, from this, Grothendieck generated an imposing amount of new category-theoretic ideas. In this context, McLarty gives a short history of sheaves and schemes as they were created in the Bourbaki milieu and shows how Grothendieck's breadth of vision and capacity for working with very many simple ideas led him to recreate algebraic geometry.

## Concluding Remarks

The history of a topic as large as algebra is no more finished than the topic itself. There are developments in algebra that followed from the work described here, and there are developments that took place at the same time that are not discussed here at all. That is as it should be. Any readable book offers only a selection of what could be said. But the absence of a terminus for this book, welcome as that is on many grounds, makes the drawing of conclusions rather tentative.

Three aspects stand out and might suggest avenues for further research. One is the tendency exhibited by the mathematicians considered here to look for concepts with which to organize their work. This is, of course, one of the characteristic features of modern algebra, but it can be refined in at least two ways. On the one hand, as Edwards and Neumann have shown here and elsewhere, Kronecker, while noted for his "constructive" mathematics, was also a highly conceptual thinker with a developed program. On the other hand, the craft of mathematics—the sheer ability to find a result through hard calculation—does not disappear. It is enough to note, with Curtis, that Noether's arithmetical considerations are today's cohomological arguments. The balance between concept and craft is an endlessly fascinating one.

The second aspect is the somewhat cooperative, somewhat competitive dimension of these algebraic researches. A mathematical presentation of the topics here would lead cleanly and directly to the main results. These chapters, however, highlight the significant role such social factors as community, communication, and education play in the evolution of mathematics. Mathematics, we suggest, cannot fully be understood without acknowledging these factors.

The third, and the oldest observation of the three, is how amply the mathematical work here described vindicates the opinion of Gauss that the study of number theory leads to deep and hidden connections between superficially different branches of mathematics. Such, too, was the opinion of Kronecker, and ultimately the opinion of Grothendieck. We can surely be confident that the on-going work of mathematicians will continue to make work for historians to do.

## References

Bashmakova, Isabella and Smirnova, Galina. 2000. *The Beginnings and Evolution of Algebra.* Trans. Abe Shenitzer. N.p.: The Mathematical Association of America.

Bos, Henk J. M. 2001. *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction.* New York: Springer-Verlag.

Cardano, Girolamo. 1545/1968. *The Great Art or the Rules of Algebra.* Trans. T. Richard Witmer. Cambridge, MA: The MIT Press.

Corry, Leo. 1996. *Modern Algebra and the Rise of Mathematical Structures.* Science Networks. Vol. 17. Basel: Birkhäuser Verlag.

Edwards, Harold M. 1984. *Galois Theory.* New York: Springer-Verlag.

Euler, Leonhard. 1770–1771/1984. *Elements of Algebra.* Trans. John Hewlett. New York: Springer-Verlag.

Høyrup, Jens. 2002. *Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin.* New York: Springer-Verlag.

Klein, Jacob. 1968. *Greek Mathematical Thought and the Origin of Algebra.* Trans. Eva Brann. Cambridge, MA: The MIT Press.

Nový, Luboš. 1973. *Origins of Modern Algebra.* Leiden: Noordhoff International Publishing.

Parshall, Karen Hunger. 1989. "Toward a History of Nineteenth-Century Invariant Theory." In *The History of Modern Mathematics.* Ed. David E. Rowe and John McCleary. 2 Vols. Boston: Academic Press, Inc., 1:157-206.

Scholz, Erhard, Ed. 1990. *Geschichte der Algebra: Eine Einführung.* Mannheim: Bibliographisches Institut Wissenschaftsverlag.

Sesiano, Jacques. 1999. *Une Introduction à l'histoire de l'algèbre: Résolution des équations des Mésopotamiens à la Renaissance.* Lausanne: Presses polytechniques et universitaires romandes.

Stedall, Jacqueline A. 2003. *The Greate Invention of Algebra: Thomas Harriot's Treatise on Equations.* Oxford: Oxford University Press.

Unguru, Sabatei. 1975. "On the Need to Rewrite the History of Greek Mathematics." *Archive for History of Exact Sciences* 15, 67-114.

Van der Waerden, Bartel. 1976. "Defence of a 'Shocking' Point of View." *Archive for History of Exact Sciences* 15, 199-210.

———. 1983. *Geometry and Algebra in Ancient Civilizations.* New York: Springer-Verlag.

———. 1985. *A History of Algebra from al-Khwārizmī to Emmy Noether.* New York: Springer-Verlag.