Introduction

Henri Poincaré[1] (Nancy, 1854; Paris, 1912) dominated the mathematics and the theoretical physics of his time; he was without a doubt the most admired mathematician while he was alive, and he remains today one of the world's most emblematic scientific figures.

[1]This name comes, so it seems, from *"poing carré"* ["square fist"]: cf. Poincaré's *Œuvres*, vol. 2, p. ix, note (1).

His work, slowly assimilated over the course of the twentieth century, is of phenomenal depth, but also of extraordinary variety.[2] In mathematics, nearly all of whose existing parts he enriched, he created entirely new branches (automorphic functions, dynamical systems, algebraic topology) and he opened the way to the theory of functions of several complex variables and that of asymptotic expansions. He completely renovated celestial mechanics, discovered on this occasion deterministic chaos, found new forms of equilibrium of celestial bodies, and proposed a scenario for the formation of double stars. In physics, he is one of the fathers of the theory of special relativity, for which he envisioned consequences, even to the motion of the stars. He moreover exerted a constant influence on the spectacular development of the physics of his time: he participated in all of the great debates, provided the first correct explanation of several experiments, and even elicited new ones.[3]

His teaching has remained legendary. In his lectures on mathematical physics (then in celestial mechanics) at the Sorbonne, he sifted the principal existing theories, reformulated them, corrected them; he presented and discussed the most recent experiments and observations. These courses, almost all of which were written up[4] and then published and widely distributed, contributed much (either by themselves or by the ideas that he conceived on their occasion and which he distributed in parallel) to the definitive formulation and acceptance of new theories (of Maxwell, Lorentz, Boltzmann, ...).

In the philosophy of sciences, he was one of the major actors in the great epistemological debates of his time; in particular, his *pragmatic occasionalism* (as it is called by Gerhard Heinzmann) nourished reflections throughout the twentieth century. In addition, his popular works on science, written in an extraordinarily clear style, had an enormous success before the wide public: *Science and Hypothesis* was translated into at least 23 languages. In a few years, more than 16 000 copies were sold in France. People read it in the public gardens and the cafés.

A "universal specialist", it was he who investigated (as a mining engineer) the causes of the firedamp explosion in the shafts of Magny (1879); it was to him that not only mathematicians, but physicists turned, such as Hertz when he did not succeed in calculating the speed of the propagation of a wave along a twisted wire, or Becquerel when he found himself in disagreement with Alfred Potier over a question of rotational polarization; it was from him (along with Darboux and Appell) that the court requested an expert opinion on the scientific validity of Alphonse Bertillon's arguments in the Dreyfus affair;[5] it was he that the Académie

[2]One can find a list of Poincaré's publications as well as numerous bibliographic references concerning his life and work on the site of the Laboratoire d'Histoire des Sciences at de Philosophie–Archives Henri Poincaré (CNRS–Nancy-Université):

`http://www.univ-nancy2.fr/poincare/ENG/documents/bd1.html`

(from which we have also obtained, with gratitude, the photos on the previous page and on page 392).

[3]See volumes 9 and 10 of his *Œuvres*, and his correspondence with the physicists: this latter, collected by André Coret and Scott Walter, is in process of publication by the Archives Henri Poincaré: `http://www.univ-nancy2.fr/poincare/chp/prjregionh.html`.

[4]By students or by other audience members: Jules Blondin, Émile Borel, Jules Drach, René Baire, Albert Quiquet, and others.

[5]See the article *Autour de l'Affaire Dreyfus: Henri Poincaré et l'action politique*, by Laurent Rollet: `http://poincare.univ-nancy2.fr/Presentation/?contentId=2240`.

des sciences asked to supervise the new measurement of the meridian arc of Quito; he presided (three times) over the Bureau des Longitudes; etc.

His scientific *Œuvres*, in 10 volumes,[6] are divided as follows:

Volume 1:	Differential equations.
Volume 2:	Automorphic functions.
Volume 3:	Algebraic integration of differential equations; continuous groups; abelian integrals; residues of double integrals; integral equations.
Volume 4:	Analytic functions of one or several variables; abelian functions; trigonometric series.
Volume 5:	Number theory.
Volume 6:	Algebraic geometry, algebraic topology.
Volume 7:	Analytic mechanics; celestial mechanics; fluid masses in rotation; three-body problem; trigonometric series of celestial mechanics (work on the methods of Lindstedt and of Gyldén).
Volume 8:	Celestial mechanics and geodesy; perturbation function and periods of double integrals; shape of the Earth; theory of tides, of the Moon, and of planets; mechanical quadratures; cosmogonic hypotheses; reports on the geodesic operations at the Equator.
Volumes 9–10:	Mathematical physics; theoretical physics.

His treatises on mathematical and theoretical physics address nearly all the important themes of his era: potential theory and fluid mechanics, mathematical theory of light, electricity and optics, thermodynamics, theory of elasticity, theory of vorticity, electric oscillations, capillarity, analytic theory of heat, probability, theory of Newtonian potential (he also provided a course on the kinetic theory of gases, which was never written).

Let us add his *Lectures on Celestial Mechanics* (3 volumes), his *Lectures on the Cosmogonic Hypotheses*, and a *Course in General Astronomy* (autograph). His text *New Methods in Celestial Mechanics* is not the fruit of a course of instruction (it is a research work), but it is written up as a course.

Let us mention, finally, his works on the philosophy of science: *Science and Hypothesis*, *The Value of Science*, *Science and Method*, *Last Thoughts* (posthumous collection).

As one can see by comparing the preceding with the table of contents of the present work, we shall address here only *a part* of the themes to which Poincaré contributed (it would be impossible to be exhaustive in a single volume!), especially underlining the modernity of his ideas and some of their current repercussions. The texts, written by experts of international renown, are accessible to any first- or second-year graduate student (or even undergraduate) in mathematics or in physics. They contain several levels of reading, and researchers (in particular doctoral candidates) will also be able to find in them useful ideas for their daily work.

Here is an overview of the themes addressed in this book.

[6] Gauthier-Villars, 1916–1954. Republished by J. Gabay, 1995–2005. An 11th volume contains a few articles, an extract of his mathematical correspondence, and the *Livre du centenaire* of his birth.

Hyperbolic geometry, automorphic functions, applications to number theory (Chapters 1, 2, and 3)

Henri Poincaré made himself known on the mathematical scene in 1881–1882 with his theory of automorphic functions. It was a kind of apotheosis of 19th century mathematics, a meeting place for group theory, complex analysis, elliptic and modular functions, differential equations, Riemann surfaces, hyperbolic geometry, and quadratic forms; it would also be a privileged model for his reflections on the philosophy of science.

The essential property of automorphic functions on a domain D of $\mathbb{C}$ is a kind of generalized periodicity: $f(\frac{az+b}{cz+d}) = f(z)$ for a certain discontinuous group of projective transformations $z \mapsto \frac{az+b}{cz+d}$ mapping D to itself. We also require that f be meromorphic on D.

After the already known case of *elliptic functions*, which are automorphic functions on $\mathbb{C}$ for a group of translations (identified with a lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$), the most useful case is that of automorphic functions on a disk:[7] this is what Poincaré called *"Fuchsian functions"*. Only very particular cases were known. By interpreting the projective transformations of the disk as motions of Lobachevsky's non-Euclidean plane (the hyperbolic plane), Poincaré managed to geometrically construct *all* the discontinuous groups of projective transformations of the disk ("*Fuchsian groups*"); he associated to them *"theta-Fuchsian"* functions (today called *automorphic forms*), which he obtained in the form of series (today known under the name of *Poincaré series*); and he thence derived *all* the Fuchsian functions as quotients of such series.

When D is no longer a disk (or a half-plane), Poincaré speaks of *"Kleinian"* functions, and the corresponding groups ("*Kleinian groups*") are obtained from hyperbolic geometry in dimension 3. Felix Klein, dissatisfied with the adjectives *"Fuchsian"* (he thought it gave too much honor to Fuchs) and *"Kleinian"* (which he considered as a consolation prize), replaced them with *"automorphic"* (a term borrowed from Cayley).[8]

The automorphic functions made it possible to obtain the famous *uniformization theorem*: every planar algebraic curve $P(x, y) = 0$, where P is an irreducible polynomial, can be parametrized in the form $x = f(z)$, $y = g(z)$, where f and g are rational functions (the case of genus 0 curves), or elliptic functions with the same lattice (curves of genus 1), or Fuchsian functions of the same group (curves of genus ≥ 2).[9] An equivalent geometric statement is that every compact, connected Riemann surface is analytically isomorphic to the Riemann sphere, or to the quotient of the complex plane by a lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, or to the quotient of the disk by a Fuchsian group.

But Poincaré's true motivation was integrating linear differential equations with algebraic coefficients, that is,

$$\frac{d^n u}{dx^n} + a_{n-1}(x, y)\frac{d^{n-1}u}{dx^{n-1}} + \cdots + a_0(x, y)u = 0,$$

[7] Or on a half-plane: one passes between them by a projective transformation.

[8] Poincaré responded to Klein by these words of Faust, which could serve as an epigraph to his philosophy: *"Name ist Schall und Rauch,"* that is, "[the] *name is* [nothing but] *sound and smoke."*

[9] This theorem, discovered without rigorous proof in 1881 by Poincaré and Klein, would be extended to analytic curves in 1907, by Poincaré and (independently) by Koebe (each of them giving a rigorous proof).

where the a_i are rational functions and where x and y are connected by an algebraic relation $P(x, y) = 0$. One can construct a Fuchsian (or rational, or elliptic, for the genera 0 and 1) parametrization $x = f(z), y = g(z)$ such that the solutions are meromorphic functions of z having particular transformation properties (*"zeta-Fuchsian"* functions): a projective transformation of the Fuchsian group of f and g translates into a change of basis in the space of solutions.

Automorphic functions and forms and their groups are tools used constantly today in various areas of mathematics and even in physics (they are involved in the calculation of collision amplitudes in string theory), and the current theory of automorphic representations (in full bloom, centered on the *Langlands program*) is a distant heir.

Chapter 1 presents the *Poincaré disk*, whose hyperbolic geometry is the key that led Poincaré to the construction of Fuchsian groups: it shows the extent to which this object is natural, rich, and ubiquitous in the mathematics of today.

Chapter 2 recounts the discovery of Fuchsian functions and groups, following and explaining Poincaré's own narrative about it (one will find there confirmation that mathematicians transform coffee into theorems, and a very interesting testimony on the role of the unconscious in mathematical discovery), all while evaluating its current aspects.

Chapter 3 presents the application of Poincaré series to analytic number theory, some examples of the resulting progress, and some new hopes (perhaps a connection with the question of *twin prime numbers!*).

Ordinary differential equations, dynamical systems (Chapters 4 and 5)

Most differential equations cannot be solved explicitly. Before Poincaré, one was therefore content to study solutions in a neighborhood of a point. Poincaré himself developed this approach in his doctoral thesis (1879), and he returned to it several times: his research (parallel to his study of trigonometric series of celestial mechanics) led him to set up the foundation, in 1885 and 1886, of the classical theory of asymptotic expansions.

But Poincaré also understood that there would always be the need to know the global behavior of integral curves. Yet for this, there were no tools. He would therefore create them (*Mémoire sur les courbes définies par une équation différentielle* [*Memoir on curves defined by a differential equation*], first part: 1881). He considered a differential equation $dx/X = dy/Y$, where X, Y are polynomials in x, y. To take into account the behavior at infinity, he compactified the (x, y)-plane into a sphere by central projection. Two orbits cannot meet except at a singular point (a point where X and Y vanish): Poincaré showed that generically there are only three types: the *nodes*, where infinitely many orbits meet; the *saddles*, where two orbits cross; and the *foci*, where the orbits wind around in spirals. Under certain conditions, there also exist *centers*, which are surrounded by periodic orbits; even though they are non-generic, centers are important in mechanics (they correspond to a form of stability): Poincaré therefore studied the conditions for their existence (center-focus problem, Poincaré–Lyapunov theorem).

Poincaré showed that for a generic field (X, Y) on the sphere, the numbers of nodes, saddles, and foci are connected by the relation $N - S + F = 2$, analogous to Euler's relation between the numbers of vertices, edges, and faces of a convex polyhedron. He noticed that the analogy persists on arbitrary algebraic surfaces (corresponding to differential equations of the form $P(x, y, y') = 0$, where P is a polynomial). Later, Heinz Hopf (1926) would show that it also persists on manifolds of all dimensions (the Poincaré–Hopf formula)—Poincaré having meanwhile generalized Euler's formula to polyhedra (convex or not) of arbitrary dimension and invented the necessary topological tools.

One of Poincaré's key ideas is that of a transverse section ("*arc without contact*"): this is an arc of a curve that is not tangent to the field (X, Y) at any point; any orbit that meets it necessarily crosses it. Poincaré reduced the study of an orbit to that of its successive points of intersection with such an arc ("*theory of consequents*"). This allowed him to prove that an orbit on the sphere can only have one of three fates: either it lands on a singular point, or it closes on itself (periodic orbit), or it winds asymptotically around an isolated periodic orbit (which Poincaré called a "limit cycle") or around a *"graph"* (a closed curve consisting of orbits joining singular points): this is the *Poincaré–Bendixson theorem* (because it was generalized later by Bendixson to continuously differentiable fields). This restricted the set of possible behaviors so much that Poincaré could determine the shape of integral curves for explicit examples of equations. In 1886, he partially generalized his methods to higher dimensions. The Poincaré–Bendixson theorem is no longer valid there[10] (it already was not on 2-dimensional surfaces of non-zero genus, such as the torus).

Chapter 4 offers an introductory survey of the theory of limit cycles—covering the Poincaré map, asymptotic phase, frequency locking, Poincaré–Bendixson, the problem of an upper bound for the number of limit cycles (Hilbert's 16th problem, Bautin's theory), the center-focus problem, and the connection with Abel's equations—while emphasizing recent advances.

Chapter 5 presents the Poincaré–Lyapunov center theorem, with the outline of a very beautiful geometric proof (Moussu, 1981); for this, one learns how to "see in $\mathbb{C}^2$" by tomography. In it, one can also find the connection with integrability in the sense of Liouville, degenerate centers, and the "hyperbolic" case.

Celestial mechanics and related problems (Chapters 6–10)

From the start of his research on curves defined by differential equations (1881), Poincaré mentioned possible applications to celestial mechanics, in particular the question of the stability of an idealized solar system over an indefinite time period. Up until then, it was thought that this problem would be solved as soon as someone had succeeded in expressing the solutions of the N-body problem by absolutely convergent trigonometric series, because it was believed that the sum of such a series was necessarily bounded. Poincaré made the observation that it is nothing of the sort (1882). In 1883, he proved, by a very simple geometric argument, the existence of infinite families of periodic solutions in the three-body problem with two sufficiently small masses. He judged it *"improbable"* that the trigonometric series of celestial mechanics converge for *neighboring* solutions to the periodic solutions: he had in mind the example of limit cycles that he had discovered in other dynamical systems (*"Indeed, I know completely analogous problems where convergence does not take place" Œuvres*, vol. 4, p. 590). But already he added at this point (without

[10] We now know that in the sphere S^3 (or in $\mathbb{R}^3$) the trajectories can wind around much more complicated objects than a periodic orbit, such as for example the famous Lorenz attractor, which is a kind of interlacing of infinitely many periodic orbits.

details, for the moment) that, even if divergent, these trigonometric series could provide excellent approximations in practice. In an 1884 article, he began studying the neighboring solutions of the periodic solutions that he had found in the three-body problem.

All this led him to the memoir *Sur le problème des trois corps et les équations de la dynamique* [*On the three-body problem and the equations of dynamics*], which was crowned with a prize of the king of Sweden and appeared in 1890.[11] Poincaré gave a new (analytic) proof of the existence of periodic solutions. Half of them are linearly unstable, and their study provided three important results:

(1) there exist *"asymptotic solutions"*, which wind around linearly unstable periodic solutions, in the future or the past: a behavior quite different from the almost periodic motions previously encountered in celestial mechanics;
(2) the expansions into the usual trigonometric series of celestial mechanics diverge in general (but he also showed what one could still extract from these, thanks to his theory of asymptotic series);
(3) there do not exist other holomorphic first integrals (quantities that are constant on *each* trajectory of the system) than those already known[12] (if there had been enough, and if they had satisfied certain conditions, a theorem of Liouville would have guaranteed that the system is integrable by quadratures after an appropriate change of coordinates).[13]

Poincaré then turned to the quantities conserved on *bundles* of trajectories, which he called *"integral invariants"*; in modern language, these are invariant measures on the phase space.[14] (He rediscovered, after Liouville and Boltzmann, the theorem on the invariance of volume in the phase space of a Hamiltonian system.) And he proved his *recurrence theorem*, in the following form: *If there exists a positive invariant measure, and if the phase space has finite measure, then the trajectories issuing from a region of non-zero measure in the phase space will pass infinitely often within this region, except for some initial positions whose probability (i.e., measure) is zero.* Poincaré showed that this theorem applies to the restricted three-body problem. (The trajectories that are asymptotic in the future, mentioned in point (1) above, are obviously among the exceptions.) With the integral invariants and his theory of consequents, Poincaré also proved the existence of infinitely

[11] The announcement of the competition, published in 1885, specified that the prize would be awarded on January 21, 1889 (the sixtieth birthday of the king of Sweden), that the memoirs must be submitted before June 1, 1888, and that they must not have been previously published. Poincaré therefore devoted himself to other work and did not take up the subject until the last moment. The publication was further delayed, after the prize had been given, because Poincaré, pressed for time, had not written the detailed proofs, and while doing so in order to answer the editors' questions he discovered and had to correct an important error (see Chapter 8 in this volume).

[12] Poincaré proved this, in his memoir, for the *restricted* three-body problem (planar motion, two bodies having circular orbits and the third a negligible mass). In the *Méthodes nouvelles de la mécanique céleste* (volume 1, 1892) he extended the result to the general three-body problem, on the condition that two among them have sufficiently small masses.

[13] These results by Poincaré do not, however, exclude the possibility of solving the N-body problem by convergent series for all values of time (which was moreover the question initially posed for the prize): such a solution was obtained (under generic conditions) by Karl Sundman in 1909 for $N = 3$, and by Qiudong Wang in 1991 for arbitrary N.

[14] Space of the states of the system (which Gibbs called *"phases"*). For a mechanical system with N degrees of freedom, it is the space of generalized coordinates and generalized velocities, or possibly a submanifold obtained by fixing the energy and the other invariants of motion.

many *doubly asymptotic "homoclinic"* solutions, that is, those which wind in both the past *and* the future around the *same* periodic solution: their study led him to the discovery of the entanglement of stable and unstable manifolds (formed of solutions that are asymptotic, respectively, in the future and in the past, for a given periodic solution) and of what is today called *deterministic chaos*. Poincaré again took up his methods and results and developed them considerably in the three volumes of his *Méthodes nouvelles de la mécanique céleste* (1892–1899), which remain today an important source of inspiration.

Poincaré started from the existence of periodic solutions, *"the sole breach from which we can try to penetrate into a place heretofore reputed to be inaccessible"* (*Méthodes nouvelles*, vol. 1, 1892). Thereafter, he continued to look for other families of such solutions. He tried (1896) a variational method, but it was better suited to the interactions in $1/r^{n+1}$ with $n > 1$ than with $n = 1$. It is only recently that we have been able to adapt such a method to the Newtonian problem (see below). In his last article on celestial mechanics (1912), he managed to prove, thanks to a theorem of geometry (the Poincaré–Birkhoff theorem), that the periodic solutions are extremely abundant, at least in the restricted three-body problem. Since a generalization of the Poincaré–Birkhoff theorem in higher dimensions was established by Charles Conley and Eduard Zehnder (1984), this abundance of periodic solutions has been found in much more general dynamical systems and is the object of active research today.

Chapter 6 presents a brief history of the work on periodic orbits of the three-body problem: the theory of the Moon by Newton (1687), who introduced the restricted three-body problem and obtained a solution that is periodic to first approximation; the theory of Hill (1877), who completed that of Newton and provided a family of rigorously periodic solutions; the work of Poincaré, which extended Hill's family of solutions, then the key ideas of his various proofs of the existence of periodic solutions of the three-body problem; finally recent examples of periodic orbits of the N-body problem, obtained by a variational approach.

Chapter 7 deals with an analogous problem to that of the existence of periodic orbits, namely the existence of closed geodesics on a deformed sphere. One will find there the essential ideas of Poincaré's two proofs (by a continuity method, then by a variational method) in dimension 2 and of George D. Birkhoff's proof (1917) valid in all dimensions. Then come the case of manifolds with more complicated topology (by Morse theory) and the question of the existence of infinitely many closed geodesics on a deformed sphere of dimension 2: Poincaré (1912), Birkhoff (1917), John Franks (1992), Nancy Hingston (1993), Victor Bangert (1993); the analogue for spheres of higher dimension remains to be done.

Chapter 8 presents some of the main results of the memoir crowned with the prize of the king of Sweden and explains Poincaré's error and his discovery, while correcting this error, of the "lattice of homoclinic intersections" (the entanglement of stable and unstable manifolds); it also explains to what extent we have been able, today, to "untangle" this lattice (work of Birkhoff, Steve Smale, Rufus Bowen). Finally, it presents two recent results that illustrate the current importance of Poincaré's discovery: on the one hand, the work of Jacques Laskar (around 1990) on chaos in the solar system, and on the other hand, a theorem of Sylvain Crovisier (2005) showing that the phenomenon of homoclinic intersections discovered by Poincaré is, in some sense, inherent in chaos.

Chapter 9 discusses Poincaré's recurrence theorem and its limitations, and notably explains why an illustration that was proposed for this theorem in popular articles does not *at all* illustrate Poincaré's true theorem.

The results obtained by Poincaré in celestial mechanics serve as guides in many other areas, where one can apply the concepts and methods that he invented. Chapter 10 gives a few current examples of this kind of transfer of ideas into fluid mechanics[15] to study different cases of chaotic advection and long-term diffusive behavior.

Among the numerous other contributions Poincaré made to celestial mechanics, not touched upon in this collection, let us mention his study of the forms of equilibrium of fluids in rotation (1885). To the forms already known (ellipsoids, annuli) Poincaré added new ones (pear-shaped[16]). He established his theorem on the "exchange of stability": under certain conditions, when two solution curves cross, the solutions of one of the branches lose stability whereas those of the other gain it (he therefore called the solution at the crossing point a *"bifurcation form"*). Poincaré deduced from this a possible scenario for the formation of certain double-star systems (which, notably, made precise a hypothesis of George Darwin, 1878): an ellipsoid, initially almost spherical, flattens as it cools, all while remaining axially symmetric; then it reaches the point where the stability passes to the ellipsoids of unequal axes; then it bifurcates toward a pear shape, widens further, and ends by separating into two distinct bodies.[17] Poincaré also applied his theorem on the exchange of stability to families of periodic orbits.

Functions of several complex variables (Chapter 11)

Poincaré's first incursion into the theory of several complex variables (1883) was his proof that a *meromorphic* function on $\mathbb{C}^2$ is the quotient of two holomorphic functions on $\mathbb{C}^2$. (Weierstrass had proved the analogous result for functions of a single variable but was unable to establish it for functions of two variables.)[18] Using the same method, he obtained in 1898 an entirely new proof of a theorem of Riemann and Weierstrass, according to which a meromorphic function of p variables, $2p$ times periodic, is a quotient of "theta functions".

In 1907, Poincaré became interested in the problem of mapping a real analytic (three-dimensional) hypersurface of $\mathbb{C}^2$ onto another by an analytic isomorphism, either locally (there is an isomorphism in a *neighborhood* of each point of the surface) or globally. He discovered that it is in general impossible, even locally (whereas for curves in $\mathbb{C}$ it is always possible locally), and that, under certain conditions, once the map is possible locally, then it is also possible globally. These results are at the source of the theory of CR-structures (for "Cauchy–Riemann"), whose involvement has subsequently been seen in a large number of problems in mathematics and physics (including the theory of general relativity).

But Poincaré's most famous contribution to the theory of functions of several complex variables is his theory of residues (1887). Since the time of Cauchy, it had been desirable to extend the residue theorem to functions of several variables,

[15] This is a just turn of events, since Poincaré had been inspired by fluid mechanics to study the "invariant integrals".

[16] Translator's Note: Poincaré used the term *piriform.*

[17] For a modern discussion of Poincaré's scenario, see for example `arXiv:astro-ph/9505008`.

[18] And it would have to wait for Pierre Cousin's thesis (1894) for the result to be extended to functions of n variables.

but after more than forty years of efforts almost no progress had been made. It was Poincaré who unblocked the situation. On the way, Poincaré obtained the conditions for a differential form of arbitrary degree to be closed (that is, for its integral over a boundary to always be zero). His reasoning leads to the generalized "Stokes formula": it is implicit in his 1887 memoir, and would be made explicit first by Volterra in 1889 (inspired by Poincaré's memoir), then (in an even simpler form) by Poincaré himself in 1899, in his *Méthodes nouvelles de la mécanique céleste* (apropos of the "invariant integrals"). Poincaré would also use a particular case of this formula in his work on topology (for calculating homology).

Chapter 11 presents the Poincaré residue, first from the cohomological viewpoint (due essentially to Poincaré and implemented by Jean Leray), then from the viewpoint of currents (more recent), and its application in contemporary mathematics. One can also see how it revived a problem already considered by Poincaré in 1885, namely a generalization of a famous theorem of Abel (*the "trace" of a rational form is rational*) and its converse.

Algebraic topology and the Poincaré conjecture (Chapter 12)

In almost all of his research, Poincaré encountered problems of topology (problems of *analysis situs*, as was said at the time). He wrote:

> *"A method that would make known to us the qualitative relations in space of more than three dimensions could, to a certain extent, render services analogous to those rendered by figures. This method can only be the* Analysis situs *of more than three dimensions.* [...] *I had need of the facts of this Science to pursue my studies on curves defined by differential equations and to extend them to differential equations of higher order and, in particular, to those of the three-body problem. I had need of them for the study of non-uniform functions of two variables. I had need of them for the study of periods of multiple integrals and the application of this study to the expansion of the perturbing function. Finally, I sensed within* Analysis situs *a means of approaching an important problem in the theory of groups, the search for discrete groups or finite groups contained in a given continuous group."*[19]

But this "Science" was in its infancy: one saw (or believed one could see) some properties, but almost nothing had been proved, for lack of reliable tools. Poincaré would create a powerful theory (algebraic topology) in two stages. In a first memoir (*Analysis situs*, 1895), he gave a definition of manifolds (in fact, analytic submanifolds of $\mathbb{R}^n$) by means of charts, fixed the precise definition of the integral of a differential form over a manifold, introduced the notion of homology, and studied its connection with the integrals of closed differential forms (again taking up the ideas of his 1887 memoir on residues of double integrals). He considered the dimensions of homology spaces, which he called the *"Betti numbers"* (because he believed they coincided with the numbers already introduced by Enrico Betti) and gave an important symmetry formula for the "Betti numbers" of *closed*[20] orientable manifolds (*Poincaré duality*). He studied decompositions of manifolds into pieces homeomorphic to polyhedra, letting himself be guided by the analogy with his

[19] Poincaré: *Œuvres*, vol. 6, p. 183.

[20] In the sense that one speaks of *closed surfaces*. More precisely, these are connected, compact manifolds without boundary.

theory (mentioned above[21]) of Fuchsian groups (where the hyperbolic plane was decomposed into polygons, images of each other by the group) and Kleinian groups (where 3-dimensional hyperbolic space was decomposed into polyhedra). This led him to define the *fundamental group* of a manifold. He observed that two homeomorphic manifolds have the same fundamental group (up to isomorphism). He noted that in the uniformization of a Riemann surface (see page 4) the Fuchsian group is nothing other than the fundamental group of the surface; from this he concluded that two *closed* manifolds of dimension 2 that have the same "Betti numbers" are homeomorphic. But he noted that this is no longer true for closed manifolds of dimension 3: he gave examples of such manifolds that have the same "Betti numbers" as the sphere S^3, but a different fundamental group. Finally, he generalized Euler's formula for polyhedra in arbitrary dimension, convex or not: the alternating sum of the numbers of faces of the different dimensions is constant, and the constant is the alternating sum of the "Betti numbers".

One of the examples Poincaré considered in his memoir shows that his "Betti numbers" are not always equal to those of Betti, and that the duality formula is not valid for the latter; but Poincaré did not realize it until Heegaard, using Betti's definition, found another counterexample to the duality formula (1898). Moreover, Heegaard pointed out a dubious assertion in the proof of the duality formula. Poincaré therefore decided to return to the question. In a first *Complément à l'Analysis situs* (1899), he redefined homology by means of triangulations (simplicial homology), thus founding algebraic topology on solid bases, and he gave a new proof of the duality formula (he also filled in the hole of his first proof). In a second *Complément* (1900), he added to his "Betti numbers" new topological invariants, the *torsion coefficients*. We recall that in his 1895 memoir, Poincaré had observed that two closed manifolds of dimension 3 can have the same "Betti numbers" without being homeomorphic. It was therefore natural to ask if two closed manifolds of dimension 3 having the same "Betti numbers" *and* the same torsion coefficients are homeomorphic. In his fifth *Complément* (1904), Poincaré showed that nothing of the sort is true: he constructed an example of a 3-dimensional manifold having the same "Betti numbers" and the same torsion coefficients as the sphere S^3 but a different fundamental group.[22] Poincaré concluded that there remained one question to address: does there exist a manifold of dimension 3 whose fundamental group is trivial (that is, all closed curves in this manifold can be continuously contracted to a point), and which is not, however, homeomorphic to S^3? The hypothesis that one does not exist became famous under the name of the *Poincaré conjecture*. Grigori Perelman (2003) proposed a proof of this conjecture, and, even more generally, of *Thurston's geometrization conjecture*, according to which eight homogeneous geometries suffice to construct all 3-dimensional manifolds (the only one of the eight whose fundamental group is trivial being the sphere).[23] Roughly, the strategy, elaborated by Richard Hamilton in 1982, consists in letting the metric evolve according to a differential equation connected to the curvature and analogous to the equation of heat propagation, which tends to homogenize the metric, and the idea is to show

[21] See the section on hyperbolic geometry and automorphic functions.

[22] This Poincaré *dodecahedral space* has recently (2003) been envisioned as a cosmological model.

[23] The verification of Perelman's proof is still in progress, but the ideas that he introduced on this occasion have already earned him the Fields Medal (which he moreover refused) in August 2006.

that one can asymptotically approach only the eight geometries of Thurston. But the curvature can blow up at certain places, and Perelman showed how one can repair the process by surgery and restart it, until the metric converges (in a certain sense).

Chapter 12 presents the outline of Hamilton's strategy and Perelman's proof.

Partial differential equations of mathematical physics (Chapter 13)

In 1886, Poincaré became the title-holder of the chair of *Mathematical physics and probability* at the Sorbonne. At the same time, he began research on the mathematical aspects of the theory of electric potential and the analytic theory of heat (1887).

Chapter 13 describes several contributions Poincaré made to the mathematical theory of the partial differential equations of physics, notably his solution to the *Dirichlet problem* (the existence of harmonic functions on a given domain with prescribed boundary conditions) by the *sweeping method* (1887, 1890); the Poincaré inequality (1890, 1894); the first rigorous proof of the existence of the entire sequence of eigenvalues of the Laplacian on an arbitrary bounded domain in $\mathbb{R}^3$ (with corresponding eigenfunctions zero on the boundary), which he obtained as zeroes of a certain analytic function (1894); and finally, the general solution of the telegraph equation (1893).

Probability (Chapters 14 and 15)

When, in his memoir *Sur le problème des trois corps et les équations de la dynamique*, Poincaré had proved his recurrence theorem (see page 7), he had specified that the initial conditions of non-recurrent trajectories had zero *probability. "This word has of itself no meaning: I will give a precise definition in my Memoir,"* he insisted in a report on his 1891 memoir. In fact, it seems that this was the question, and more precisely at the time of the revision of his memoir (see Chapter 3 of Anne Robadey's thesis[24]), which led Poincaré to reflect on the notion of probability and to forge a personal approach to the question. He observed, notably, that any positive and finite "integral invariant" (invariant measure) on the entire phase space provides a valid notion of probability.

It was in 1894 that Poincaré chose to teach probability. His course would be published in 1896; a modified and expanded edition would appear in 1912. It is from this course by Poincaré that we receive the expressions *"characteristic function"* (in the probabilistic sense) and *"normal law"*. Darboux, in his *Éloge historique d'Henri Poincaré*[25] (1913) wrote:

> *"This treatise by Poincaré does not appear to me to have been regarded for its full value. I am certain it will figure proudly next to the masterpieces of Laplace and Bertrand. I especially point out its very fine introduction on the laws and the definition of chance, its chapters on the probability of the continuous, where Poincaré sheds light on a famous paradox proposed by Bertrand; also those which he dedicated to the theory of errors and*

[24] *Différentes modalités de travail sur le général dans les recherches de Poincaré sur les systèmes dynamiques* [*Different ways of approaching the notion of genericity in Poincaré's research on dynamical systems*] (Paris, January 3, 2006), available at: `oai:hal.archives-ouvertes.fr:tel-00011380_v1`.

[25] *Œuvres* of Henri Poincaré, volume 2, p. xxxiv.

to the famous law of Gauss. Bertrand limited himself to criticism and demolition. Poincaré started to rebuild."

Chapter 14 presents these contributions and other original aspects of Poincaré's course, notably the relations that he introduced between probability theory and group theory.

Chapter 15 deals more particularly with *geometric probabilities.* Starting with the Buffon needle problem, it presents a theorem of Poincaré and gives two relatively recent applications of it: the first shows (following Edelman and Kostlan, 1995) how this theorem makes it possible to recover very simply an estimate (due to Mark Kac, 1943) of the average number of zeroes of a random real polynomial of large degree; the second, due to Michel Mendès France (1987), is an illustration of the subtle connection that exists between "disorder" and "confusion".

Continuous groups (Lie groups) and their algebras (Chapter 16)

In 1899, the date of Sophus Lie's death, Poincaré published an article on *"continuous groups"* (Lie groups) and their algebras: he embeds every finite-dimensional real Lie algebra in an algebra of symbolic polynomials, in which the Lie bracket takes the form of a simple commutator bracket ($[A, B] = AB - BA$), and which is nothing other than what is today called the *universal enveloping algebra* of the Lie algebra. Its existence constitutes the *Poincaré–Birkhoff–Witt theorem.* Poincaré derived from it a conceptually simple proof of Lie's third fundamental theorem (local version), according to which every finite-dimensional real Lie algebra is isomorphic to that of a (germ of a) Lie group.

Chapter 16 relates the historical context of this contribution of Poincaré to the theory of Lie groups and presents Poincaré's proof, certain developments that occurred during the 20th century, and finally a recent (2003) analogue of the Poincaré–Birkhoff–Witt theorem within the framework of Leibniz algebras, which are a generalization of Lie algebras.

The principle of relativity and the Poincaré group (Chapter 17)

The discovery of the relativity of time is like a surprising puzzle. In 1886, Poincaré became interested in the philosophical work of Auguste Calinon on the foundations of mechanics, and in particular on the quantitative notion of time; Poincaré also emphasized a qualitative problem, that of simultaneity. In 1893, he entered the Bureau des Longitudes. Measuring a longitude amounted to determining the Paris (or Greenwich) time to compare it with the local time: at the time, one used for this a telegraph by submarine cables.[26] The proximity between the philosophical problem and the practical problem led him, in 1898, to a philosophical article entitled *La mesure du temps* [*The measurement of time*], in which Poincaré explains that the notion of simultaneity for distant phenomena presupposes the choice of a procedure of synchronization by exchanging signals. In 1900, he perceived that the formal time variable that appears in the formulas of Lorentz transformations has a very simple physical interpretation: it is the time that one measures in the given reference frame if one synchronizes the watches in

[26]Einstein would also be confronted with practical problems of synchronization: having entered the Patent Office of Bern in 1902, he there had to assess the patent requests concerning electrical synchronization of city clocks. See Peter Galison's book *Einstein's Clocks, Poincaré's Maps: Empires of Time*, 2003.

it by an exchange of signals at the speed of light in a vacuum, ignoring the motion of the reference frame. Moreover, from 1895 Poincaré had been led (mainly by analysis of the Michelson–Morley experiment) to extend the principle of relativity to electromagnetism and even to all of physics. In 1904–1905, he obtained results that would be recorded in his article entitled *Sur la dynamique de l'électron*: the Lorentz transformations form a "continuous group" (a Lie group), whose Lie algebra he writes; he derives the formula for the addition of velocities, the transformations of the electromagnetic field and of the sources, the invariance of $\vec{E}^2 - \vec{B}^2$, $\vec{E} \cdot \vec{B}$, and $x^2 + y^2 + z^2 - c^2t^2$. He suggests the idea that by replacing time with an imaginary variable the Lorentz transformations can be interpreted as rotations in a four-dimensional space; the idea that the principle of relativity, being valid for all of physics, imposes a modification of the law of gravitation, while also limiting the choices; the application to the problem of the precession of Mercury's perihelion (but these corrections from special relativity only explain a part of this); the idea of gravitational waves, citing as a potentially observable consequence the acceleration of the orbital motions of celestial bodies.[27]

Chapter 17 presents some of these contributions by Poincaré. It gives a proof (due to Jean-Marc Lévy-Leblond) of the formulas for Lorentz transformations, under very simple and general hypotheses: space is homogeneous and isotropic, the transformations form a group (the Poincaré condition), and there exist causal chains. It then explains the role of the Poincaré group (group of space-time displacements) in quantum physics (Wigner's theorem) and mentions the fact that it is essentially the only possible symmetry group for the quantum theory of fields (theorems of Coleman–Mandula and Haag–Łopuszański–Sohnius). Finally, it presents an argument explaining why it is natural that Galilean electromagnetism was not discovered before that of special relativity.

Applied physics (Chapter 18)

Poincaré was interested in many questions of applied physics: telegraphy (he solved the telegraph equation in 1893, and from 1904 to 1910 he taught at the École Professionnelle des Postes et Télégraphes), electrotechnics, propagation of waves... To explain the phenomena observed in Gouy's experiment on polarization by diffraction, he made the first correct asymptotic calculation of diffraction by a screen with a sharp edge (1892, 1896).

In 1909, he calculated the diffraction of a Hertzian wave by the Earth: he showed that the amplitude decreases exponentially in the shadow zone, which agreed with the hypothesis that the waves transmitted by Marconi from one end of the Atlantic to the other were not transmitted by diffraction but by reflections on the ocean and the ionosphere. Poincaré's calculation inaugurated a new field of research, that of exponentially small effects in the solutions of differential equations.

Chapter 18 presents Poincaré's calculation, then Fock's subsequent method, confirming Poincaré's result; it then describes the method of Kruskal and Segur (1985)—which (in contrast with those of Poincaré and Fock) does not require first obtaining an explicit expression for the solutions—and it illustrates this method by

[27] It was actually by this means (acceleration of the orbital motions in a binary pulsar) that the existence of gravitational waves was verified, which earned the Nobel Prize in physics for Joseph H. Taylor and Russell A. Hulse in 1993. But the correct calculations result from Einstein's general theory of relativity.

an example (the Korteweg–de Vries wave equation, perturbed by a small term of order 5).

Philosophy of science (Chapter 19)

Poincaré seems to have always been interested in the philosophy of science. The unexpected involvement of hyperbolic geometry in his research on automorphic functions (1880) could only reinforce his interest in the foundations of geometry: questions of Euclidean geometry concerning systems of curvilinear triangles were interpreted simply in terms of hyperbolic geometry, and this interpretation allowed him to find the solution. In such cases, there is every interest in taking the point of view of hyperbolic geometry, not because it is more "true" than Euclidean geometry, but because it is more *convenient* (*"we can no more say that Euclidean geometry is true and the geometry of Lobachevsky false* [or the converse]*, than we could say that Cartesian coordinates are true and polar coordinates false"*[28]). On the other hand, when one studies the motions of solid bodies (beginning with the motions of our own limbs), the most convenient description uses the language of Euclidean geometry, for *"experience teaches us that the various possible motions of these bodies are connected almost exactly by the same relations"* as Euclidean displacements;[29] *"we can state mechanical facts by relating them to a non-Euclidean space, which would be a less convenient coordinate system, but one just as legitimate as our ordinary space; the statement would thus become much more complicated, but it would remain possible."*[30] This is what would be done by beings who had been educated in a world where the most convenient geometry was non-Euclidean, and who had been transported into our world. (Chapters III and IV of *Science and Hypothesis* present a suggestive illustration of this.) Thus, the fundamental hypotheses of geometry are neither inevitable frameworks of thought nor experimental facts: they are convenient choices, taking observations into account.

The period was also a time of critically examining the foundations of mechanics: Mach's *Mechanik* appeared in 1883, Calinon's *Étude critique sur la mécanique* in 1885... Poincaré was closely interested. (Later, in 1897 and 1901, he would publish two articles on the foundations of mechanics, whose essential points he would repeat in *Science and Hypothesis* in 1902.) From mechanics, he passed to all of physics. In his teaching of physics, each time he would draw up a *"tableau d'ensemble"* of all the available theories; for *"it would be dangerous to limit oneself to one of them; one would thus risk feeling a blind, and consequently misleading, confidence in one's place. One must therefore study all of them, and it is their comparison that may above all be instructive."*[31] This obviously does not exclude pruning them; one should only preserve the *"fruitful"* hypotheses.

Poincaré also extended his reflections to mathematical analysis, arithmetic, and logic. Poincaré started publishing in the *Revue de métaphysique et de morale* in 1893 (the date of the journal's creation) and published one or two articles in it nearly every year until the end of his life. The essence of these articles (and

[28] Poincaré, *Sur les hypothèses fondamentales de la géométrie* [*On the Fundamental Hypotheses of Geometry*], 1887.

[29] Ibid.

[30] *Science and Hypothesis*, Chapter VI.

[31] Preface to the *Théorie mathématique de la lumière* [*Mathematical theory of light*], course from the year 1887–1888, published in 1889.

others, published elsewhere) would form the material of his famous collections on the philosophy of science (already mentioned above), published starting in 1902.

Chapter 19 proposes a survey of Poincaré's philosophy, which Gerhard Heinzmann justly calls *pragmatic occasionalism*, since according to this point of view, experience is for our minds only the occasion to exercise its faculties and to become aware of this, and because its criterion is pragmatic (one only chooses the fruitful hypotheses). It explains and illustrates this philosophy successively in arithmetic, in geometry, and in physics. It also presents Poincaré's view on logic.

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